The convolution theorem

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Reference table in notes

TABLE 11.1: *Some useful Fourier transform pairs.*

Function	Fourier transform	Tag
$f(x,y)$	$\int\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+vy)}dxdy = \mathcal{F}(f)(u,v)$	1
$\int\int_{-\infty}^{\infty} \mathcal{F}(f)(u,v)e^{i2\pi(ux+vy)}dudv = f(x,y)$	$\mathcal{F}(f)(u,v)$	2
$\frac{\partial f}{\partial x}(x,y)$	$u\mathcal{F}(f)(u,v)$	3
$0.5\delta(x+a,y)+0.5\delta(x-a,y)$	$\cos 2\pi au$	4
$\cos 2\pi ax$	$0.5\delta(u+a,v)+0.5\delta(u-a,v)$	5
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$	6
$box_2(x,y)$	$\frac{\sin u}{u}\frac{\sin v}{v}$	7
$f(ax,by)$	$\frac{\mathcal{F}(f)(u/a,v/b)}{ab}$	8
$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(x-i,y-j)$	$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(u-i,v-j)$	9
$f(x-a,y-b)$	$e^{-i2\pi(au+bv)}\mathcal{F}(f)$	10
$f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$	$\mathcal{F}(f)(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta)$	11
$(f * g)(x,y)$	$\mathcal{F}(f)\mathcal{F}(g)(u,v)$	12

The convolution theorem

- **Convolution** in the spatial domain is **multiplication** in the frequency domain (and vice versa)

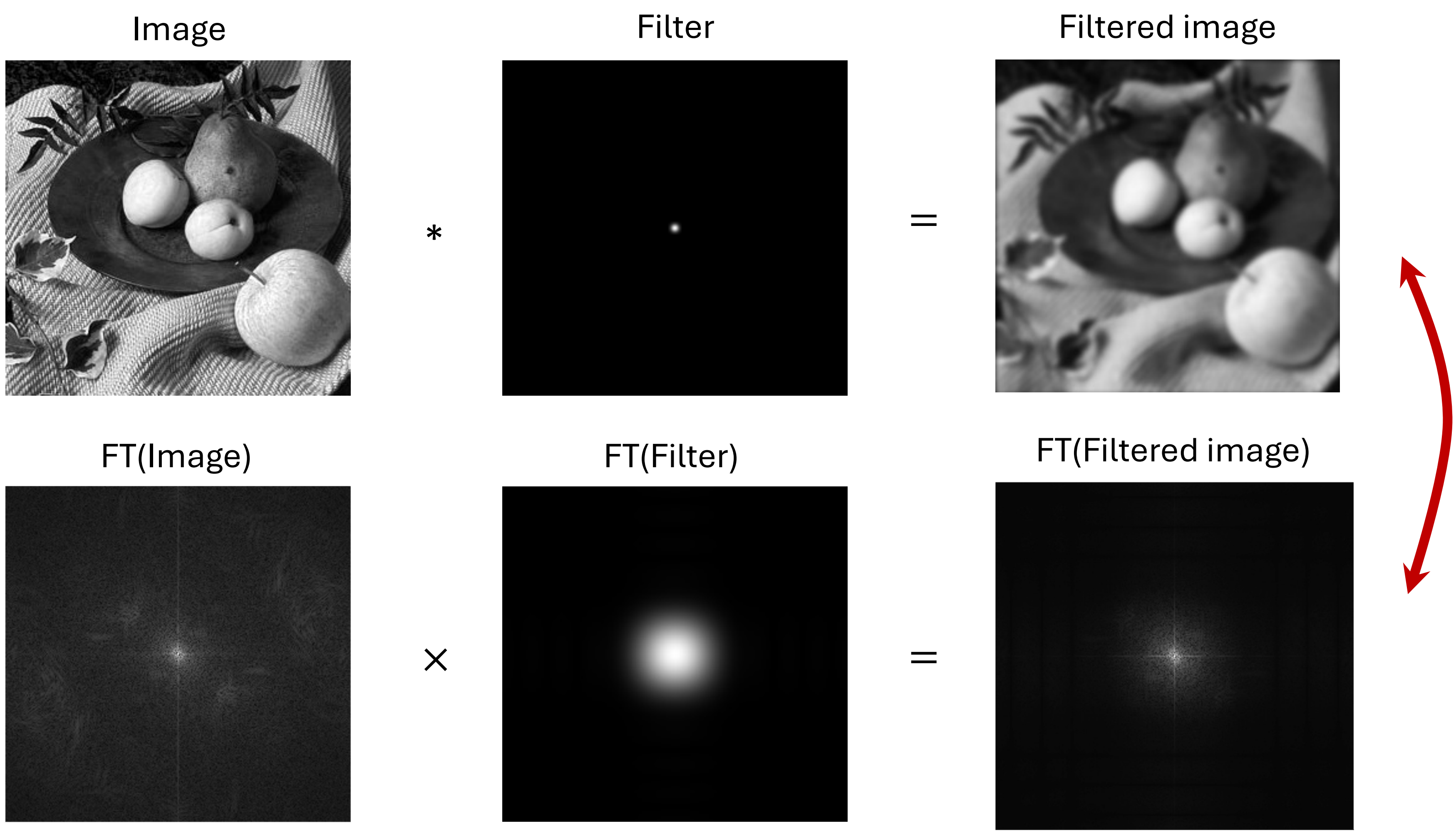
- The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{FG\} = \mathcal{F}^{-1}\{F\} * \mathcal{F}^{-1}\{G\}$$

2D convolution theorem example



Easy to prove

$$
\begin{aligned}
\mathcal{F}(f * g) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \right) \exp\left[-i2\pi u t\right] dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) g(\tau) \right) \exp\left[-i2\pi u t\right] dt d\tau \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \exp\left[-i2\pi u t\right] dt g(\tau) \right) d\tau \\
&= \int_{-\infty}^{\infty} \left[\mathcal{F}(f)\right] \exp\left[-i2\pi u \tau\right] g(\tau) d\tau \\
&= \left[\mathcal{F}(f)\right] \int_{-\infty}^{\infty} g(\tau) \exp\left[-i2\pi u \tau\right] d\tau \\
&= \left[\mathcal{F}(f)\right] \left[\mathcal{F}(g)\right].
\end{aligned}
$$

Swap integration bounds

Move g out of integral

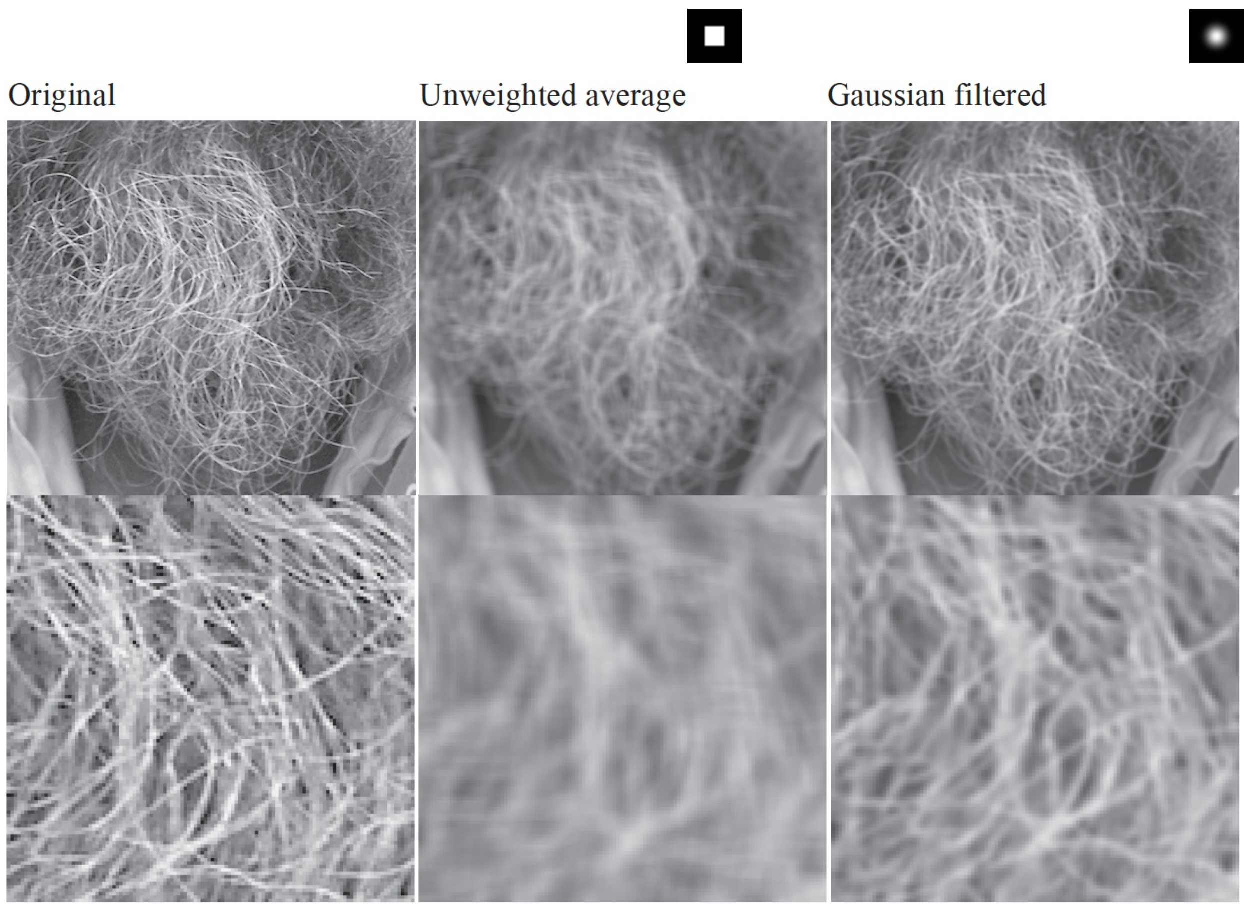
Shift result from before

Convolution theorem

- Suppose f and g both consist of N pixels
- What is the complexity of computing $f * g$ in the spatial domain?
  - $O(N^2)$
- And what is the complexity of computing $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\mathcal{F}\{g\}\}$?
  - $O(N \log N)$ using FFT
- Thus, convolution of an image with a large filter can be more efficiently done in the frequency domain

Mystery 1

Why does filtering with a Gaussian give a nice smooth image, but filtering with a box filter gives artifacts?



Gaussian case

- Convolution theorem:

  FT (filtered) = FT(gaussian)FT(image)

- FT(gaussian)=another gaussian

- Weight falls off smoothly at higher frequencies

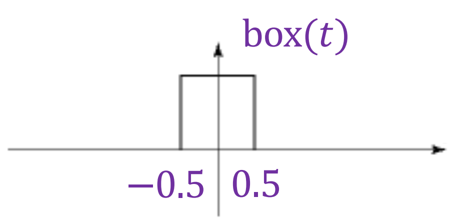
Box filter

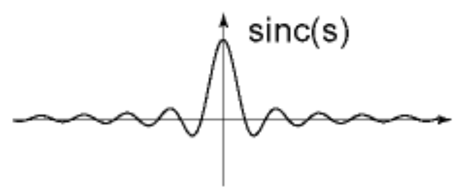
- Convolution theorem:

FT (filtered)= FT(box)*FT(image)

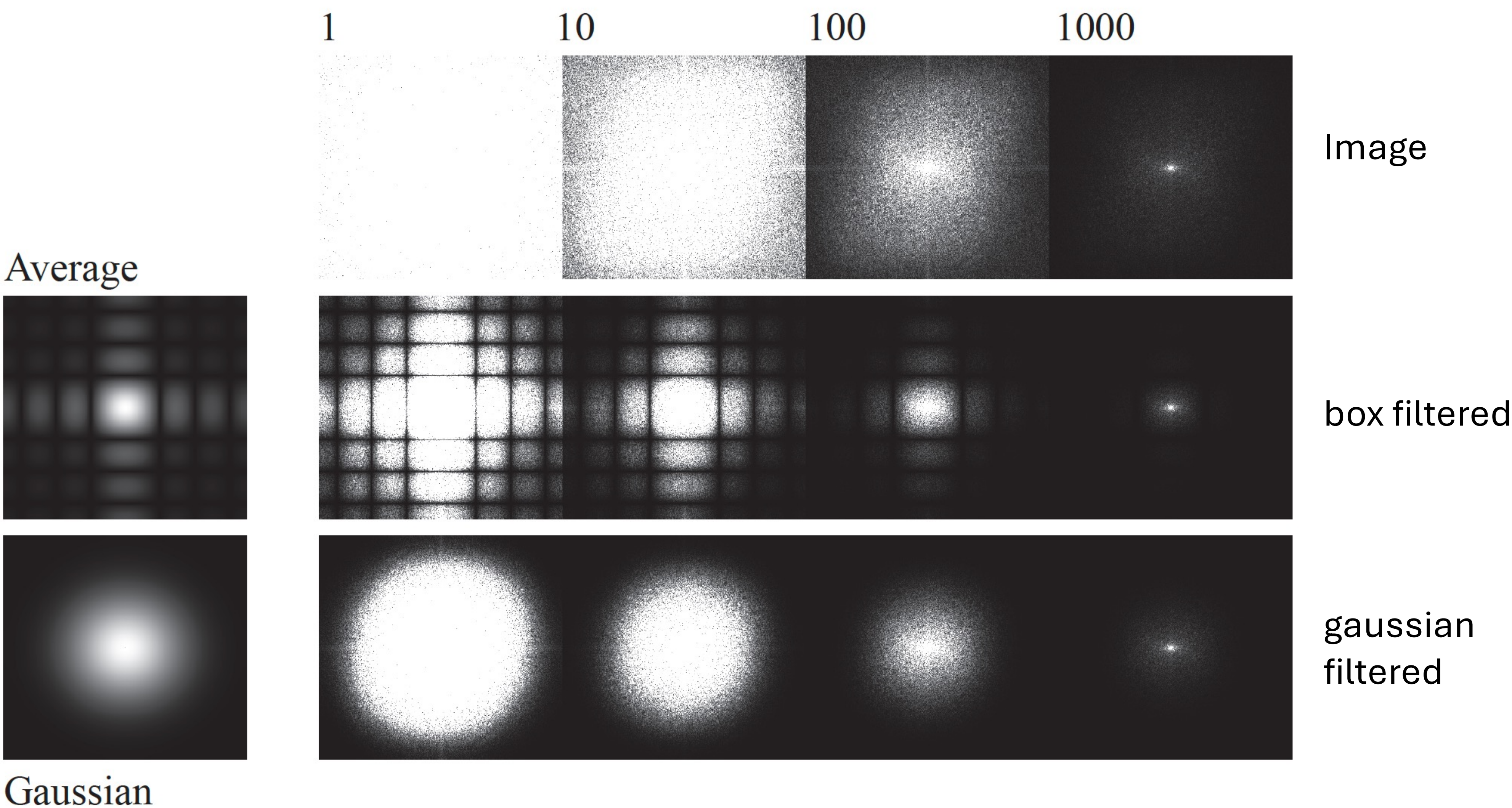
- FT(box)=Sinc

$$\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$



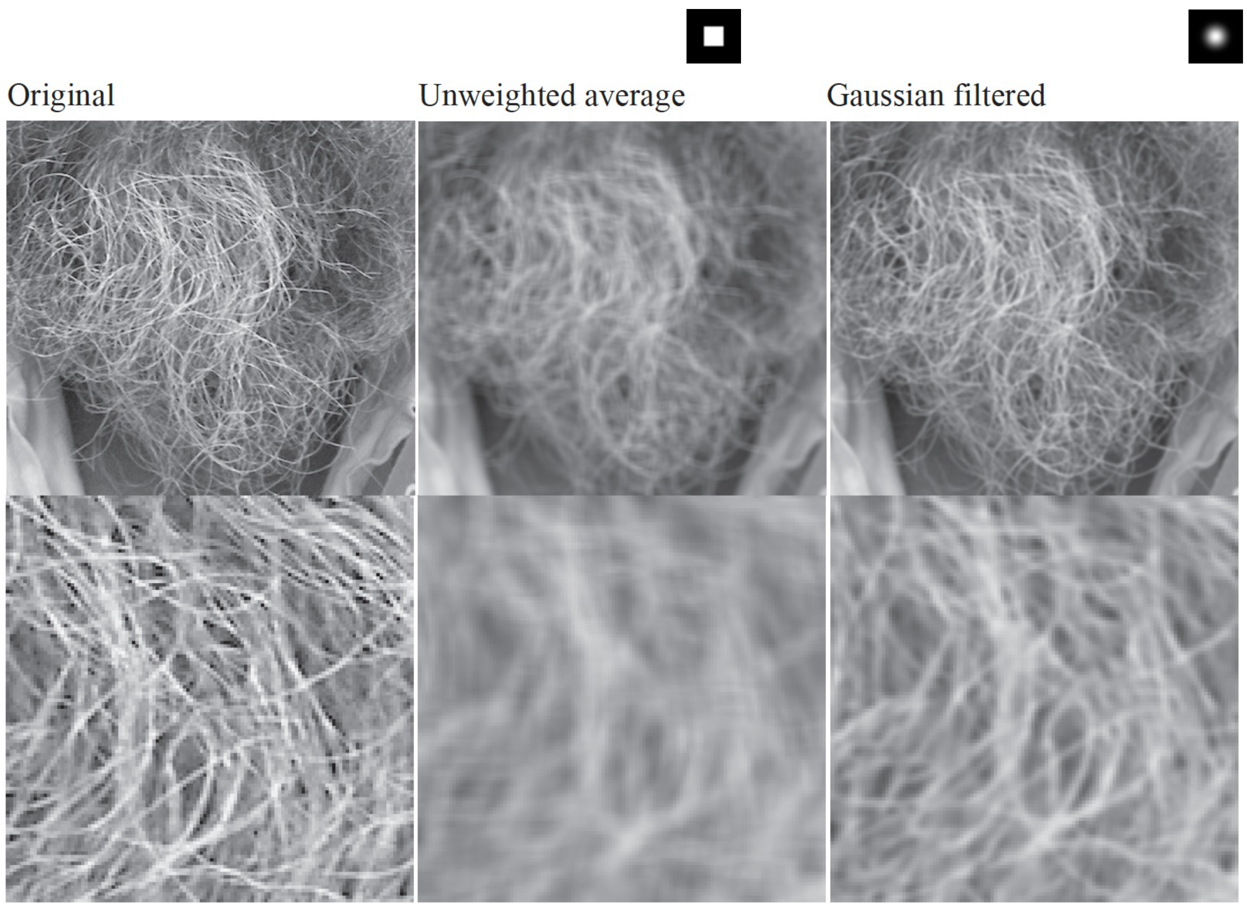


Magnitudes


1
10
100
1000
Image
Average
box filtered
gaussian
filtered
Gaussian

Mystery 1, Solved

Why does filtering with a Gaussian give a nice smooth image, but filtering with a box filter gives artifacts?



Things to think about...

12.1. Is $f * g * h$ the same as $g * f * h$? (use the convolution theorem).

12.2. Convolution in the Fourier domain is equivalent to what in the signal domain?

12.3. Section 12.1 has: "But if an image is going to be heavily smoothed, it will lose a lot of detail, and the detailed form of the smoother might not matter much. " Explain.

12.4. Section 12.1 has: "Imagine you have a filter $f(x, y)$ that detects a small pattern. Then (say) $f(x/10, y/10)$ will detect a larger version of this pattern." Explain.

12.5. Finding a pattern in a smoothed and downsampled version of the image is largely equivalent to finding a large version of the pattern in the original image. Explain.

12.6. Will ringing affect a gradient estimate?