Quick and dirty Fourier theory - II

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1D Fourier transform

- Define a set of basis functions:

$$\psi_u(t) = e^{i2\pi ut}, \qquad u \in (-\infty, \infty)$$

- Compare

Integer

$$e^{i2k\pi t} = \cos(2k\pi t) + i\sin(2k\pi t)$$

1D Fourier transform

- Define a set of basis functions:

$$\psi_u(t) = e^{i2\pi ut}, \qquad u \in (-\infty, \infty)$$

- Inner product for complex functions is given by:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dt$$

- Orthonormality:

$$\langle \psi_{u_1}, \psi_{u_2} \rangle = \delta(u_1 - u_2) = \begin{cases} ? & \text{if} \quad u_1 = u_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t) dt = f(0)$$

1D Fourier transform

- Represent $f(t)$ as
  - a weighted combination
    - of the basis functions $\psi_u(t) = e^{i2\pi ut}$ with weights $F(u)$:

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ut} du$$

  - Each weight $F(u)$ given by the inner product of f and ψ_u:

$$F(u) = \langle f, \psi_u \rangle = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

1D Fourier transform

- Forward transform:

$$f(t) \xrightarrow{\mathcal{F}} F(u)$$

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

- Existence for FT's is tricky.
  - if $\int_{-\infty}^{\infty} |f(t)|^2 \, dt$ is finite, FT exists
  - it does exist for many more functions, but machinery gets quite complicated

1D Fourier transform

- Forward transform:

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

- Inverse transform:

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ut} du$$

Fourier transform pairs $f(t) \leftrightarrow F(u)$

1D Fourier transform

- Maps a real function to complex function
  - so has magnitude and phase
  - or real and imaginary components

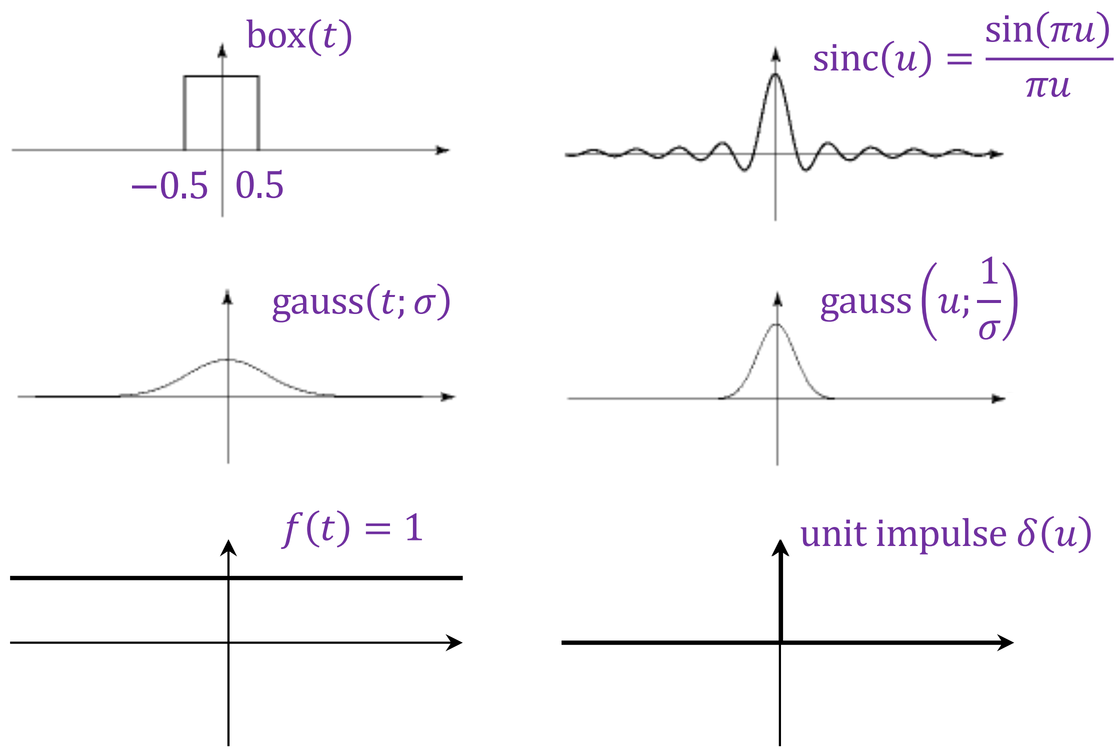
- Important properties:
  - Energy preservation (Parseval's theorem):

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(u)|^2 \, du$$

  - Linearity:

$$\mathcal{F}\{af_1 + bf_2\} = a\mathcal{F}\{f_1\} + b\mathcal{F}\{f_2\}$$

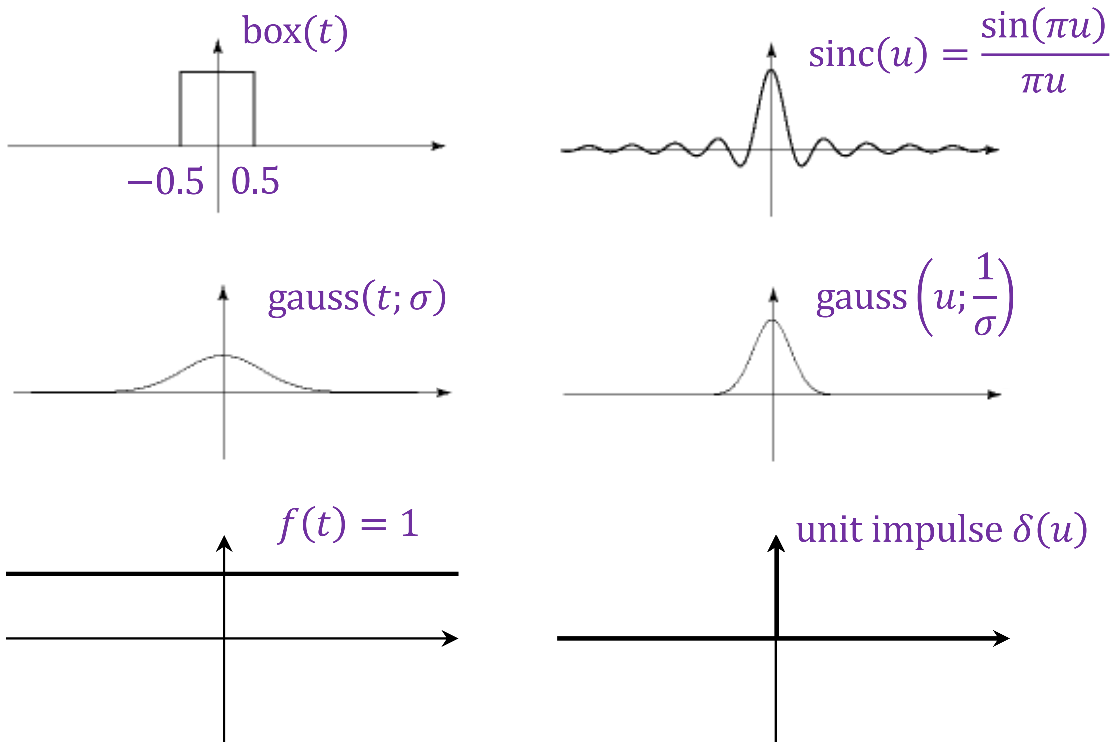
Important Fourier transform pairs



*The last one is formal since these functions don't meet the mathematical requirements for FT

Important Fourier transform pairs

Notice that when
f has narrower support,
FT(f) has broader, and
Vice versa!



*The last one is formal since these functions don’t meet the mathematical requirements for FT

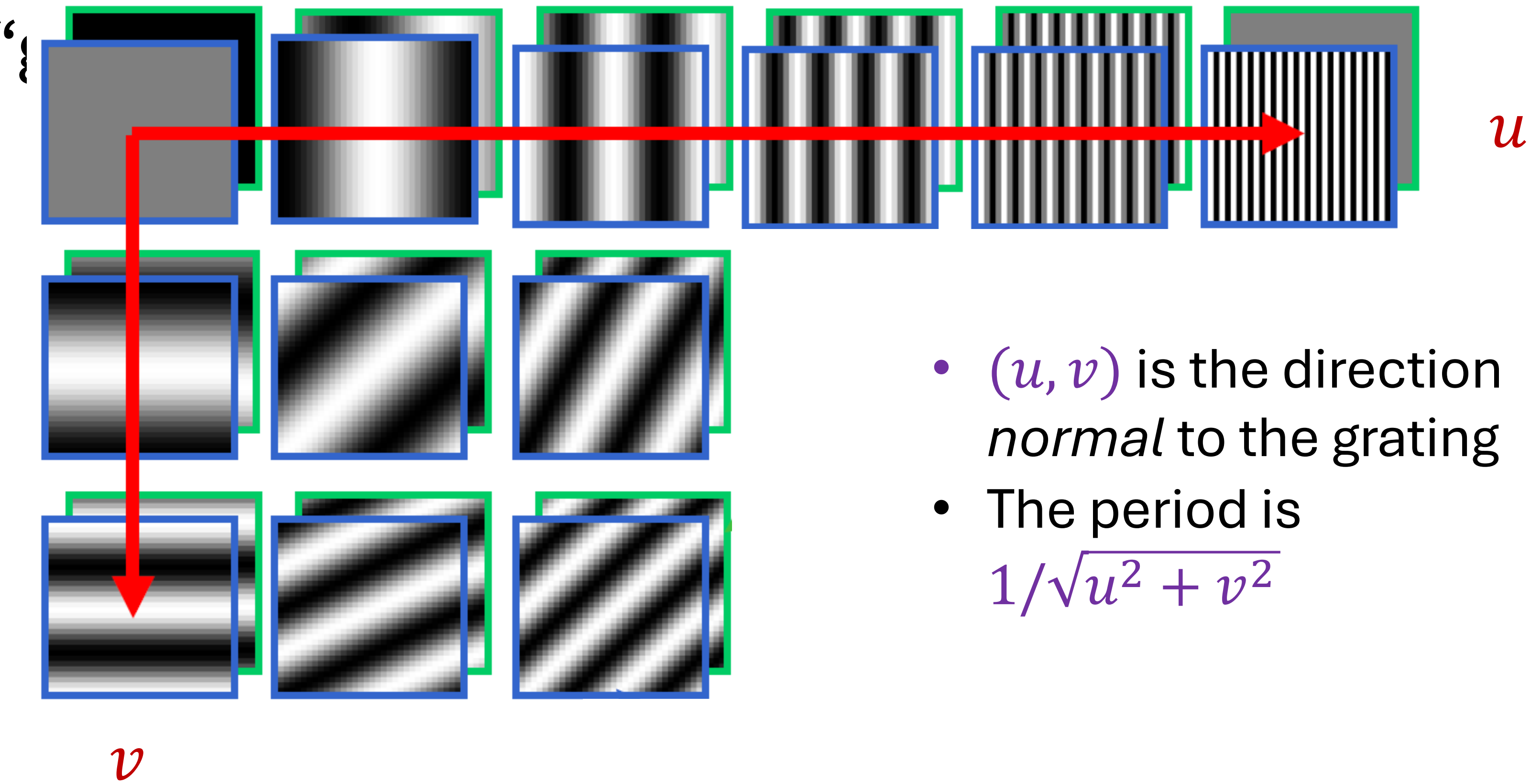
2D Fourier transform

- 2D basis functions:

$$
\begin{aligned}
\psi_{u,v}(x,y) &= e^{i2\pi ux} e^{i2\pi vy} \\
&= e^{i2\pi(ux+vy)} \\
&= \cos 2\pi(ux+vy) + \\
&\quad i \sin 2\pi(ux+vy)
\end{aligned}
$$

2D Fourier transform

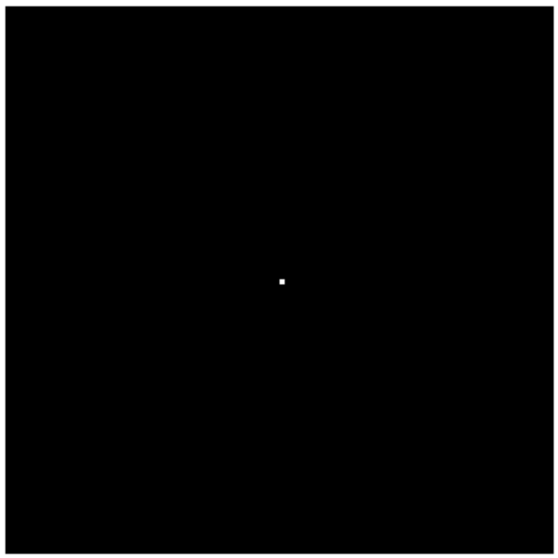
- 2D basis functions are oriented sinusoidal "



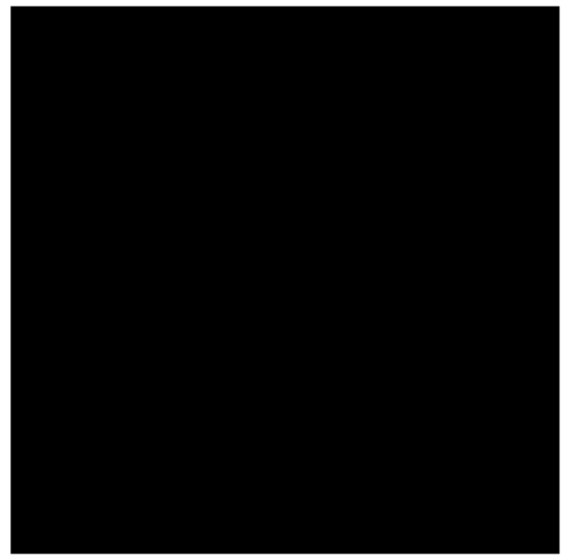
- (u, v) is the direction *normal* to the grating
- The period is $1/\sqrt{u^2 + v^2}$

Basis function examples

(u, v)

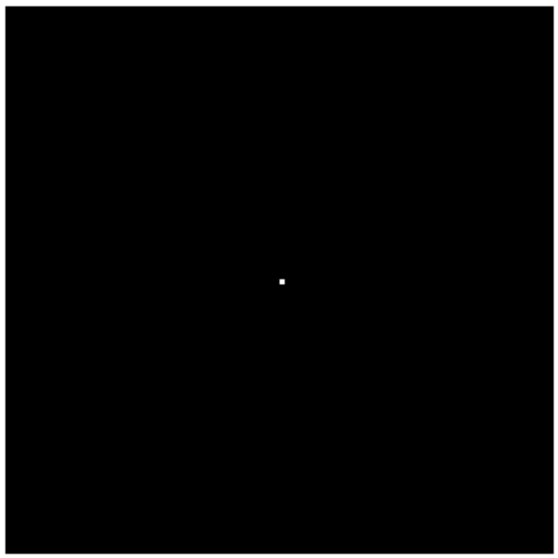


Real component

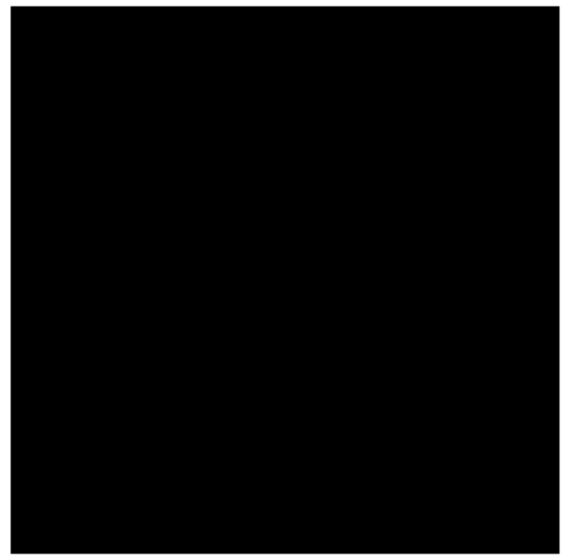


Basis function examples

(u, v)

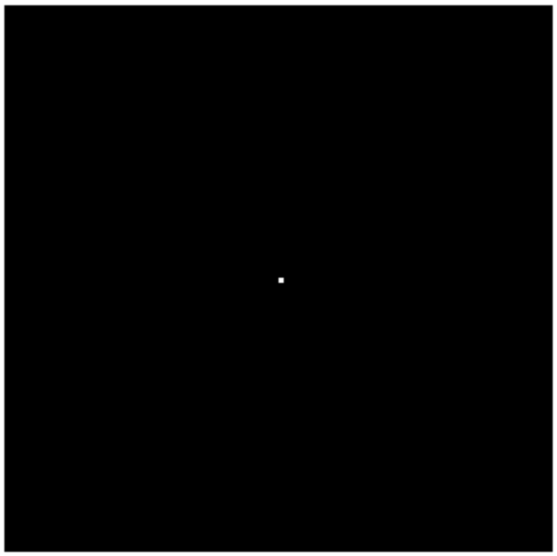


Real component

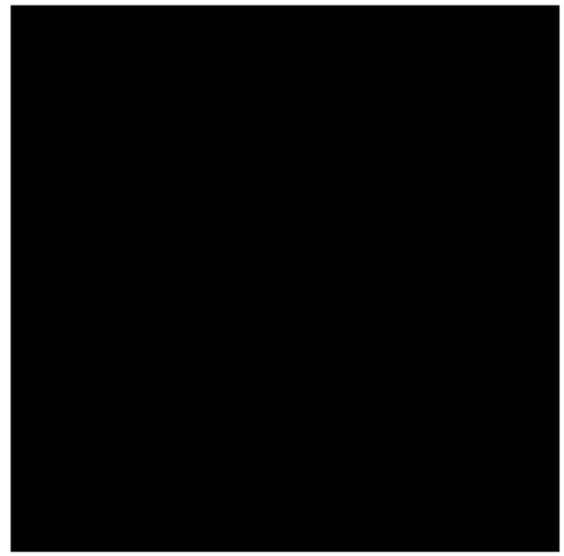


Basis function examples

(u, v)

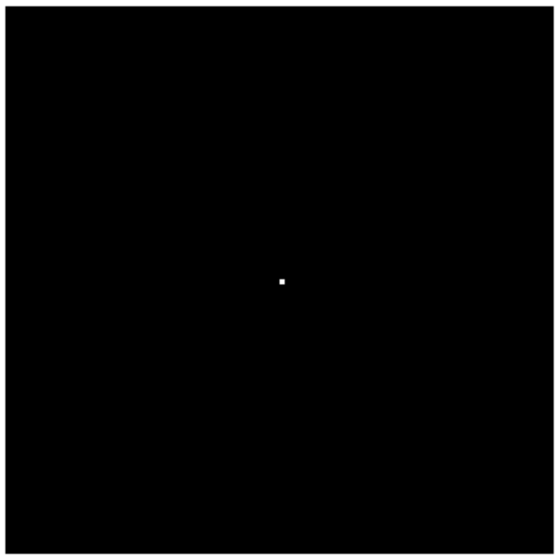


Real component

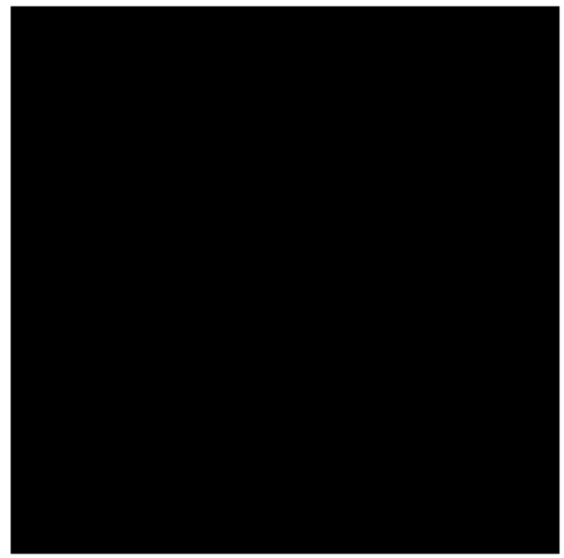


Linear combination of basis functions

(u, v)



Real component



2D Fourier transform

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+vy)}dx\,dy$$

- Output is 2D and complex-valued:

$$F(u,v) = \mathrm{Re}(F(u,v)) + i\,\mathrm{Im}(F(u,v))$$

  - Magnitude spectrum: $|F(u,v)| = \sqrt{\mathrm{Re}(F(u,v))^2 + \mathrm{Im}(F(u,v))^2}$
  - Phase angle spectrum: $\tan^{-1}\frac{\mathrm{Im}(F(u,v))}{\mathrm{Re}(F(u,v))}$

2D Fourier transform

- This is a linear operator applied to the function

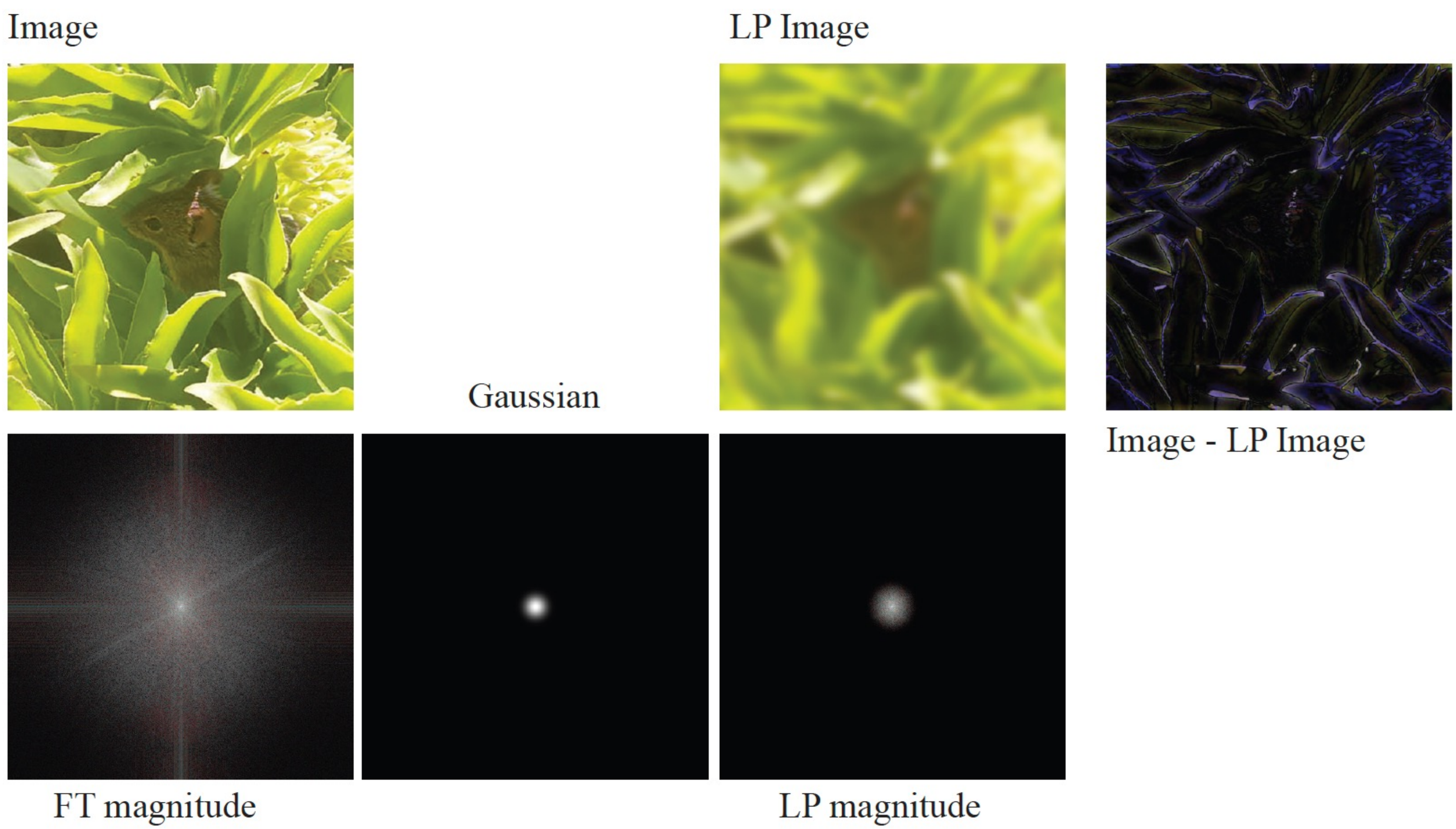
- Can get discrete approximation by:
  - discretize a 2D function
  - straighten into a vector
  - multiply by appropriate matrix
  - unstraighten vector

- Matrix comes from basis functions
  - matrix multiplication can be made fast
    - FFT=Fast Fourier Transform

Trick – low pass filter

- Multiply FT magnitude by Gaussian
- Inverse FT
- High frequencies are suppressed

Smoothing by FT



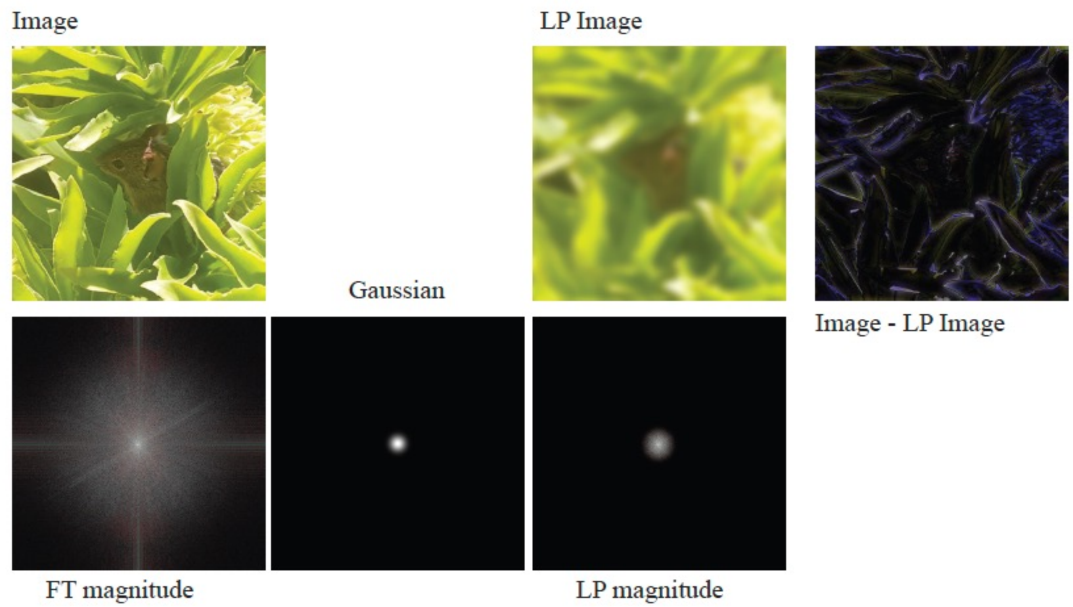
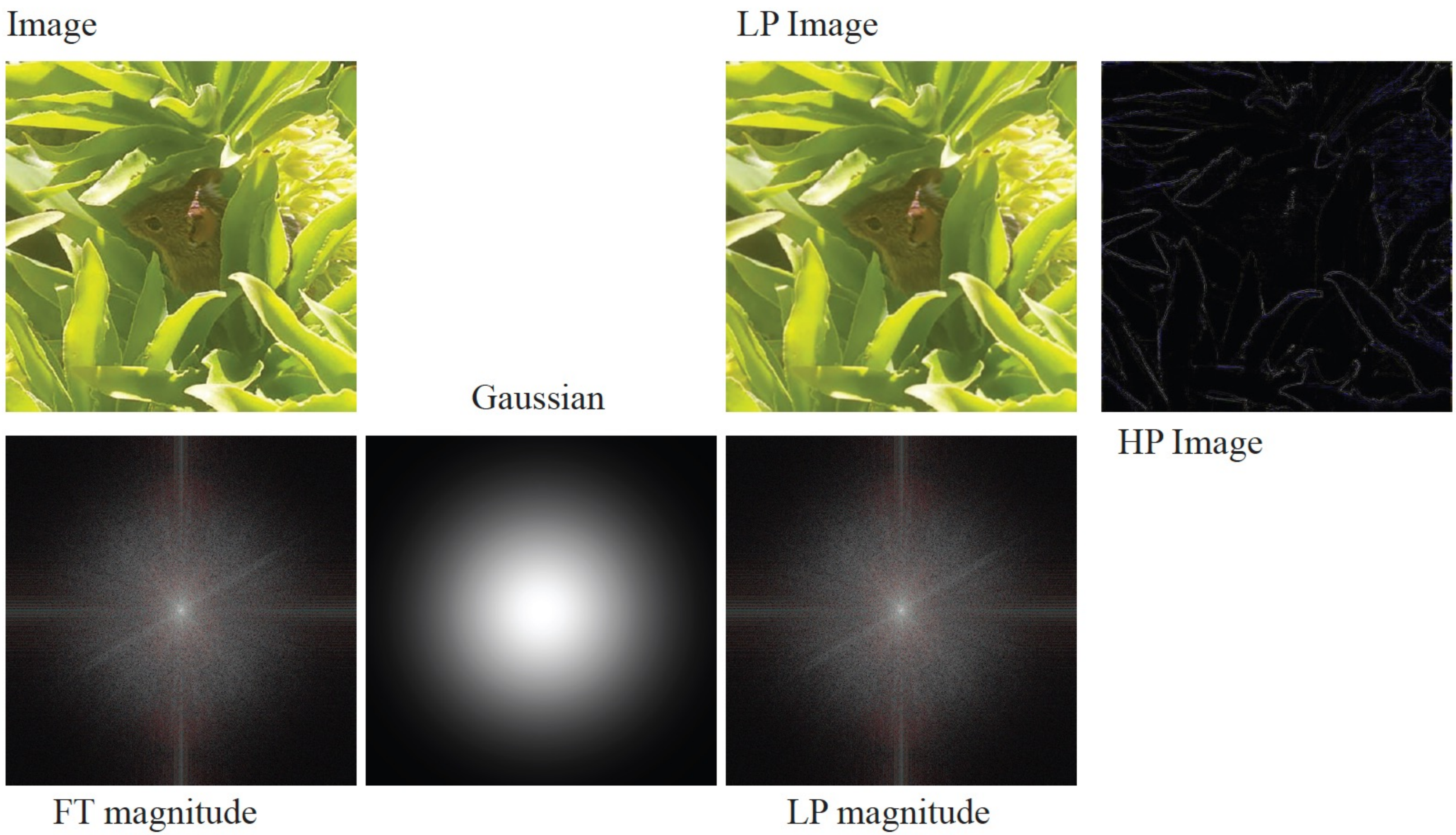


FIGURE 6.2: *On the* **top left**, *the image of a four striped grass mouse with the log magnitude of its Fourier transform on the* **bottom left**. **Center left**, *the gaussian with* $\sigma = 10$ *in* u, v *space. This is multiplied by the weights, and the log magnitude of the result appears* **center right**. **Above** *this is the image obtained by inverting the Fourier transform – equivalently, the low pass filtered image.* **Far left** *shows the high pass filtered image, obtained by subtracting the low pass filtered image from the original. I have* not *shown the log magnitude of the high pass filtered image, because scaling makes the result quite difficult to interpret (it doesn't look filtered). The low pass filtered version is heavily blurred, because only the lowest spatial frequencies appear in the result. Note the high pass filtered version contains what is missing from the low pass version, so has few large values which appear at edges.* Image credit: *Figure shows my photograph, taken at Kirstenbosch and Long Beach respectively.*

Smoothing with FT



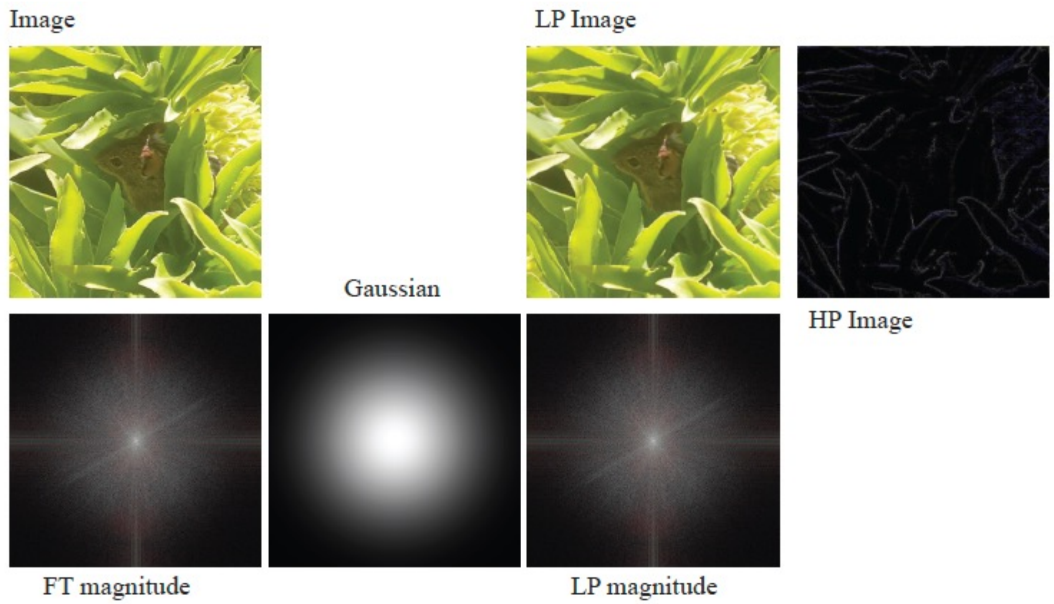


FIGURE 6.3: *On the* **top left**, *the image of a four striped grass mouse with the log magnitude of its Fourier transform on the* **bottom left**. **Center left**, *the gaussian with* $\sigma = 100$ *in* u, v *space. This is multiplied by the weights, and the log magnitude of the result appears* **center right**. **Above** *this is the image obtained by inverting the Fourier transform – equivalently, the low pass filtered image.* **Far left** *shows the high pass filtered image, obtained by subtracting the low pass filtered image from the original. I have* not *shown the log magnitude of the high pass filtered image, because scaling makes the result quite difficult to interpret (it doesn't look filtered). The low pass filtered version is less heavily blurred than that in Figure 6.2, because only the lowest spatial frequencies appear in the result. Note the high pass filtered version contains what is missing from the low pass version, so has very few large values which appear at edges.* Image credit: *Figure shows my photograph, taken at Kirstenbosch and Long Beach respectively.*

Phase vs. magnitude

- Phase from one image, magnitude from another

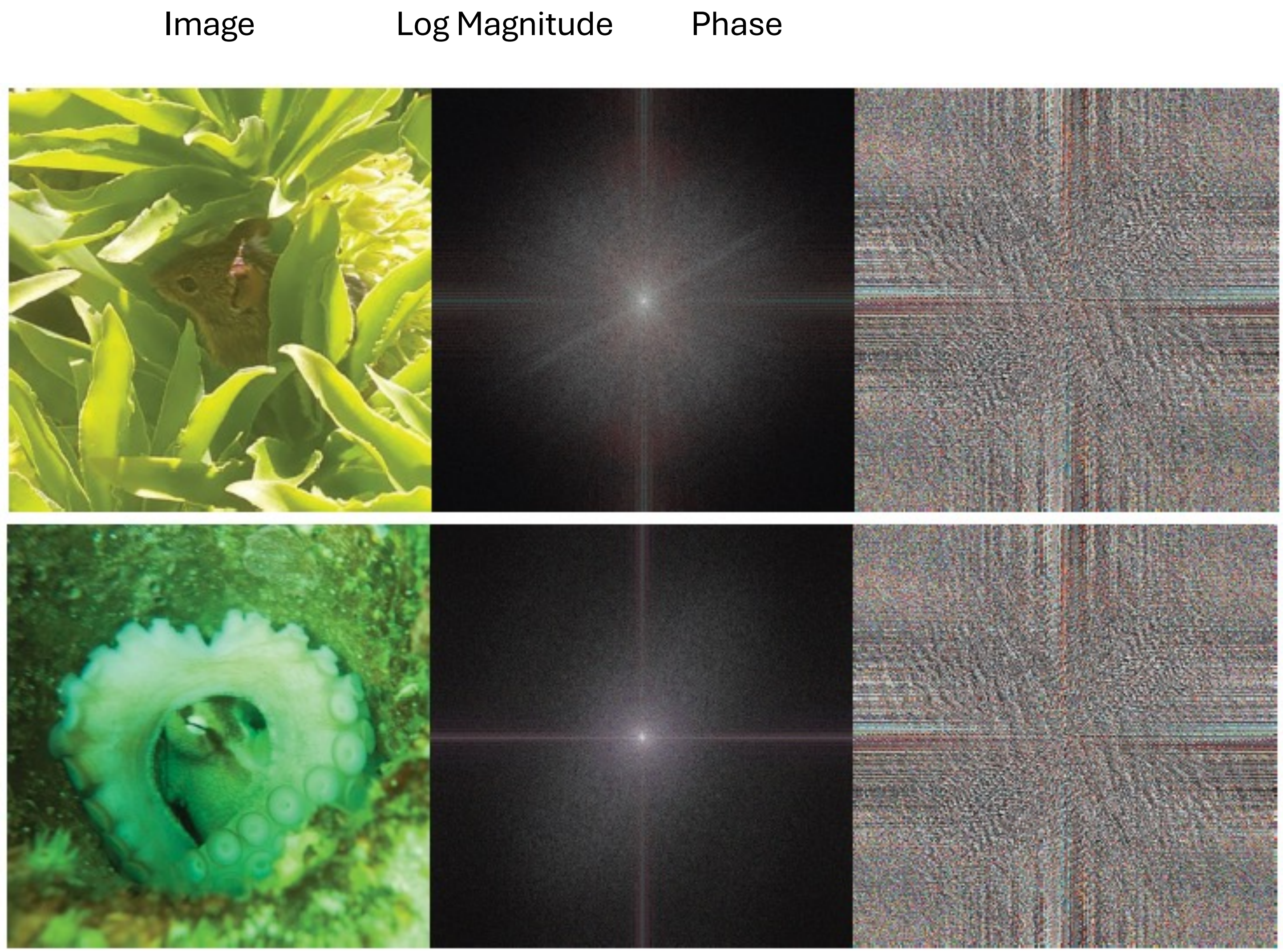
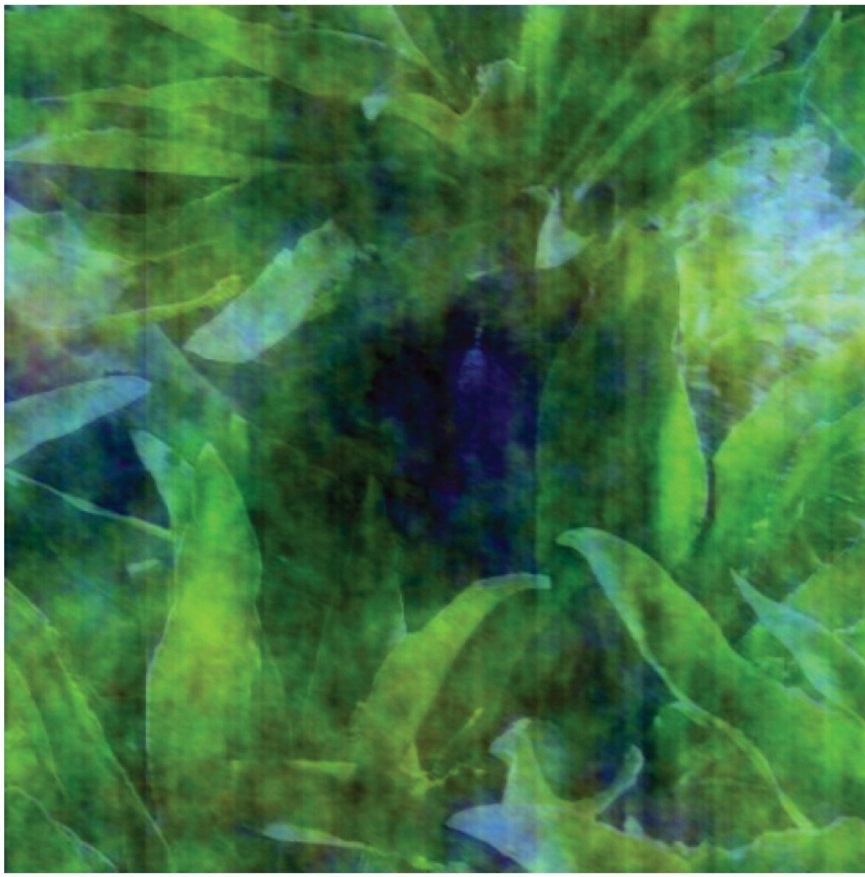
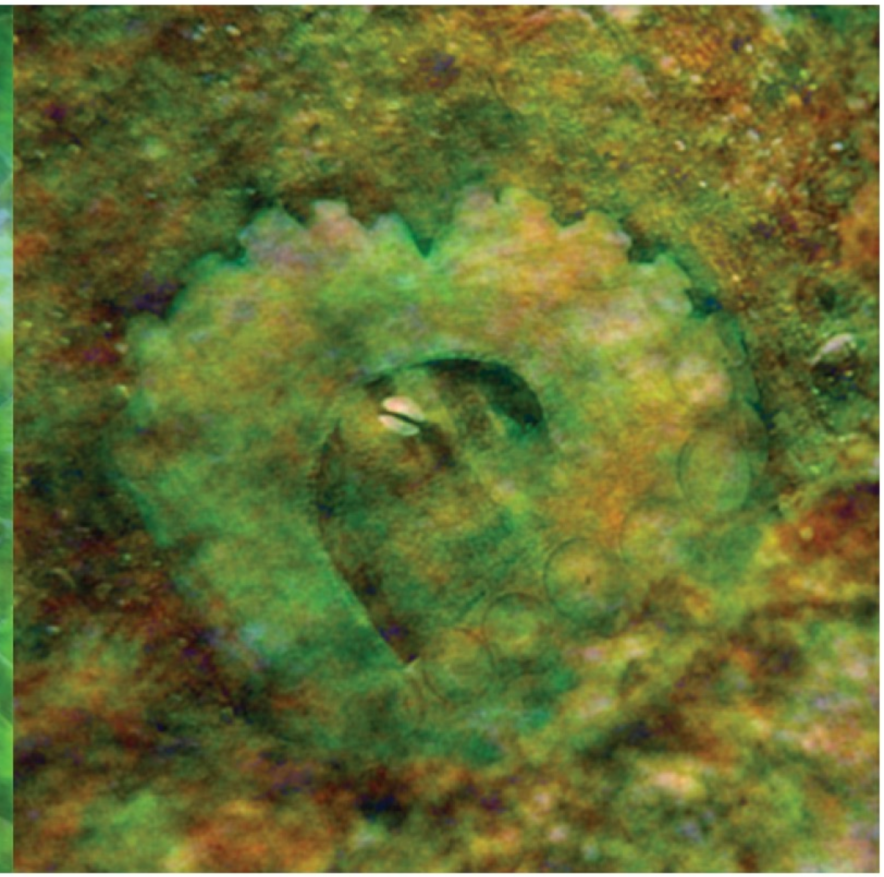


Image
Log Magnitude
Phase

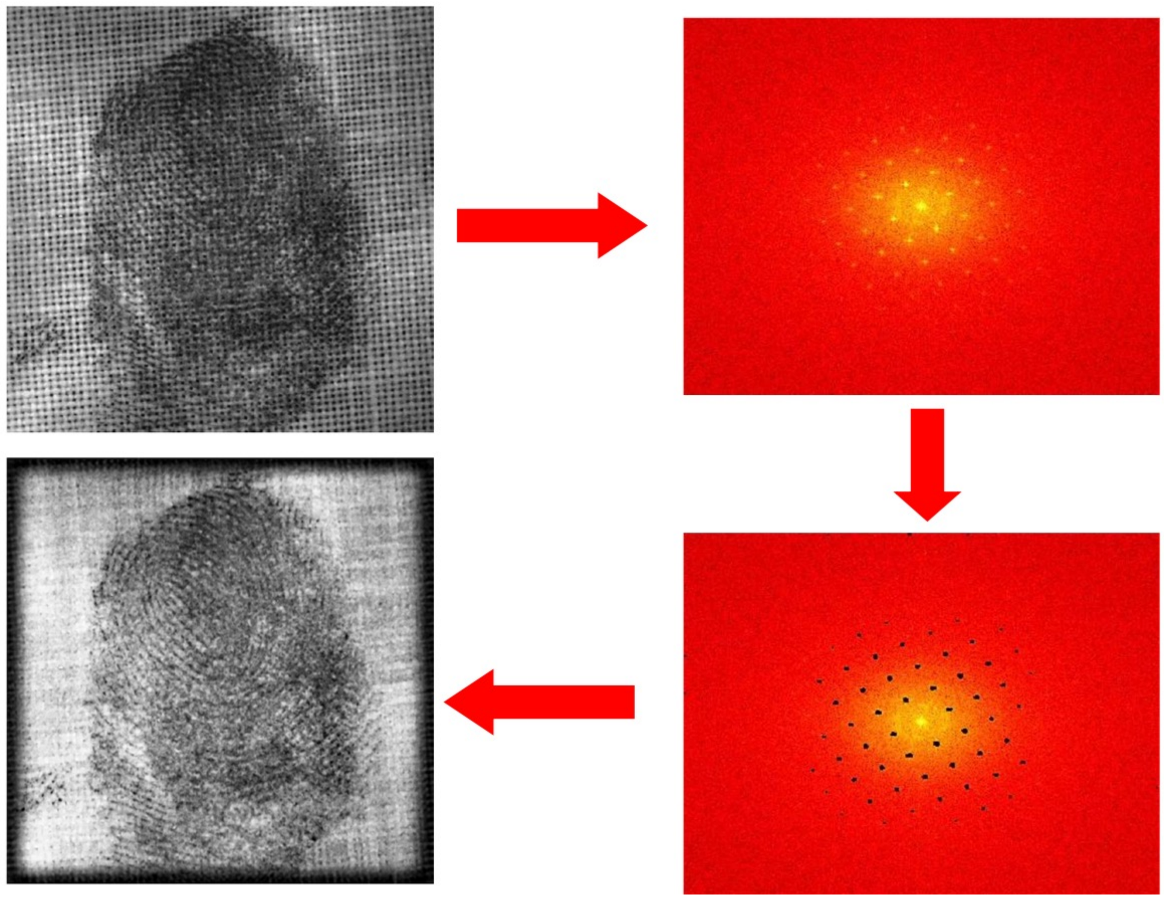
Mouse phase, octopus mag



Octopus phase, mouse mag



Application: Removing periodic patterns



Magnitude image

Remove peaks

Source: A. Zisserman

Important effect

- “wider” function has “narrower” Fourier transform
- “narrower” function has “wider” Fourier transform

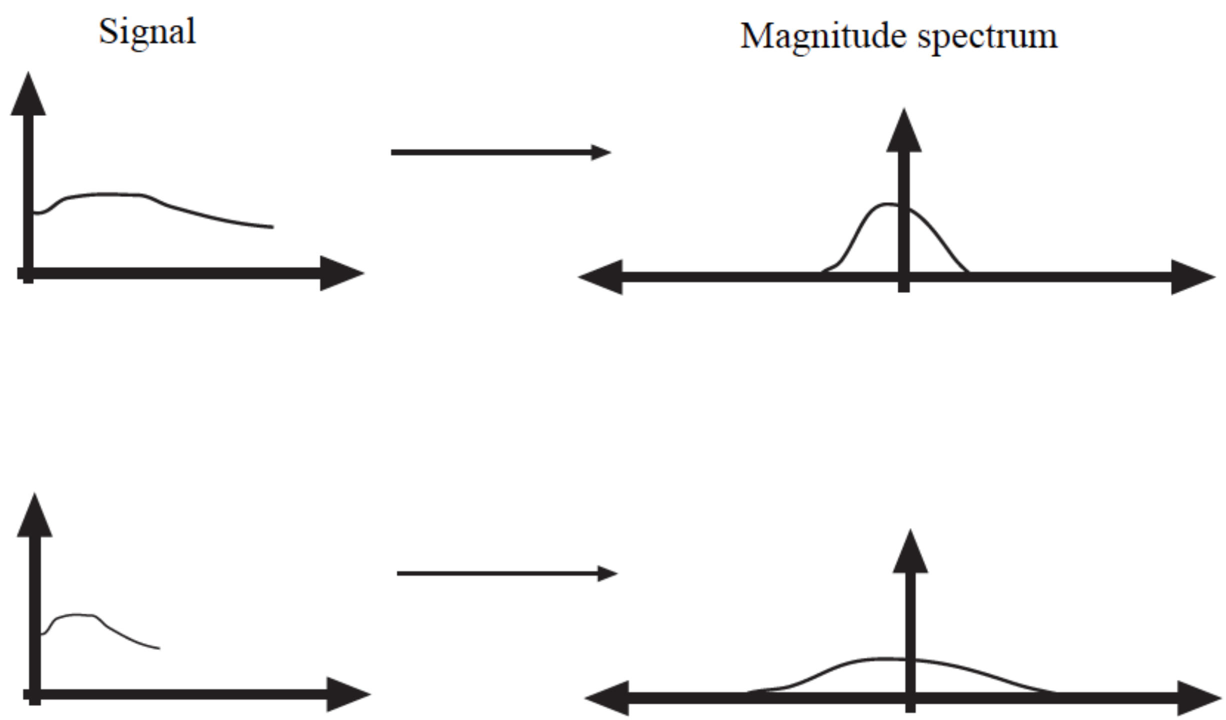


FIGURE 7.1: **Top** *shows* $f(t)$ *and its magnitude spectrum, and* **bottom** $f(2t)$ *and its magnitude spectrum. Notice how narrowing the function broadens the Fourier transform (from top to bottom); or broadening it narrows the Fourier transform (from bottom to top).*

Things to think about...

11.6. Section 11.2.1 describes the real and complex components as $e^{i2\pi(ux+vy)}$ sinusoids on the x, y plane. Check that each term is constant when $ux + vy$ is constant.

11.7. Section 11.2.1 describes the real and complex components as $e^{i2\pi(ux+vy)}$ sinusoids on the x, y plane. Check that the frequency of each sinusoid is $\sqrt{u^2 + v^2}$.

11.8. In Section 11.2.4, I say the fact that the magnitude spectra of images tends to be similar is related to the property that pixels mostly look like their neighbors. Explain this relationship briefly.

11.9. In Section 11.2.6, I say: "change one pixel in an image, and you change the whole Fourier transform". Explain.