Geometric transformations

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Notation

There are a number of important and useful geometric transformations of the plane that can be applied to images. Image transformations are implemented in the same way as subsampling: by scanning the pixels of the target and modifying them using interpolates of pixels from the source. This means it is important that transformations are invertible. Adopt the convention that a point $\mathbf{x} = (x, y)$ is mapped by a transformation to the point $\mathbf{u} = (u, v) = (u(x, y), v(x, y))$, and $\mathbf{u} = (u, v)$ is mapped to $\mathbf{x} = (x, y)$ by the inverse. In vector notation, $\mathbf{x}$ is mapped to $\mathbf{u}$, and so on. Write $\mathbf{A}$ for a 2×2 matrix, whose i, j'th component is a_{ij}.

Translation

Definition: 4.1 *Translation*

Translation maps the point (x, y) to the point $(u, v) = (x + t_x, y + t_y)$ for two constants t_x and t_y. Here $(x, y) = (u - t_x, v - t_y)$. In vector notation,

$$\mathbf{u} = \mathbf{x} + \mathbf{t} \text{ and } \mathbf{x} = \mathbf{u} - \mathbf{t}.$$

Useful Fact: *Translation preserves lengths and angles. Choose two points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*. The squared distance from* $\mathbf{x}_1$ *to* $\mathbf{x}_2$ *is* $(\mathbf{x}_1 - \mathbf{x}_2)^T(\mathbf{x}_1 - \mathbf{x}_2)$*; but for a translation* $(\mathbf{u}_1 - \mathbf{u}_2) = (\mathbf{x}_1 - \mathbf{x}_2)$*. A similar argument shows that angles are preserved (***exercises** *).*

Definition: 4.2 *Rotation*

Rotation takes the point (x, y) to the point

$$(u, v) = x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta.$$

Here θ is the angle of rotation, rotation is anti-clockwise, and

$$(x, y) = u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta.$$

Write $\mathcal{R}$ for a 2×2 rotation matrix (a matrix where $\mathcal{R}^T\mathcal{R} = \mathcal{I}$ and $\det(\mathcal{R}) = 1$); then

$$\mathbf{u} = \mathcal{R}\mathbf{x} \text{ and } \mathbf{x} = \mathcal{R}^{-1}\mathbf{u} = \mathcal{R}^T\mathbf{u}.$$

Useful Fact: *Rotation preserves lengths and angles. Choose two points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*. The squared distance from* $\mathbf{x}_1$ *to* $\mathbf{x}_2$ *is* $(\mathbf{x}_1 - \mathbf{x}_2)^T(\mathbf{x}_1 - \mathbf{x}_2)$*; but for a rotation* $(\mathbf{u}_1 - \mathbf{u}_2) = \mathcal{R}(\mathbf{x}_1 - \mathbf{x}_2)$ *and* $\mathcal{R}^T\mathcal{R} = \mathcal{I}$*. A similar argument shows that angles are preserved (***exercises** *).*

Definition: 4.3 *Euclidean transformations*

A Euclidean transformation is a rotation and translation, so $(u(x,y), v(x,y)) = (x\cos\theta - y\sin\theta + t_x, v(x,y) = x\sin\theta + y\cos\theta + t_y)$. Euclidean transformations preserve lengths and angles (and so areas) and are sometimes referred to as rigid body transformations. Here $(x,y) = ((u-t_x)\cos\theta + (v-t_y)\sin\theta, -(u-t_x)\sin\theta + (v-t_y)\cos\theta)$. In vector notation, for $\mathcal{R}$ a rotation,

$$\mathbf{u} = \mathcal{R}\mathbf{x} + \mathbf{t} \text{ and } \mathbf{x} = \mathcal{R}^T(\mathbf{u} - \mathbf{t}).$$

Useful Fact: *Euclidean transformations preserve lengths and angles (you can think of a Euclidean transformation as a rotation followed by a translation).*

Definition: 4.4 *Uniform scaling*

For **uniform scaling**, $(u, v) = (sx, sy)$ for $s > 0$. Here $(x, y) = (1/su, 1/sv)$. In vector notation,

$$\mathbf{u} = s\mathbf{x} \text{ and } \mathbf{x} = (1/s)\mathbf{u}.$$

Useful Fact: *Uniform scaling preserves angles, but not lengths (*exercises*). Uniform scaling preserves ratios of lengths (*exercises*)*

Definition: 4.5 *Non-uniform scaling*

For **non-uniform scaling**, $(u, v) = (sx, ty)$ for s and t both positive, and so $(x, y) = (1/su, 1/tv)$. Write $\mathsf{diag}\left((s, t)\right)$ for the matrix with s and t on the diagonal. In vector notation,

$$\mathbf{u} = \mathsf{diag}\left((s, t)\right)\mathbf{x} \text{ and } \mathbf{x} = \mathsf{diag}\left((1/s, 1/t)\right)\mathbf{u}.$$

Useful Fact: *Non-uniform scaling will usually change both lengths and angles.*

Definition: 4.6 *Affine transformations*

Affine transformations are better written in vector notation. Write $\mathcal{A}$ for an invertible 2×2 matrix, and $\mathbf{t}$ for some constant vector. Then

$$\mathbf{u} = \mathcal{A}\mathbf{x} + \mathbf{t} \text{ and } \mathbf{x} = \mathcal{A}^{-1}(\mathbf{u} - \mathbf{t}).$$

Useful Fact: *Affine transformations will usually change both lengths and angles.*

Definition: 4.7 *Projective transformations*

Projective transformations involve quite inefficient notation if one does not know homogenous coordinates (Section **??**), and writing them in vector form is clumsy. Write p_{ij} for the i, j'th component of a 3×3 matrix $\mathcal{P}$ that is invertible. Then

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{p_{11}x+p_{12}y+p_{13}}{p_{31}x+p_{32}y+p_{33}} \\ \frac{p_{21}x+p_{22}y+p_{23}}{p_{31}x+p_{32}y+p_{33}} \end{bmatrix}.$$

The inverse transformation is obtained by applying the inverse of $\mathcal{P}$ to $\mathbf{u}$ according to the recipe above. For a vector representation, write

$$\mathcal{P} = \begin{bmatrix} \mathbf{p}_1^T & p_{13} \\ \mathbf{p}_2^T & p_{23} \\ \mathbf{p}_3^T & p_{33} \end{bmatrix}$$

for a 3×3 array with inverse $\mathcal{Q}$. Then

$$\mathbf{u} = \begin{bmatrix} \frac{\mathbf{p}_1^T\mathbf{x}+p_{13}}{\mathbf{p}_3^T\mathbf{x}+p_{33}} \\ \frac{\mathbf{p}_2^T\mathbf{x}+p_{23}}{\mathbf{p}_3^T\mathbf{x}+p_{33}} \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} \frac{\mathbf{q}_1^T\mathbf{u}+q_{13}}{\mathbf{q}_3^T\mathbf{u}+q_{33}} \\ \frac{\mathbf{q}_2^T\mathbf{u}+q_{23}}{\mathbf{q}_3^T\mathbf{u}+q_{33}} \end{bmatrix}$$

This definition means that, if $\mathcal{P} = \lambda\mathcal{Q}$ for some $\lambda \neq 0$, then $\mathcal{P}$ and $\mathcal{Q}$ implement the same projective transformation.

- All the transformations are special cases of projective transformations

Things to think about

4.1. Which transformations preserve angles?
4.2. Which transformations preserve lengths?
4.3. Could there be a family of transformations that preserves lengths, but not angles? Why?
4.4. Assume that the 2×2 matrix $\mathcal{N}$ has the property $\mathcal{N}^T \mathcal{N} = \mathcal{I}$ and $\det(\mathcal{N}) = -1$. Check that there is some rotation $\mathcal{R}$ such that $\mathcal{N} = \mathcal{R}\mathsf{diag}\left((1, -1)\right)$.
4.5. What happens if you apply $\mathsf{diag}\left((1, -1)\right)$ to an image?
4.6. Figure 4.1 shows two image coordinate systems. What transformation takes the coordinates of a point in the left-hand coordinate system to the coordinates of the same point in the right-hand coordinate system?
4.7. Write $\mathcal{R}$ for a rotation matrix. Show that the transformation that takes $\mathbf{x}$ to $\mathcal{R}(\mathbf{x} - \mathbf{t}) + \mathbf{t}$ is a rotation about the point $\mathbf{t}$.