Representing lines and planes

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Never do this without good reason

Useful Fact: *Do not represent a line as the set of points* (x, y) *where* $y = ax + b$ *or a plane as the set of points* (x, y, z) *where* $z = ax + by + c$*. Vertical lines – where* x *is constant – and vertical planes – where* $ax + by + c$ *is a constant – cannot be represented in this form.*

There are missing lines!

Lines and planes

Useful Fact: *Represent a line as the set of points* $\mathbf{x} = (x_1, x_2)^T$ *where* $ax_1 + bx_2 + c = 0 = \mathbf{a}^T\mathbf{x} + c$ *and a plane as the set of points* (x_1, x_2, x_3) *where* $ax_1 + bx_2 + cx_3 + d = 0 = \mathbf{a}^T\mathbf{x} + d$*. A tuple* (a, b, c) *corresponds to a line as long as not all elements are zero. A tuple* (a, b, c, d) *corresponds to a plane, as long as not all elements are zero.*

Useful Fact: *The line represented by* $(\lambda a, \lambda b, \lambda c) = (\lambda \mathbf{a}, \lambda c)$ *is the same as the line represented by* $(a, b, c) = (\mathbf{a}, c)$ *for* $\lambda \neq 0$*. This means that many tuples represent the same line. Choosing to avoid this ambiguity by requiring one element of the tuple to 1 means that you cannot represent some collection of lines. For example, the family* $(u, v, 1)$ *omits any line through the origin* **exercises**

A convenient repn for lines

Useful Fact: *If you represent a line by the tuple* $(\cos\theta, \sin\theta, r)$*, where* $0 \leq r < 0$ *and* $0 \leq \theta < \pi$*, then all lines are represented, and there is exactly one* (θ, r) *that corresponding to a given line***exercises** . *For this representation,* r *is the perpendicular distance from the line to the origin and* θ *is the orientation of the line (meaning that the vector* $(\sin\theta, \cos\theta)^T$ *points along the line).*

Distance to a line, normal

Useful Fact: *The perpendicular distance from a point* $\mathbf{x}$ *to a line* $(\mathbf{a}, c)$ *is given by*

$$abs(\mathbf{a}^T\mathbf{x} + c) \qquad \textit{if} \qquad \mathbf{a}^T\mathbf{a} = 1.$$

In my experience, this fact is useful enough to be worth memorizing.

Useful Fact: *The normal of a line represented as* $(a, b, c) = (\mathbf{a}, c)$ *is given by*

$$\frac{\mathbf{a}}{\sqrt{\mathbf{a}^T\mathbf{a}}}$$

Things to think about...

13.1. What is the normal of a plane represented by $(\mathbf{a}, d)$?

13.2. Why does the family of lines given by $(u, v, 1)$ omit any line through the origin?

13.3. What lines are missing from the family of lines given by $(1, v, w)$?

13.4. You are given a line (a, b, c). Show there is exactly one (θ, r) in $0 \leq r < 0$ and $0 \leq \theta < \pi$ so that the line can be represented as $(\cos\theta, \sin\theta, r)$.

13.5. You are given a line (a, b, c). Show there are exactly two (θ, r) in $0 \leq r < 0$ and $0 \leq \theta < 2\pi$ so that the line can be represented as $(\cos\theta, \sin\theta, r)$. Interpret these two solutions.