B-splines - I

- We obtain a set of blending functions by a recursive definition, with “switches” at the base of the recursion
- Curve:

$$X(t) = \sum_{k=0}^{n} P_k B_{k,d}(t)$$

- where d (called the “order”) is:

$$2 \le d \le n + 1$$

B-Spline Blending Functions

- Knots
  - idea: parameter values where curve segments meet, as in Hermite example

$$\left(t_0, t_1, \ldots, t_{n+d}\right)$$

where $t_0 \le t_1 \le \ldots \le t_{n+d}$

- Blending functions

$$B_{k,1}(t) = \begin{cases} 1 & t_k \le t \le t_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{k,d}(t) = \left(\frac{t - t_k}{t_{k+d-1} - t_k}\right) B_{k,d-1}(t) + \left(\frac{t_{k+d} - t}{t_{k+d} - t_{k+1}}\right) B_{k+1,d-1}(t)$$

de Boor Algorithm

knot vector: [0 0 0 0 1 4 5 5 5 5]

Cubic ($d = 3$, $k = 4$)

- Evaluate at t = 2

  $\mathbf{p}_{4,3} = 1/3\ \mathbf{p}_{4,2} + 2/3\ \mathbf{p}_{3,2}$

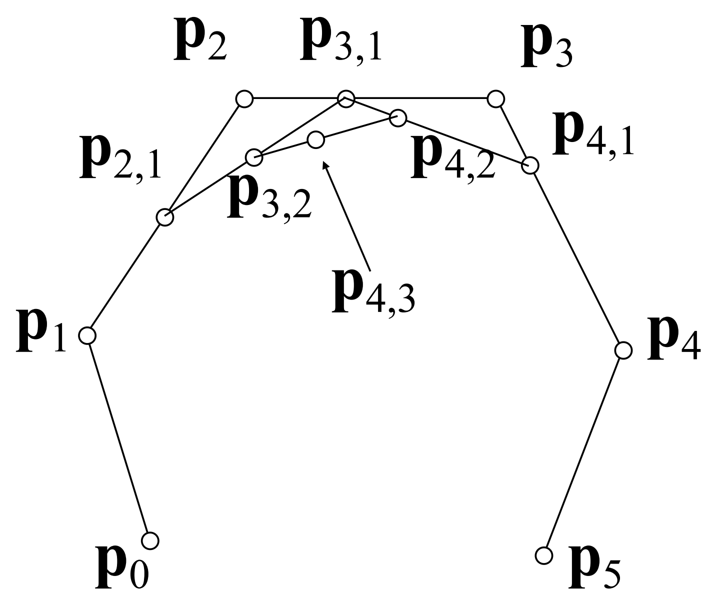
  $\mathbf{p}_{4,2} = 1/4\ \mathbf{p}_{4,1} + 3/4\ \mathbf{p}_{3,1}$

  $\mathbf{p}_{3,2} = 2/4\ \mathbf{p}_{3,1} + 2/4\ \mathbf{p}_{2,1}$

  $\mathbf{p}_{4,1} = 1/4\ \mathbf{p}_{4,0} + 3/4\ \mathbf{p}_{3,0}$

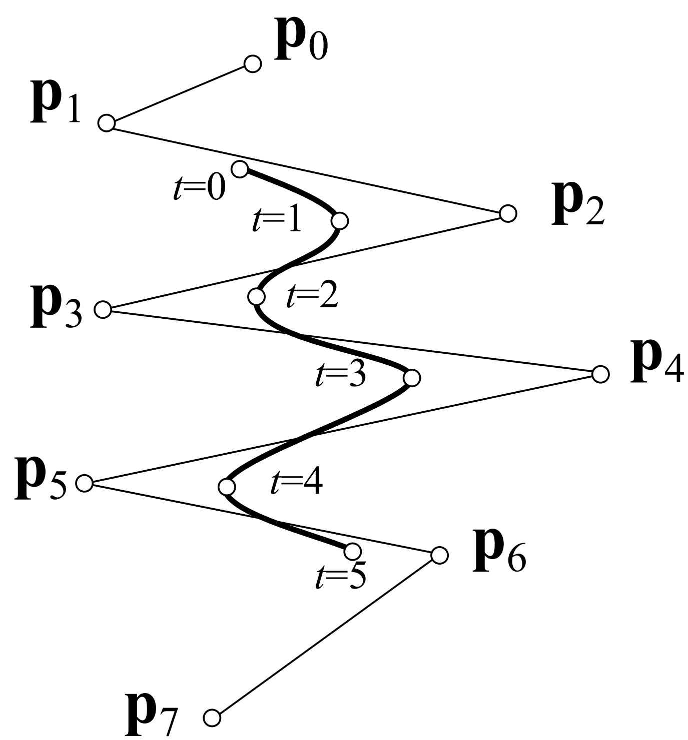
  $\mathbf{p}_{3,1} = 2/5\ \mathbf{p}_{3,0} + 3/5\ \mathbf{p}_{2,0}$

  $\mathbf{p}_{2,1} = 2/4\ \mathbf{p}_{2,0} + 2/4\ \mathbf{p}_{1,0}$



$$\mathbf{p}_{i,l} = \frac{t - t_i}{t_{k+i-l} - t_i}\mathbf{p}_{i,l-1} + \frac{t_{k+i-l} - t}{t_{k+i-l} - t_i}\mathbf{p}_{i-1,l-1}$$

Uniform B-Splines

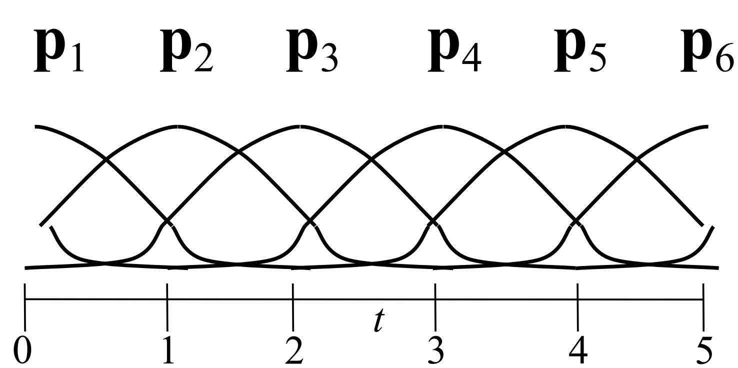


- Notation
  - d = degree of polynomial
  - k = order of polynomal = $d + 1$
  - E.g. cubic: d = 3, k = 4
- Segment $i \le t < i+1$ uses $k = d + 1$ control points $\mathbf{p}_i$ to $\mathbf{p}_{i+d}$

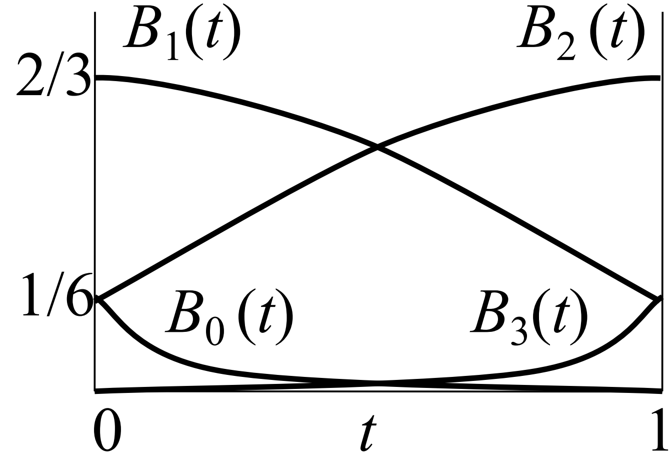
$$\mathbf{p}(\mathrm{t}) = \Sigma_{j=0}^{3} B_j(t \bmod 1)\mathbf{p}_{i+j}$$

- Normalized basis function $N_{i,d}(t)$
- $N_{i,d}(t) = B_{\mathrm{floor}(t-i)}(t \bmod 1)$ if $i \le t < i+d+1$
  - Otherwise its zero
- Knot vector
  - e.g. [0,1,2,3,4,5,6,7]
  - in general $[t_0, t_1, \ldots, t_{n+k}]$

$$\mathbf{p}(t) = \sum_{i=0}^{n} N_{i,k}(t)\mathbf{p}_i$$



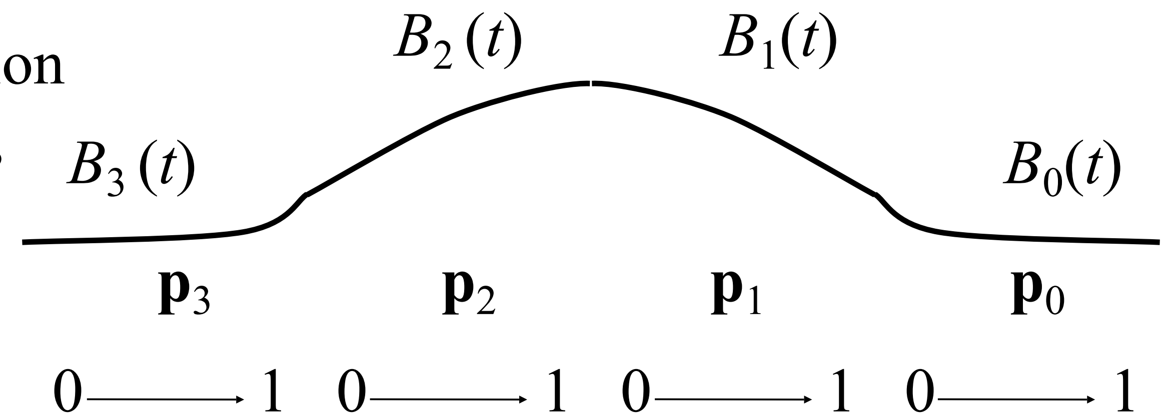
B-Spline Basis



$$
\begin{aligned}
\mathbf{p}(t) = &(-1/6t^3 + 1/2t^2 - 1/2t + 1/6)\mathbf{p}_0 + \\
&(\ 1/2t^3 - t^2 + 2/3)\mathbf{p}_1 + \\
&(-1/2t^3 + 1/2t^2 + 1/2t + 1/6)\mathbf{p}_2 + \\
&(\ 1/6t^3)\mathbf{p}_3
\end{aligned}
$$

$= B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$

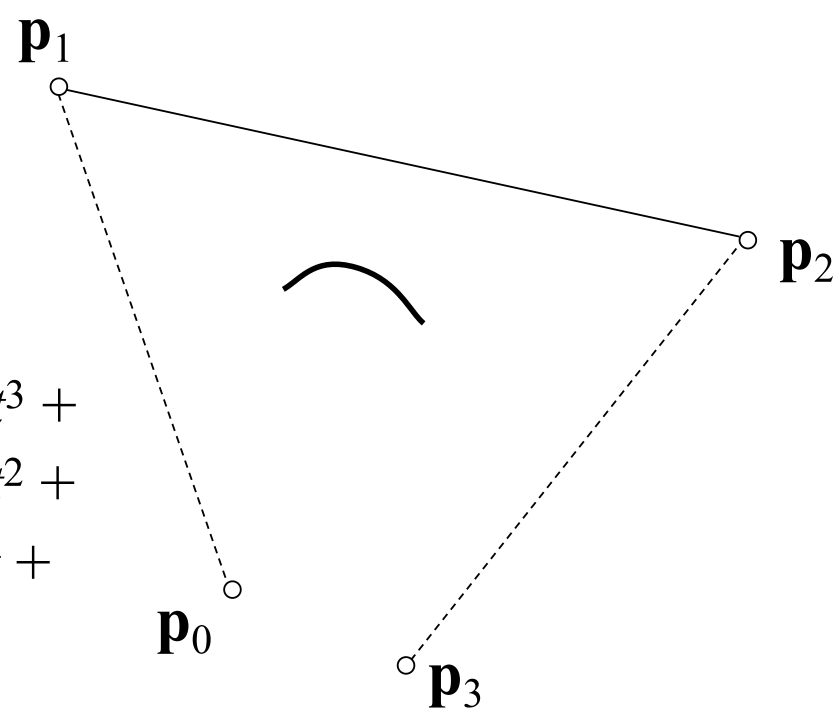
- Piecewise cubic approximation of a Gaussian bump function
- Progressively weights points along spline



Catmull-Clark Subdivision

CS 319
Advanced Topics in Computer Graphics
John C. Hart

B-Spline Segment

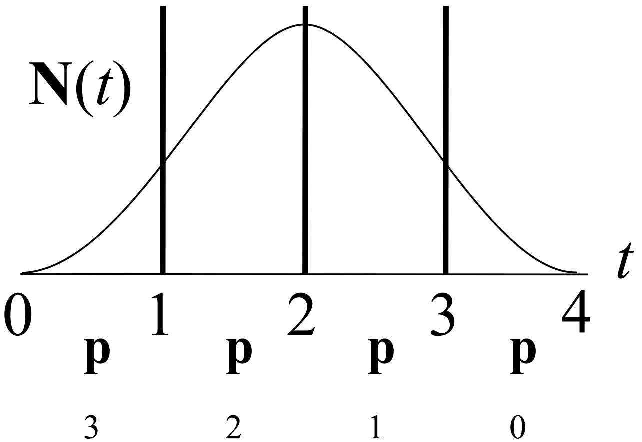


$$\begin{aligned}\mathbf{p}(t) = {} & (-1/6\mathbf{p}_0+1/2\mathbf{p}_1-1/2\mathbf{p}_2+1/6\mathbf{p}_3)t^3 + \\ & (\ 1/2\mathbf{p}_0 - \mathbf{p}_1+1/2\mathbf{p}_2)t^2 + \\ & (-1/2\mathbf{p}_0 + 1/2\mathbf{p}_2)t + \\ & 1/6\mathbf{p}_0+2/3\mathbf{p}_1+1/6\mathbf{p}_2\end{aligned}$$

but makes more sense as…

$$\begin{aligned}\mathbf{p}(t) = {} & (-1/6t^3 + 1/2t^2 - 1/2t + 1/6)\mathbf{p}_0 + \\ & (\ 1/2t^3 - t^2 + 2/3)\mathbf{p}_1 + \\ & (-1/2t^3 + 1/2t^2 + 1/2t + 1/6)\mathbf{p}_2 + \\ & (\ 1/6t^3)\mathbf{p}_3\end{aligned}$$

Uniform Global Basis

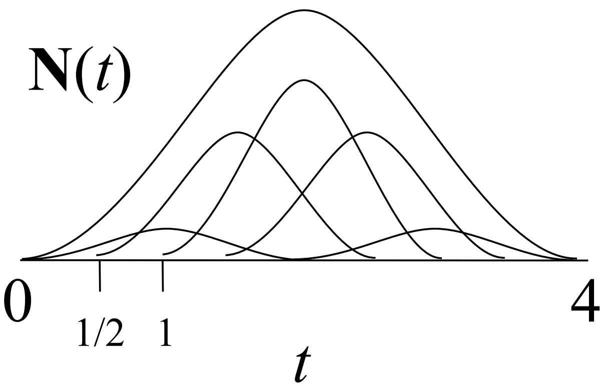


- Let $N(t)$ be a global basis function for our uniform cubic B-splines
- $N(t)$ is piecewise cubic

$$N(t) = \begin{cases} \frac{1}{6}t^3 & = \frac{1}{6}t^3 & \text{if } t < 1 \\ -\frac{1}{2}(t-1)^3 + \frac{1}{2}(t-1)^2 + \frac{1}{2}(t-1) + \frac{1}{6} & = -\frac{1}{2}t^3 + 2t^2 - 2t + \frac{2}{3} & \text{if } t < 2 \\ \frac{1}{2}(t-2)^3 - (t-2)^2 + \frac{2}{3} & = \frac{1}{2}t^3 - 4t^2 + 10t - \frac{22}{3} & \text{if } t < 3 \\ -\frac{1}{6}(t-3)^3 + \frac{1}{2}(t-3)^2 - \frac{1}{2}(t-3) + \frac{1}{6} & = -\frac{1}{6}t^3 + 2t^2 - 8t + \frac{32}{3} & \text{otherwise} \end{cases}$$

$$\mathbf{p}(t) = N(t)\,\mathbf{p}_3 + N(t+1)\,\mathbf{p}_2 + N(t+2)\,\mathbf{p}_1 + N(t+3)\,\mathbf{p}_0$$

B-Spline Wavelets



- We can make the uniform B-spline basis function as the sum of smaller copies of itself

$$N(t) = 1/8\, N(2t) + \tfrac{1}{2}\, N(2t-1) + \tfrac{3}{4}\, N(2t-2) + \tfrac{1}{2}\, N(2t-3) + 1/8\, N(2t-4)$$

- For example, for the $t < 1$ case, we have

$$N(t) = 1/8\, N(2t) + \tfrac{1}{2}\, N(2t-1)$$

- For the $t < 1/2$ half of this domain, we get

$$1/6\, t^3 = 1/8 \cdot 1/6 (2t)^3 + 0$$

- For the other half, $1/2 < t < 1$, we get

$$1/8\,(-\tfrac{1}{2}(2t)^3 + 2(2t)^2 - 2(2t) + 2/3) + \tfrac{1}{2} \cdot 1/6\,(2t-1)^3$$
$$= -\tfrac{1}{2}t^3 + t^2 - \tfrac{1}{2}t + 2/24 + 1/12\,(8t^3 - 12t^2 + 6t - 1)$$
$$= 1/6\, t^3$$

- For the rest of the cases, you're on your own

B-Spline Splitting

- Given a cubic B-spline

$\mathbf{p}(t) = N(t)\,\mathbf{p}_3 + N(t+1)\,\mathbf{p}_2 + N(t+2)\,\mathbf{p}_1 + N(t+3)\mathbf{p}_0$

what are the four control points for the first half of the curve?

- Weight on $\mathbf{p}_0$:

$N(t+3) = 1/8\,N(2t+6) + ½\,N(2t+5) + ¾\,N(2t+4) + ½\,N(2t+3) + 1/8\,N(2t+2)$

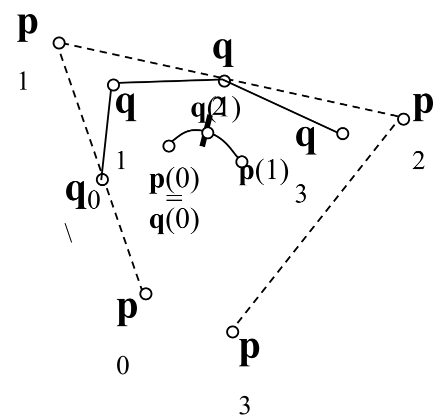
- Weight on $\mathbf{p}_1$:

$\mathrm{N}(t+2) = 1/8\,N(2t+4) + ½\,N(2t+3) + ¾\,N(2\mathrm{t}+2) + ½\,N(2t+1) + 1/8\,N(2t)$

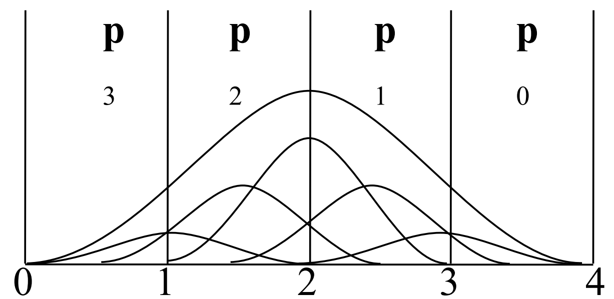
- Find $\mathbf{q}(t)$ st $\mathbf{q}(2t) = \mathbf{p}(t)$ for $0 <= t <= ½$

$\mathbf{q}(2t) = N(2t+3)\mathbf{q}_0 + N(2t+2)\mathbf{q}_1 + N(2t+1)\mathbf{q}_2 + N(2\mathrm{t})\mathbf{q}_3$

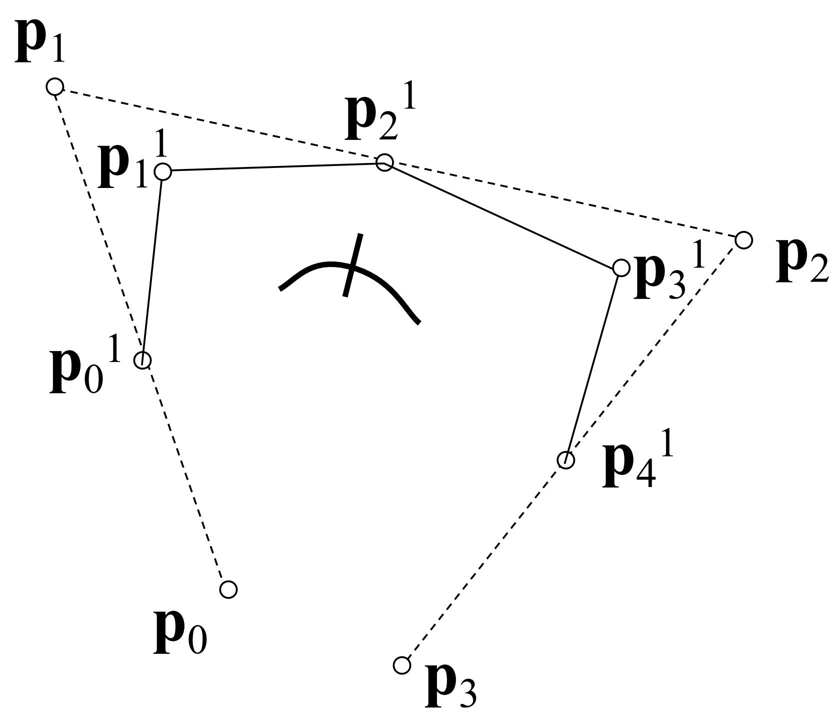
- Weight on $\mathbf{q}_0$ same as on $½\,\mathbf{p}_0 + ½\,\mathbf{p}_1$
- Weight on $\mathbf{q}_1$ includes $1/8\,\mathbf{p}_0 + ¾\,\mathbf{p}_1$ + more…
- Work out the rest on your own



$N(t) = 1/8\,N(2t) + ½\,N(2t-1) + ¾\,N(2t-2) + ½\,N(2t-3) + 1/8\,N(2t-4)$



B-Spline Splitting

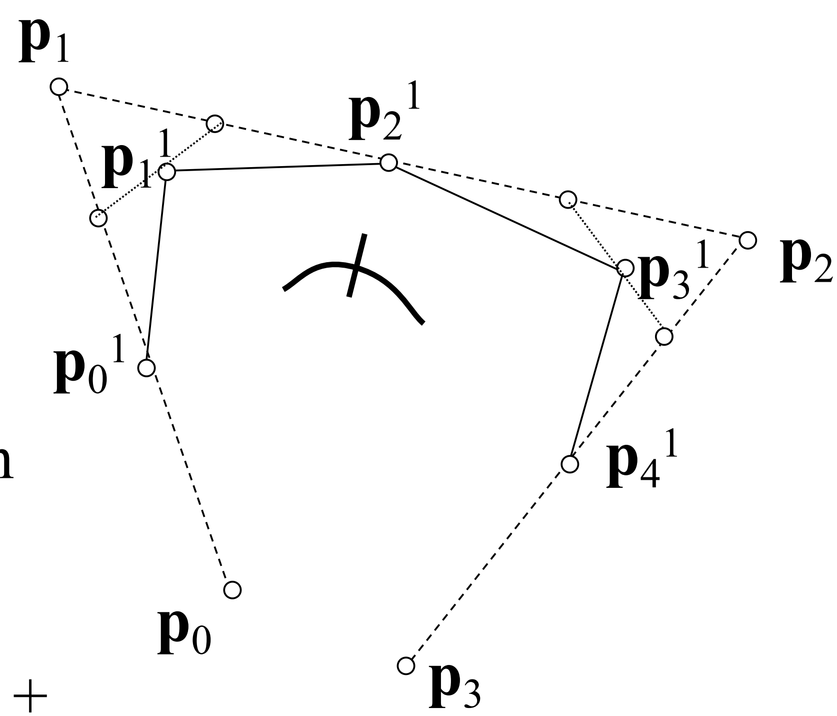


$$\begin{bmatrix}\mathbf{p}_0^1\\ \mathbf{p}_1^1\\ \mathbf{p}_2^1\\ \mathbf{p}_3^1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}4 & 4 & 0 & 0\\ 1 & 6 & 1 & 0\\ 0 & 4 & 4 & 0\\ 0 & 1 & 6 & 1\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{p}_1^1\\ \mathbf{p}_2^1\\ \mathbf{p}_3^1\\ \mathbf{p}_4^1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}1 & 6 & 1 & 0\\ 0 & 4 & 4 & 0\\ 0 & 1 & 6 & 1\\ 0 & 0 & 4 & 4\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{p}_0^1\\ \mathbf{p}_1^1\\ \mathbf{p}_2^1\\ \mathbf{p}_3^1\\ \mathbf{p}_4^1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}4 & 4 & 0 & 0\\ 1 & 6 & 1 & 0\\ 0 & 4 & 4 & 0\\ 0 & 1 & 6 & 1\\ 0 & 0 & 4 & 4\end{bmatrix}\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}$$

Curve Subdivision



- Edge points

  $\mathbf{p}_2{}^1{}_i = 1/2\ \mathbf{p}_i + 1/2\ \mathbf{p}_{i+1}$

  – edge points at midpoint between vertices

- Vertex points

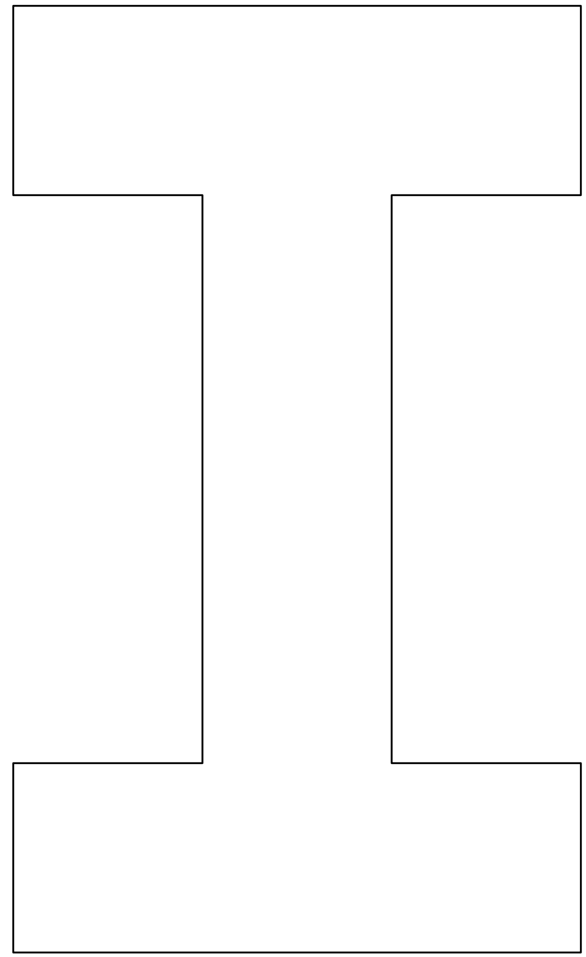
  $\mathbf{p}_{2i}{}^1{}_{+1} = 1/8\ \mathbf{p}_i + 3/4\ \mathbf{p}_{i+1} + 1/8\ \mathbf{p}_{i+2}$

  – midpoint between midpoints between old vertices and new edge points

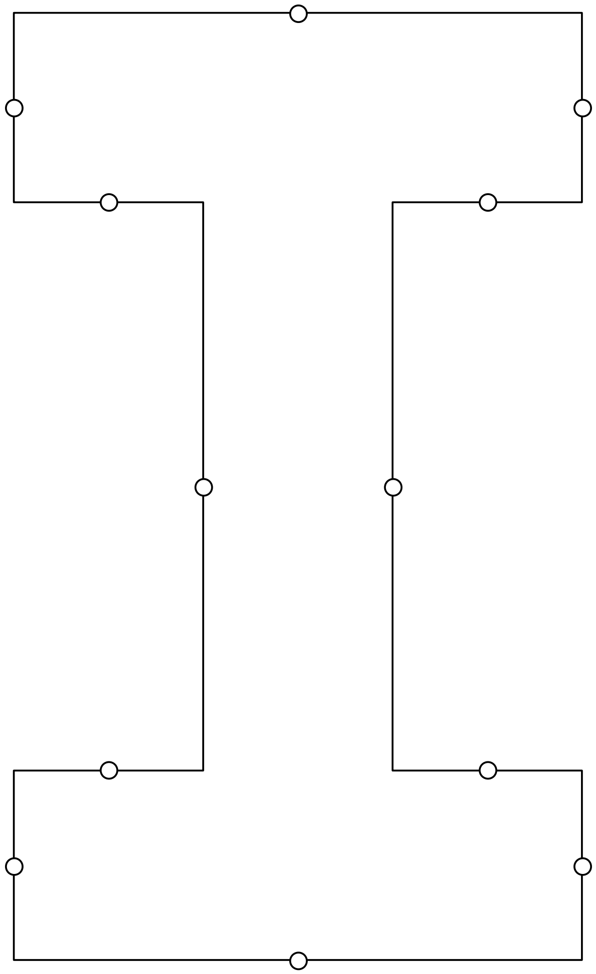
  $= 1/2\ (1/2\ \mathbf{p}_2{}^1{}_i + 1/2\ \mathbf{p}_{i+1}) + 1/2\ (1/2\ \mathbf{p}_{i+1} + 1/2\ \mathbf{p}_{2i}{}^1{}_{+1})$

$$\begin{bmatrix} \mathbf{p}_0^1 \\ \mathbf{p}_1^1 \\ \mathbf{p}_2^1 \\ \mathbf{p}_3^1 \\ \mathbf{p}_4^1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/8 & 3/4 & 1/8 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/8 & 3/4 & 1/8 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Coarse Polygon

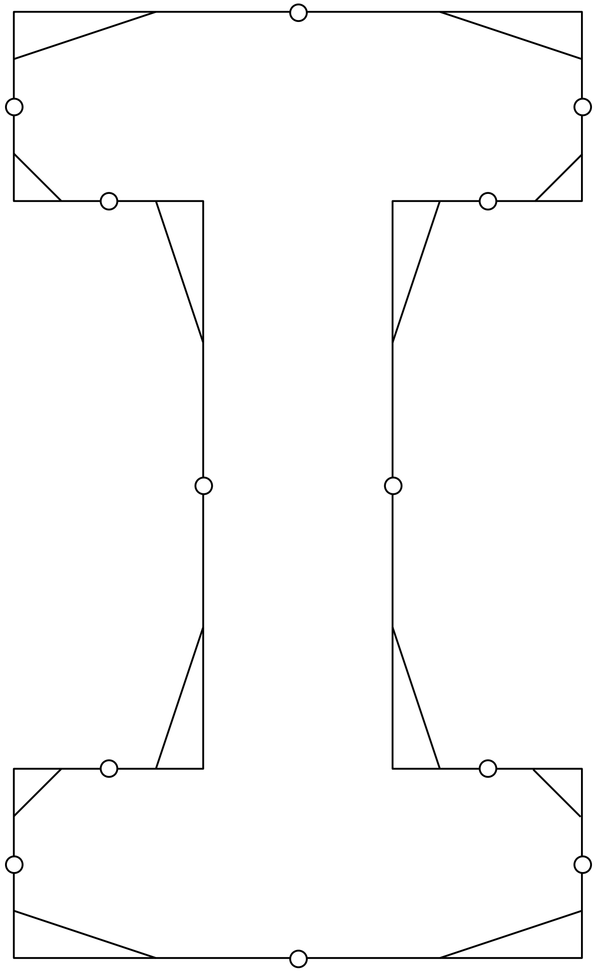


Add Edge Midpoints

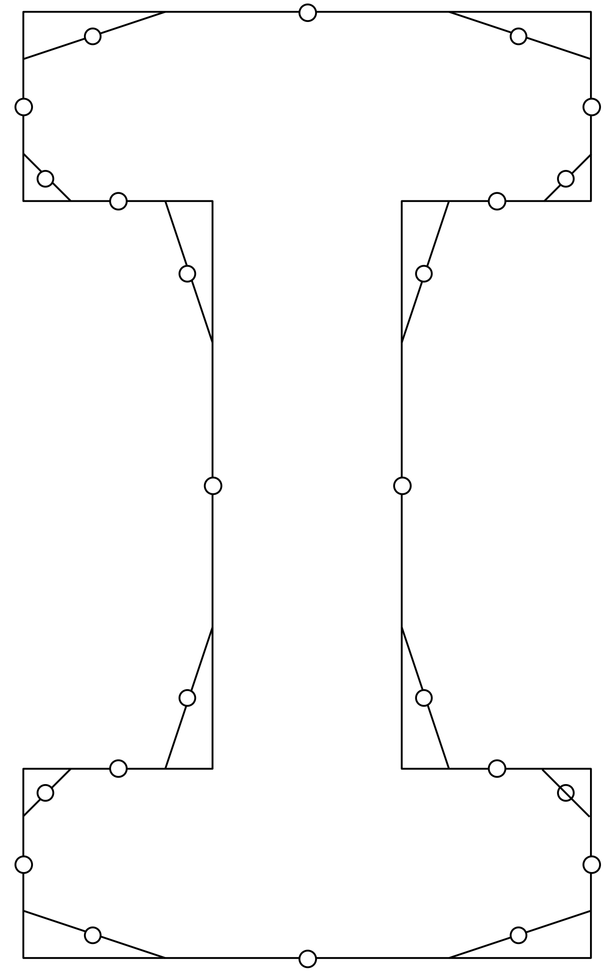


Add Struts

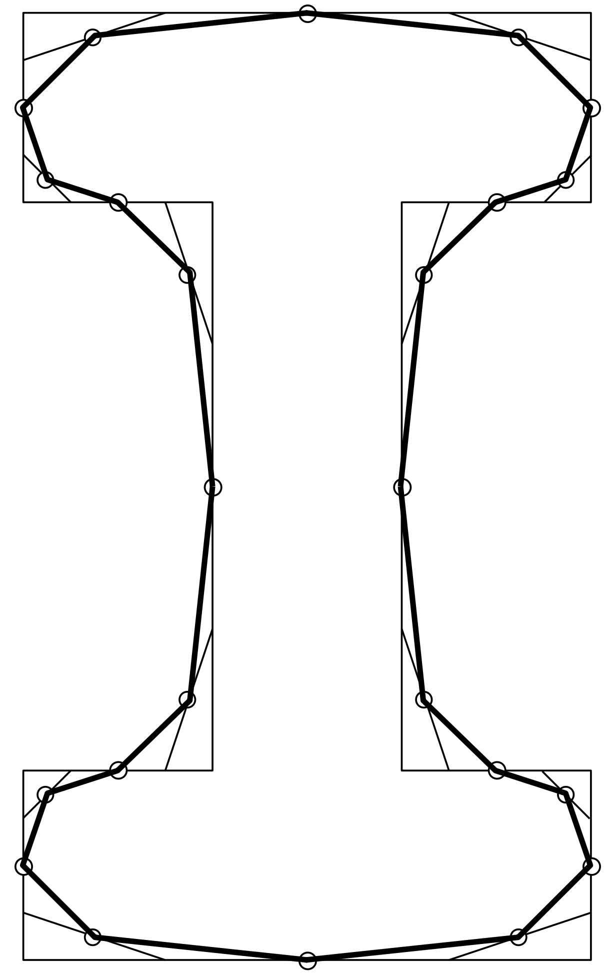
- Struts connect midpoints of segments from vertices to edge midpoints
- One strut per vertex



Add Strut Midpoints



Connect Midpoints

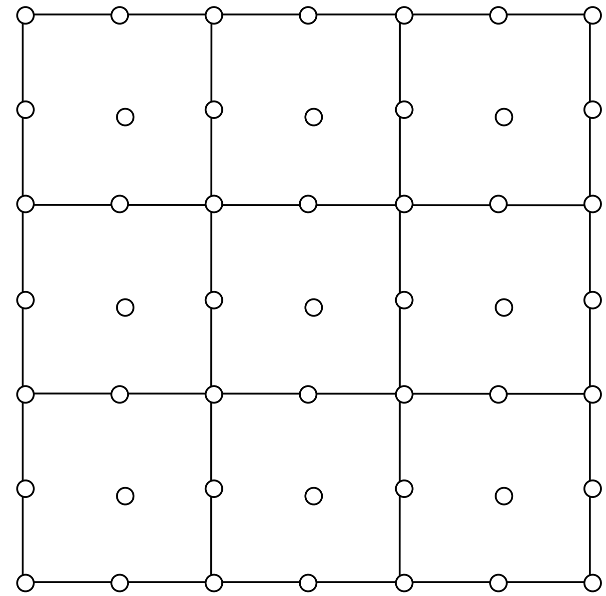


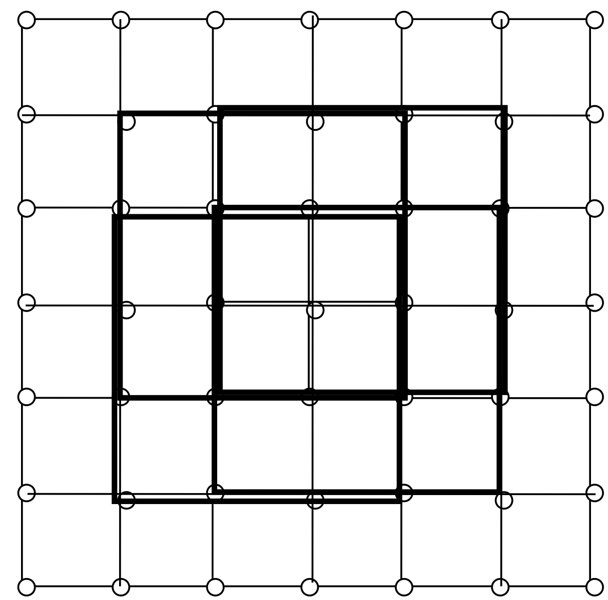
B-Spline Patches

- Tensor product of two curves

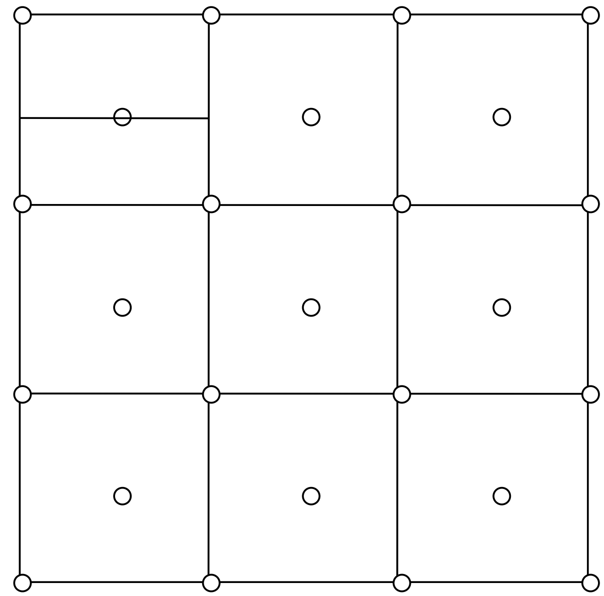
$$\mathbf{p}(s,t) = \sum_{j=0}^{n} \sum_{i=0}^{n} N_j^n(s) N_i^n(t) \mathbf{p}_{ij}$$

- Need to subdivide control points to create four sub-patches
- Need to generate new control points
  - vertex points (replacing control points)
  - edge points
  - face points





Face Points



- Approximate edge points as midpoint of control points

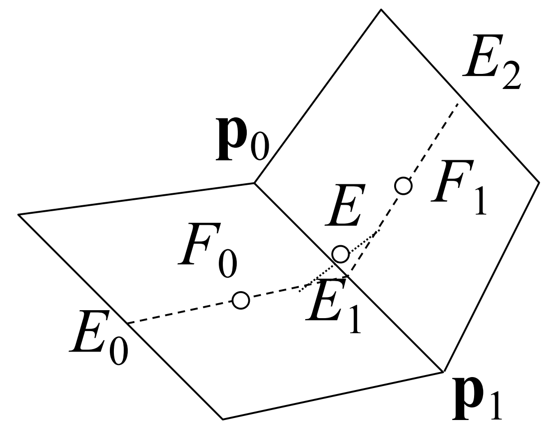
  $E = 1/2\ \mathbf{p} + 1/2\ \mathbf{p}$

- Face point is midpoint of approximate edge points

  $F = 1/2\ E + 1/2\ E$

  $= 1/4\ \mathbf{p} + 1/4\ \mathbf{p} + 1/4\ \mathbf{p} + 1/4\ \mathbf{p}$

Edge Points



- Face points are midpoints between approx. edge points
- Approx. edge point is midpoint between control points
- Actual edge point is midpoint between midpoints between approx edge point and face points

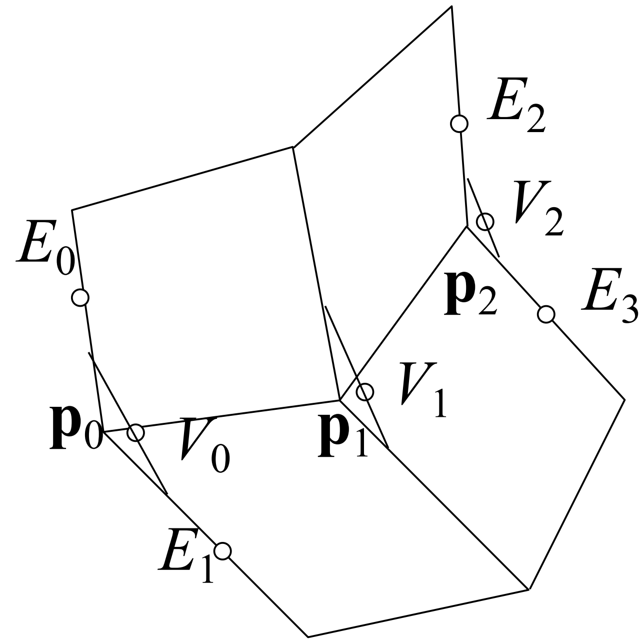
$$
\begin{aligned}
E &= 1/2\,(1/2\,(1/2\,E_0 + 1/2\,E_1) + 1/2\,E_1) \\
&+ 1/2\,(1/2\,E_1 + 1/2\,(1/2\,E_1 + 1/2\,E_2)) \\
&= 1/2\,(1/2\,F_0 + 1/2\,(1/2\,\mathbf{p}_0 + 1/2\,\mathbf{p}_1)) + \\
&\quad 1/2\,(1/2\,(1/2\,\mathbf{p}_0 + 1/2\,\mathbf{p}_1) + 1/2\,F_1) \\
&= 1/4\,(F_0 + \mathbf{p}_0 + \mathbf{p}_1 + F_1)
\end{aligned}
$$

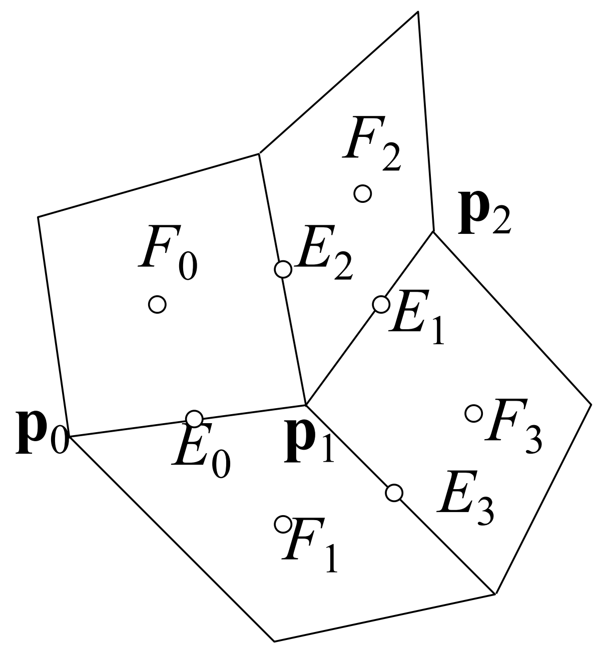
Vertex Points

$$V_0 = 1/4\ E_0 + 1/2\ \mathbf{p}_0 + 1/4\ E_1$$

$$V_2 = 1/4\ E_2 + 1/2\ \mathbf{p}_2 + 1/4\ E_3$$

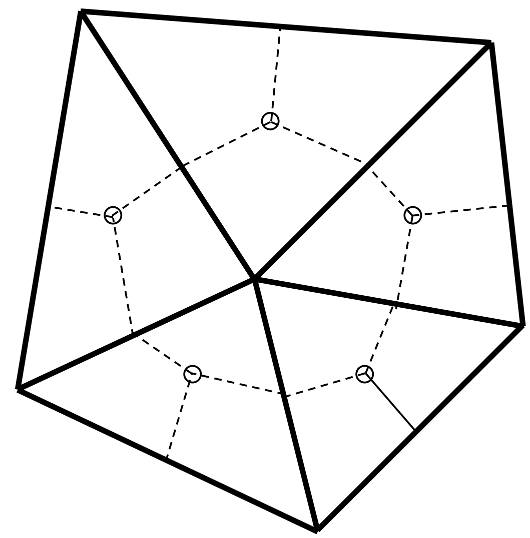
$$\begin{aligned} V &= 1/2\ (1/2\ (1/2\ V_0 + 1/2\ V_1) + 1/2\ V_1) + \\ &\quad 1/2\ (1/2\ V_1 + 1/2\ (1/2\ V_1 + 1/2\ V_2) \\ &= 1/4\ (1/4\ (F_0 + F_1 + \mathbf{p}_0 + \mathbf{p}_1) + \\ &\quad 1/4\ (F_2 + F_3 + \mathbf{p}_1 + \mathbf{p}_2) + 2\ V_1) \\ &= 1/4\ (1/4\ (F_0 + F_1 + F_2 + F_3) + \\ &\quad 1/4\ (\mathbf{p}_0 + 2\ \mathbf{p}_1 + \mathbf{p}_2) + \\ &\quad 2/4\ (E_2 + E_3 + 2\ \mathbf{p}_1)) \\ &= 1/16(F_0 + F_1 + F_2 + F_3 + \\ &\quad 2E_0 + 2E_1 + 2E_2 + 2E_3 + 4\mathbf{p}_1) \end{aligned}$$



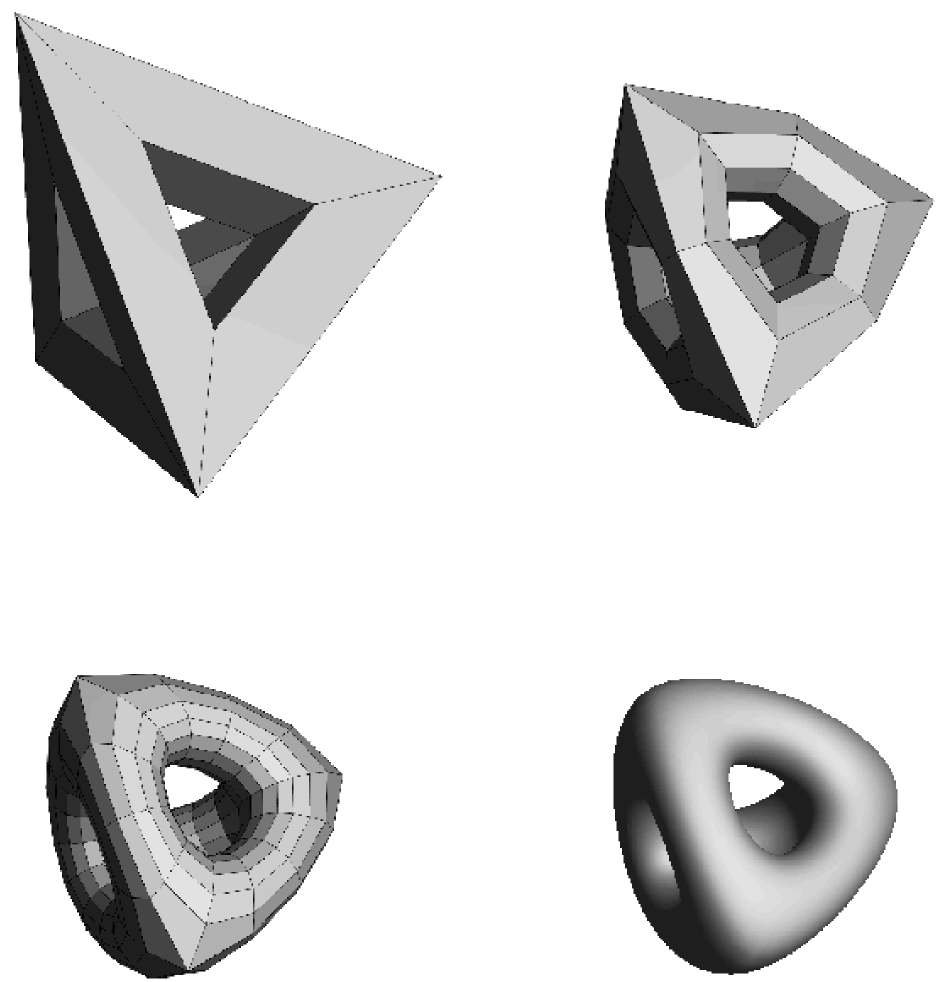


Catmull-Clark Subdiv

- Face points = average of (*n*) control points
- Edge points = average of two control points and two face points
- Vertex points = average of…
  - average of adjacent face points
  - twice the average of midpoints of adjacent edges
  - (n – 3) terms of the control point

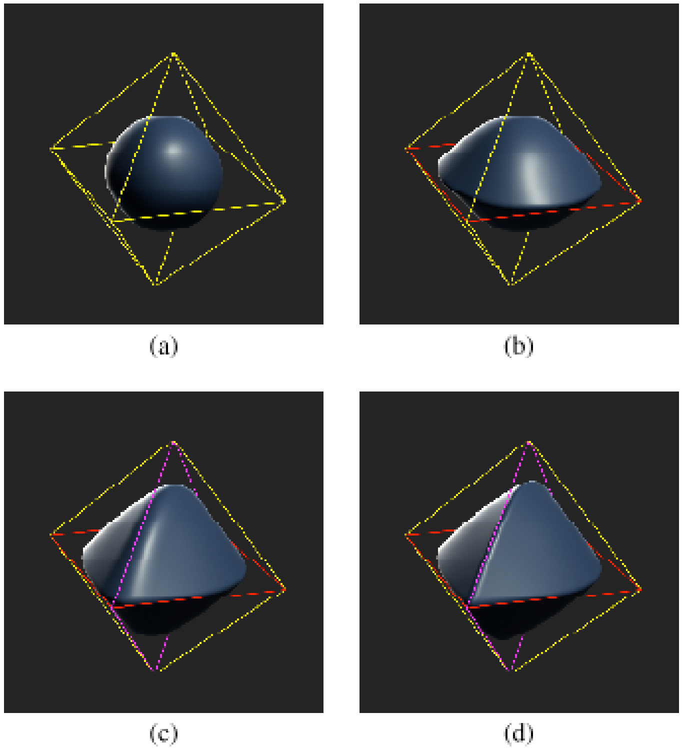


Example



Another Example

Creases



$$f^{i+1}_j = \text{Centroid of polygon}$$

$$e^{i+1}_j = (v^i + e^i_j)/2$$

- Dart vertex (one sharp edge):

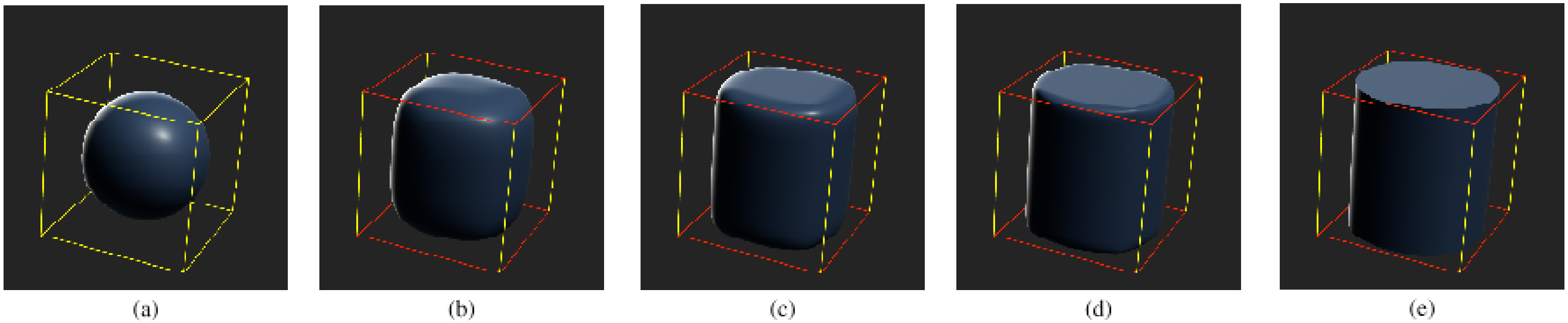
$$v^{i+1} = (n\text{-}2)/n\ v^i + 1/n^2\ \Sigma_j\ e^i_j + 1/n^2\ \Sigma_j\ f^{i+1}_j$$

- Crease vertex (two sharp edges):

$$v^{i+1} = (e^i_j + 6v^i + e^i_k)/8$$

- Corner vertex (three or more sharp edges)

$$v^{i+1} = v^i$$



Rational B-Splines

- Quotient of B-splines

  $\mathbf{p}(t) = (\Sigma\, w_i\, \mathbf{p}_i\, N_i(t))/(\Sigma\, w_i\, N_i(t))$
- B-spline in 4-D homogenous space
- Projected back into 3-D via homogenous division
- Weight values affect "tension" near control points
- Weights can also define control points at infinity
- Capable of generating conic sections (e.g. circular arcs)