Tracking in 1D - the simplest Kalman filter

- **Assume**:
  - prior is ~~linear~~ Normal
  - State is 1D - x_i
  - dynamics are

$$x_{i+1} = a_i x_i + b_i + \xi_i$$

$$\xi_i \sim N(0, \sigma_{d_i})$$

  - Measurement is

$$y_{i+1} = c_{i+1} x_{i+1} + d_{i+1} + \zeta_{i+1}$$

$$\zeta_{i+1} \sim N(0, \sigma_{m_i})$$

Prior:

after meas!

we have

$$P(x_i \mid y_0 \cdots y_i) \quad \text{is} \quad N(\mu_i^+, \sigma_i^+)$$

want

$$P(x_{i+1} \mid y_0 \cdots y_i)$$ ← prediction Note meas only up to i

$$x_{i+1} = a_i x_i + b_i + \xi_i$$

$\xi_i \sim N(0, \sigma_{d_i})$

i.e. take a Normal r.v., mult by const, add const, add normal r.v.

$\Rightarrow$ a) x_{i+1} is normal

b) $$x_{i+1} \sim N\left(\underbrace{a_i \mu_i^+ + b_i}_{\mu_{i+1}^-}, \underbrace{\sqrt{a^2 \sigma_i^{+2} + \sigma_{d_i}^2}}_{\sigma_{i+1}^-}\right)$$

Now we want

$$P(x_{i+1} \mid y_0 \cdots y_{i+1})$$

$$\propto P(y_{i+1} \mid x_{i+1}) \, P(x_{i+1} \mid y_0 \cdots y_i)$$

(by Markov assumption)

work w/ logs

$$\log P(x_{i+1} \mid y_0 \cdots y_{i+1})$$

$$= -\frac{1}{2\sigma_{m_{i+1}}^2}\left(y_{i+1} - c_{i+1} x_{i+1} - d_{i+1}\right)^2 - \frac{1}{2\sigma_{i+1}^{-2}}\left[x_{i+1} - \mu_{i+1}^{-}\right]^2$$

$$+ \text{ const s}$$

quadratic in x_{i+1}, <u>so</u>

$P(x_{i+1} \mid y_0 \cdots y_{i+1})$ is normal

$$N(\mu_{i+1}^{+}, \sigma_{i+1}^{+})$$

Now we can use pattern matching

$$\log\left(\text{Normal}(m, s)\right) = -\frac{x^2}{2s^2} + \frac{mx}{s^2} + \text{const.}$$

multiply out, etc.

$$\log p(x_{i+1} | y_0 \cdots y_{i+1}) =$$

$$-\frac{x_{i+1}^2}{2}\left[\frac{c_{i+1}^2}{\sigma_{m,i+1}^2} + \frac{1}{\sigma_{i+1}^{-2}}\right]$$

$$+\left[\frac{(c_{i+1}(y_{i+1}-d))}{\sigma_{m,i+1}^2} + \frac{\bar{\mu}_{i+1}}{\sigma_{i+1}^{-2}}\right]x_{i+1} + \cdots \left(\text{No } x_{i+1}\right)$$

matching yields

$$\sigma_{i+1}^{+} = \sqrt{\frac{\sigma_{m,i+1}^{2}\,\sigma_{i+1}^{-2}}{c_{i+1}^{2}\,\sigma_{i+1}^{-2} + \sigma_{m,i+1}^{2}}}$$

<u>notice</u> $\quad \dfrac{\sigma_{m,i+1}^{2}}{\sigma_{i+1}^{-2}}$ v. small $\longrightarrow \dfrac{\sigma_{m,i+1}}{c_{i+1}}$

"<u>go with meas.</u>"

v. large $\longrightarrow \sigma_{i+1}^{-}$

"go <u>with dyn</u>"

$$\mu_{i+1}^{+} = \frac{\sigma_{i+1}^{-2}\left(c_{i+1}\left(y_{i+1} - d_{i+1}\right)\right) + \mu_{i+1}^{-}\,\sigma_{m,i+1}^{2}}{c_{i+1}^{2}\,\sigma_{i+1}^{-2} + \sigma_{m,i+1}^{2}}$$

$\dfrac{\sigma_{i+1}}{\sigma_m}$ large $\longrightarrow \dfrac{y_{i+1} - d_{i+1}}{c_{i+1}}$ (go with meas)

small $\longrightarrow \mu_{i+1}^{-}$ (go w/ dyn)

All this extends to multiple ... quite cleanly (Derivation easy, but tedious).

Common, useful recipe:

- Choose dynamics
- Now from μ_i^+, Σ_i^+
  - → predict μ_{i+1}^-, Σ_{i+1}^-
  - → use this to choose measurement (gating).
  - → use meas to get μ_{i+1}^+, Σ_{i+1}^+

Duck hunting:

think about the duck movie
→ how to drop a K.F. on that?

1) ~~Dynam~~ Duck model

- circular blob with center $\underline{x}$,

  and $\log(\text{radius}) = l$

(why log? - usual for scales)

- size doesn't change much?

$$l_{i+1} = l_i + \varepsilon_i \qquad \varepsilon_i \sim N(0, \sigma^2_{duck})$$

- seems to have smooth movement?

$$x_{i+1} = x_i + v_i \Delta_t + \xi_i \qquad \xi_i \sim N(0, \cdot\cdot)$$

$$v_{i+1} = v_i + a_i \Delta_t + \zeta_i \qquad \zeta_i \sim N(0, \cdot\cdot)$$

$$a_{i+1} = \eta_i \sim N(0, \cdot\cdot)$$

measurement:

- → make an estimate of center, radius
- → possibly weighted from μ_i^-, Σ_i^-
- → $cest_i = x_i + \text{noise}$

  $rest_i = l_i + \text{noise}$

This will likely deal with parts falling off duck

Issues:

- Gating may bias est.
  - multiple gates?
- Dynamic model is shakey
  - it's just a smoother
  - (sometimes) mult. dynamic models.

~~B~~ Issues :

- Birth + death of tracks.

  → rely on a detector ?

  → rely on spatial gates ?

  ~~de~~

- Crossings :

  - sometimes, dynamics will sort this out.

  - otherwise, you have a nasty discrete inference problem

Nonlinear measurem...

$$y_i = f(x_i) + \xi_i$$

options:

1) linearize about μ_i^-

(Extended Kalman filter)

but this can lead to mischief around zero

eg: $f(x) = x^2$

2) Particle Filter