The Kalman Filter in 1D

D.A. Forsyth, UIUC

A 1D Problem

- Drop a measuring device on a cable down a hole
  - where is it?
- Setup:
  - measurement of depth x
  - actual distance down the hole θ
  - known $p(\theta)$ which will be normal, $N(\theta_c; \sigma_c^2)$
  - known $p(x|\theta)$ which will be normal, $N(c\theta; \sigma_m^2)$
- Q: what is $p(\theta|x)$?

A 1D problem, II

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}$$

(Bayes rule), so that:

$$p(\theta|x) \propto p(x|\theta)p(\theta)$$

And:

$$\begin{aligned} \log p(\theta|x) &= \log p(x|\theta) + \log p(\theta) + K \\ &= -\frac{(c\theta - x)^2}{2\sigma_m^2} - \frac{(\theta - \theta_c)^2}{2\sigma_c^2} + K' \end{aligned}$$

A 1D problem, III

$$\log p(\theta|x) = -\frac{(c\theta - x)^2}{2\sigma_m^2} - \frac{(\theta - \theta_c)^2}{2\sigma_c^2} + K'$$

$$= -\frac{\theta^2}{2}\left[\frac{\sigma_m^2 + c^2\sigma_c^2}{\sigma_m^2\sigma_c^2}\right] + \theta\left[\frac{\theta_c\sigma_m^2 + cx\sigma_c^2}{\sigma_m^2\sigma_c^2}\right] + K''$$

A 1D problem, IV

- Now *IF* $p(\theta|x)$ is normal (say $N(\mu_t; \sigma_t^2)$)
- Then

$$\log p(\theta|x) = -\frac{(\theta - \mu_t)^2}{2\sigma_t^2} + K'''$$

$$= -\frac{\theta^2}{2\sigma_t^2} + \theta\frac{\mu_t}{\sigma_t^2} + K'''$$

Pattern match

$$\log p(\theta|x) = -\frac{(c\theta - x)^2}{2\sigma_m^2} - \frac{(\theta - \theta_c)^2}{2\sigma_c^2} + K'$$

$$= -\frac{\theta^2}{2}\left[\frac{\sigma_m^2 + c^2\sigma_c^2}{\sigma_m^2\sigma_c^2}\right] + \theta\left[\frac{\theta_c\sigma_m^2 + cx\sigma_c^2}{\sigma_m^2\sigma_c^2}\right] + K''$$

$$\log p(\theta|x) = -\frac{(\theta - \mu_t)^2}{2\sigma_t^2} + K'''$$

$$= -\frac{\theta^2}{2\sigma_t^2} + \theta\frac{\mu_t}{\sigma_t^2} + K'''$$

A 1D Problem, V

$$\mu_t = \frac{\theta_c \sigma_m^2 + cx\sigma_c^2}{\sigma_m^2 + c^2\sigma_c^2}$$

$$\sigma_t^2 = \frac{\sigma_c^2 \sigma_m^2}{\sigma_m^2 + c^2\sigma_c^2}$$

- Important checks:
  - what happens if measurement is unreliable?
  - what happens if prior is very diffuse?

Summary, with change of notation

Useful Fact: 9.2 *The parameters of a normal posterior with a single measurement*

Assume we wish to estimate a parameter θ. The prior distribution for θ is normal, with known mean μ_π and known standard deviation σ_π. We receive a single data item x_1 and a scale c_1. The likelihood of x_1 is normal with mean $c_1\theta$ and standard deviation $\sigma_{m,1}$, where $\sigma_{m,1}$ is known. Then the posterior, $p(\theta|x_1, c_1, \sigma_{m,1}, \mu_\pi, \sigma_\pi)$, is normal, with mean

$$\mu_1 = \frac{c_1 x_1 \sigma_\pi^2 + \mu_\pi \sigma_{m,1}^2}{\sigma_{m,1}^2 + c_1^2 \sigma_\pi^2}$$

and standard deviation

$$\sigma_1 = \sqrt{\frac{\sigma_{m,1}^2 \sigma_\pi^2}{\sigma_{m,1}^2 + c^2 \sigma_\pi^2}}.$$

Now a second measurement arrives…

- We know that $p(\theta|x)$ is normal
  - think of this as the prior

- We know that $p(x_1|\theta)$ is normal
  - think of this as the likelihood

- So:
  - the posterior $p(\theta|x_1, x)$ must be normal

  - and we can update as before!