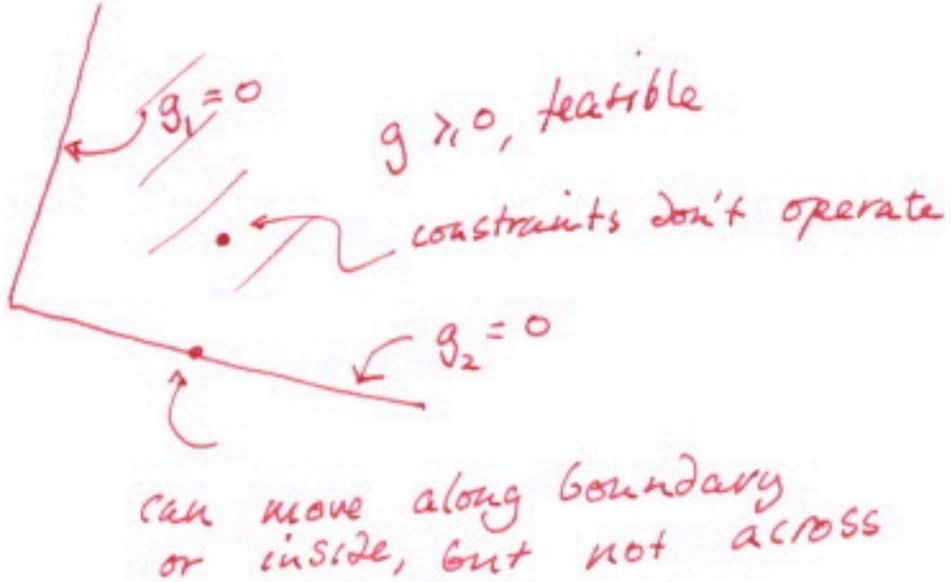
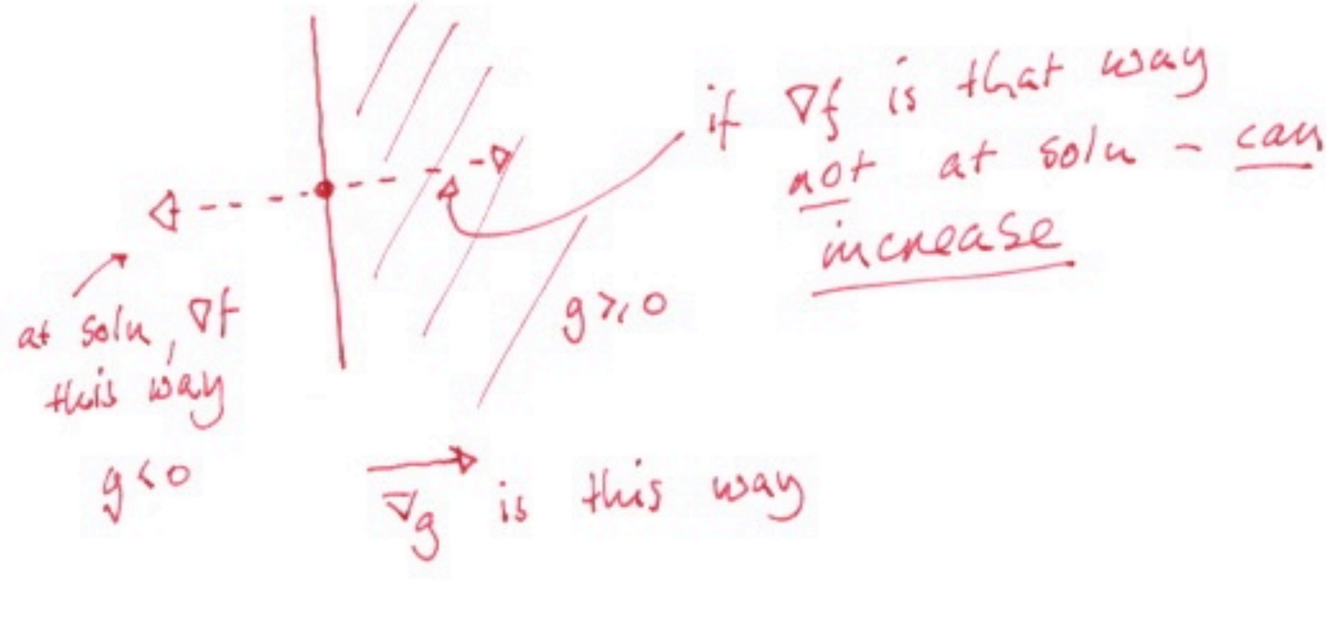
Lagrangians and inequalities

[use l.p. to visualize]

max $f(x)$

st $g(x) \geq 0$





all this means that at a soln. of

$$\max \; f(x)$$
$$\text{st} \quad g(x) \geq 0$$

$$\nabla f + \lambda^T J_g = 0$$

jacobian; matrix of derivatives.

<u>AND</u> $\lambda \geq 0$

<u>AND</u> $g \geq 0$

<u>AND</u> if $g_i = 0$, $\lambda_i > 0$
$g_i > 0$, $\lambda_i = 0$

<u>Sidenote:</u> a great deal of ~~[illegible]~~ confusion can result caused by different conventions.
various signs will change if:

- $\min f$ instead of $\max f$
- $g \leq 0$ instead of $g \geq 0$

you are likely to see <u>all</u> possibilities. <u>BUT</u>.
the signs are there for a <u>reason</u>. You can work it all out as above.

Now bundle in equality constraints

$$\max f \quad \text{st} \quad g \geq 0$$
$$h = 0$$

$$\mathcal{L}: \quad f - \lambda^T g - \mu^T h$$

Conditions for a solution:

$$\boxed{\begin{aligned} &\nabla_x \mathcal{L} = 0 \\ &g \geq 0 \\ &h = 0 \\ &h_i g_i = 0, \text{ for each } i \\ &\lambda \geq 0 \end{aligned}}$$

Karush-Kuhn-Tucker or KKT conditions

Notice that

$$\lambda_i g_i = 0 \quad \text{is} \equiv \text{to} \begin{cases} g_i = 0, & \lambda \geq 0 \\ g > 0, & \lambda = 0 \end{cases}$$

this is known as a $\boxed{\text{complementarity condition}}$

and is VICIOUSLY NON LINEAR
and a source of <u>much suffering</u>

We can still construct duals.

$$Q(\lambda, \mu) = \sup_x \mathcal{L}(x, \lambda, \mu)$$

(notice: if you want to min, wrt)

$$Q(\lambda, \mu) \geq f(x^*)$$

(by previous argument)

Under fairly weak conditions
(eg: convex problem, feasible set has interior point)

<u>Strong duality</u> holds.

$$\hookrightarrow \inf_{\lambda, \mu} Q(\lambda, \mu) = f(x^*).$$

This has important applications, which we will explore.

<u>Constructing duals</u> :

Linear program

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax = b, \quad x \geq 0$$

$$\mathcal{L}(x,\lambda,\mu) = c^T x - \lambda^T x - \mu^T(Ax-b)$$

$$Q(\lambda,\mu) = \sup_x \mathcal{L}(x,\lambda,\mu)$$

notice that this $= \infty$ <u>unless</u> the coeff of x in $\mathcal{L}$ is 0

<u>so</u> we must have

$$c - A^T\mu - \lambda = 0$$

for the dual to be finite

IN THIS SET, $\mathcal{L}(x,\lambda,\mu) = \mu^T b$

<u>AND</u> $\lambda \geq 0$ ← from KKT

so dual is

$$\min \quad \mu^T b$$

$$\text{st} \quad c - A^T\mu - \lambda = 0$$

$$\lambda \geq 0$$

KKT for linear program

$$\nabla_x \mathcal{L} = 0 = c - A^T\mu - \lambda$$

$$x \geq 0$$
$$Ax - b = 0$$

these are primal feasibility conditions

$$\lambda \geq 0$$

these are dual feasible conds.

[complementarity]

so we have (x, λ, μ) is
ⓐ primal feas
ⓑ dual feas
ⓒ complementarity

$\Updownarrow$

(x, λ, μ) satisfy KKT

$\Updownarrow$

(x, λ, μ) are soln

This opens a new world of algorithms

Was: Simplex work in primal.

New

- work with primal feasible x
  dual infeasible λ, μ
  complementarity; improve λ, μ.

- work with primal infeasible x
  dual feasible λ, μ
  complementarity; ~~fix~~ improve x

- work w/ feasible x
  dual feasible λ, μ
  not comp.; improve comp