Stochastic gradient descent

Many learning problems look like

$$\arg\min_{w} \; L(w, x_i, y_i) + \lambda \|w\|_2$$

loss on examples

some regularize typically 2-norm or 1-norm.

eg. SVM.

loss is $\sum_i$ hinge loss (w, example i)

eg. logistic regression

loss is $\sum_i$ exp loss.

In these cases, two important obs.

1) We don't so much care about

$$E = \sum_{i \in \text{examples}} L(w, \text{example}_i) \quad \text{as about}$$

for empirical loss

$$\sum_{i \in \underline{\text{everything}}} L(w, \text{example}_i). = J$$

for true loss

this means that the min of the optimization problem w^*, may not be best w (which is w^{opt})

$$w^{opt} = \arg\min J$$

$$w^* = \arg\min E$$

error incurred by exact optimization

$\Rightarrow$ This means inexact opt might be OK.

2) Exact optimization could be <u>hugely</u> expensive

(millions of examples; millions of feats)

<u>Idea:</u> Pick example (feature) at random, compute gradient for that alone, take small step.

<u>STOCHASTIC GRADIENT DESCENT</u>

- How far?

  steplengths δ_i should have property

  - $\delta_N \longrightarrow 0, \quad N \to \infty$
  - $\sum_{i=1}^{R} \delta_i \longrightarrow \infty \quad R \to \infty$

(ie. path ~~can~~ must be infinitely long, but steps must become infinitely small) e.g. **Logistic regression with 2-Norm**

$$\min_{w} \sum_{i} \left[\mathcal{L}_e(\omega, y, x) + \frac{\lambda}{N} \|w\|_2^2 \right] = \mathcal{E} + R$$

- for each step choose 1 example$_N$ UAR. Say ~~a~~ k'th

$$\nabla g^{(n)} = \left[\frac{x_k}{1 + e^{w^{(n)T} x_k}} - \left(\frac{y_k + 1}{2} \right) x_k \right] + \frac{2\lambda}{N} w^{(n)}$$

$$w^{(n+1)} = w^{(n)} - \delta_i \cdot g^{(n)}$$

- Notice, because we choose UAR,

$$E(g^{(n)}) = \nabla_w \left[\mathcal{E} + R \right]$$

typically, step lengths look like

$$\delta_i = \frac{1}{i}$$

(this satisfies constraints).

We get (with work)

$$\frac{1}{(w_t - w^*)^2}$$ grows <u>linearly</u> in t

<u>i.e.</u> early steps really help, late steps don't do much.

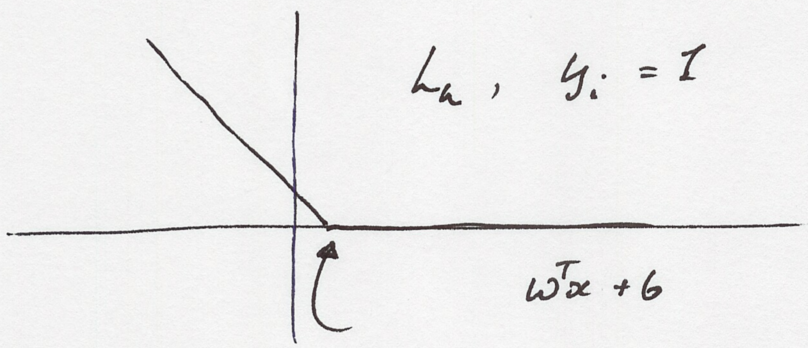
<u>Notice</u>: <u>algorithm is on-line</u>

What about SVM?

$$\sum_i \left[L_h(w, x_i, y_i) + \frac{\lambda}{N} w^T w \right]$$

which isnt differentiable.

$$L_h = \max(0, 1 - y_i(w^T x_i + b))$$



however, this is convex.

- We invoke the <u>Sub-gradient</u>

consider the graph of a function

$$(x_1, x_2, x_3, \cdot, x_n, f(x_1 \cdots x_n))$$

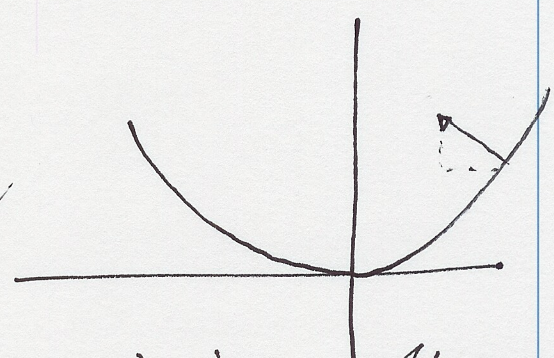
this is a surface.

- assume ~~the~~ f is differentiable
  — surface has a normal

$$N = k\left(-\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, \ldots, +1\right)$$

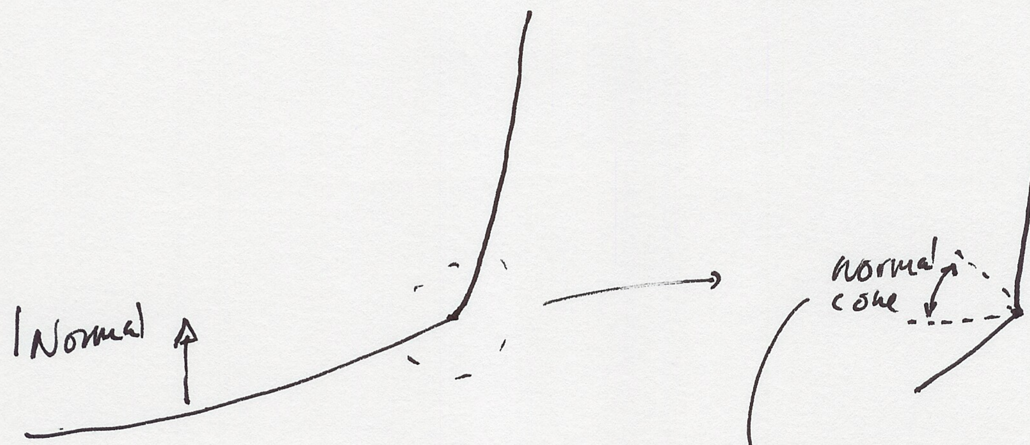
$$= k\left(-\nabla f, +1\right)$$

eg curve in the plane
is graph of fn of 1 var



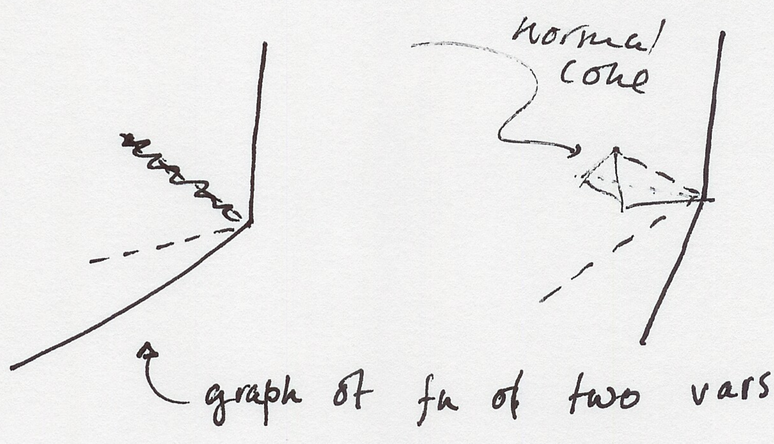
Notice we can read gradient off normal

Now, if ~~curve~~ function is not diff, we can come up w/ normal cone



all of these vectors are normals at this pt.

(• Notice, f needs to be convex so we know what interior is for this construction)



graph of fn of two vars

- read off gradient corresp to <u>any</u> element of normal cone
- this is <u>subgradient</u> ($\tilde{\nabla}$)
- moving backward along subgradient will guarantee descent for small enough step
- subgradient of diff. fn = gradient

<u>SVM</u>.

$$\tilde{\nabla}_w \mathcal{L}_h(w) = \begin{cases} 0 & \text{if } \mathcal{L}_h(w) = 0 \\ -y_i x_i & \text{otherwise} \end{cases}$$

so ~~Subgr~~ Stochastic Subgradient Descent is

- choose k'th example at random
- if right, $w^{(n+1)} = w^{(n)}\left(1 - \frac{\lambda \delta_i}{2N}\right)$

  if wrong, $w^{(n+1)} = w^{(n)} - \delta_i \left[-y_i x_i + \frac{\lambda w^{(n)}}{2N} \right]$

This is amazingly effective.

L_1 regularization

$$\|w\|_1 = \sum_i |w_i|$$

$$\|w\|_2^2 = \sum_i w_i^2$$

<u>Notice</u>: in L_2 norm, small w_i have little effect hence, not much advantage in driving to zero; in L_1 norm, effect of small w_i is substantial; they <u>tend</u> to go to zero, resulting in <u>sparsity</u> (desirable)

eg L_1 reg logistic regression

$$\min_{w} \sum_{i} L_e(w, y_i, x_i) \quad + \lambda \|w\|_1$$

- not differentiable
- subgradient unlikely to enforce sparsity

alternative (equivalent)

$$\min_{w} \sum_{i} L_e(w, y_i, x_i) + \sum_{e} h_e$$

$$-h_k \leq w_k \leq h_k$$

(Notice this is a box problem; Solve with interior point method; excellent version due to Koh et al)