A gentle introduction to Fourier analysis

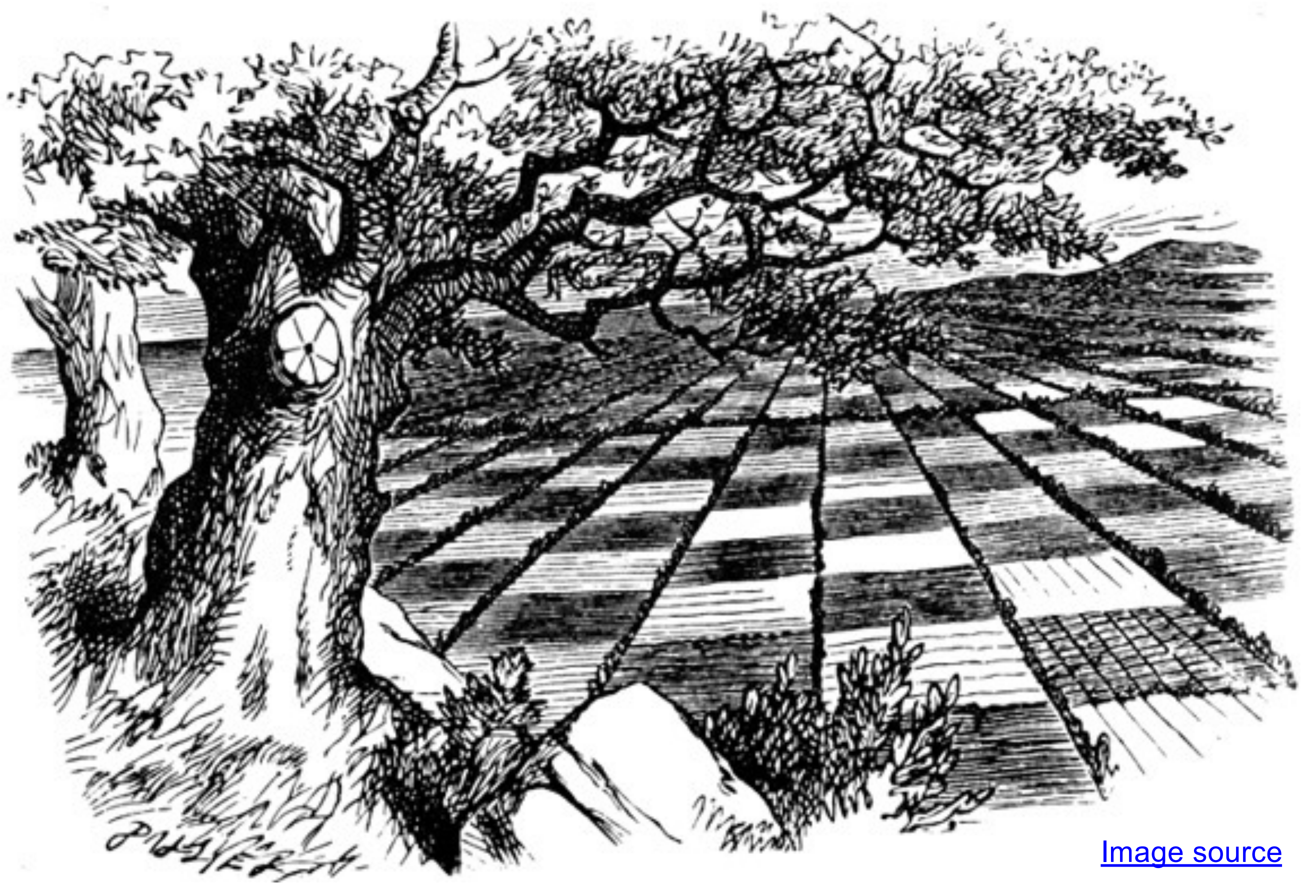


Image source

Many slides borrowed from S. Seitz, A. Efros, D. Hoiem, B. Freeman, A. Zisserman

Reference table in notes

Function	Fourier transform	Tag
$f(x,y)$	$\iint_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+vy)}dxdy = \mathcal{F}(f)(u,v)$	1
$\iint_{-\infty}^{\infty} \mathcal{F}(f)(u,v)e^{i2\pi(ux+vy)}dudv = f(x,y)$	$\mathcal{F}(f)(u,v)$	2
$\frac{\partial f}{\partial x}(x,y)$	$u\mathcal{F}(f)(u,v)$	3
$0.5\delta(x+a,y)+0.5\delta(x-a,y)$	$\cos 2\pi au$	4
$\cos 2\pi ax$	$0.5\delta(u+a,v)+0.5\delta(u-a,v)$	5
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$	6
$box_1(x,y)$	$\frac{\sin u}{u}\frac{\sin v}{v}$	7
$f(ax,by)$	$\frac{\mathcal{F}(f)(u/a,v/b)}{ab}$	8
$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(x-i,y-j)$	$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(u-i,v-j)$	9
$f(x-a,y-b)$	$e^{-i2\pi(au+bv)}\mathcal{F}(f)$	10
$f(x\cos\theta-y\sin\theta,x\sin\theta+y\cos\theta)$	$\mathcal{F}(f)(u\cos\theta-v\sin\theta,u\sin\theta+v\cos\theta)$	11
$(f*g)(x,y)$	$\mathcal{F}(f)\mathcal{F}(g)(u,v)$	12

Convolution theorem

- **Convolution** in the spatial domain translates to **multiplication** in the frequency domain (and vice versa)

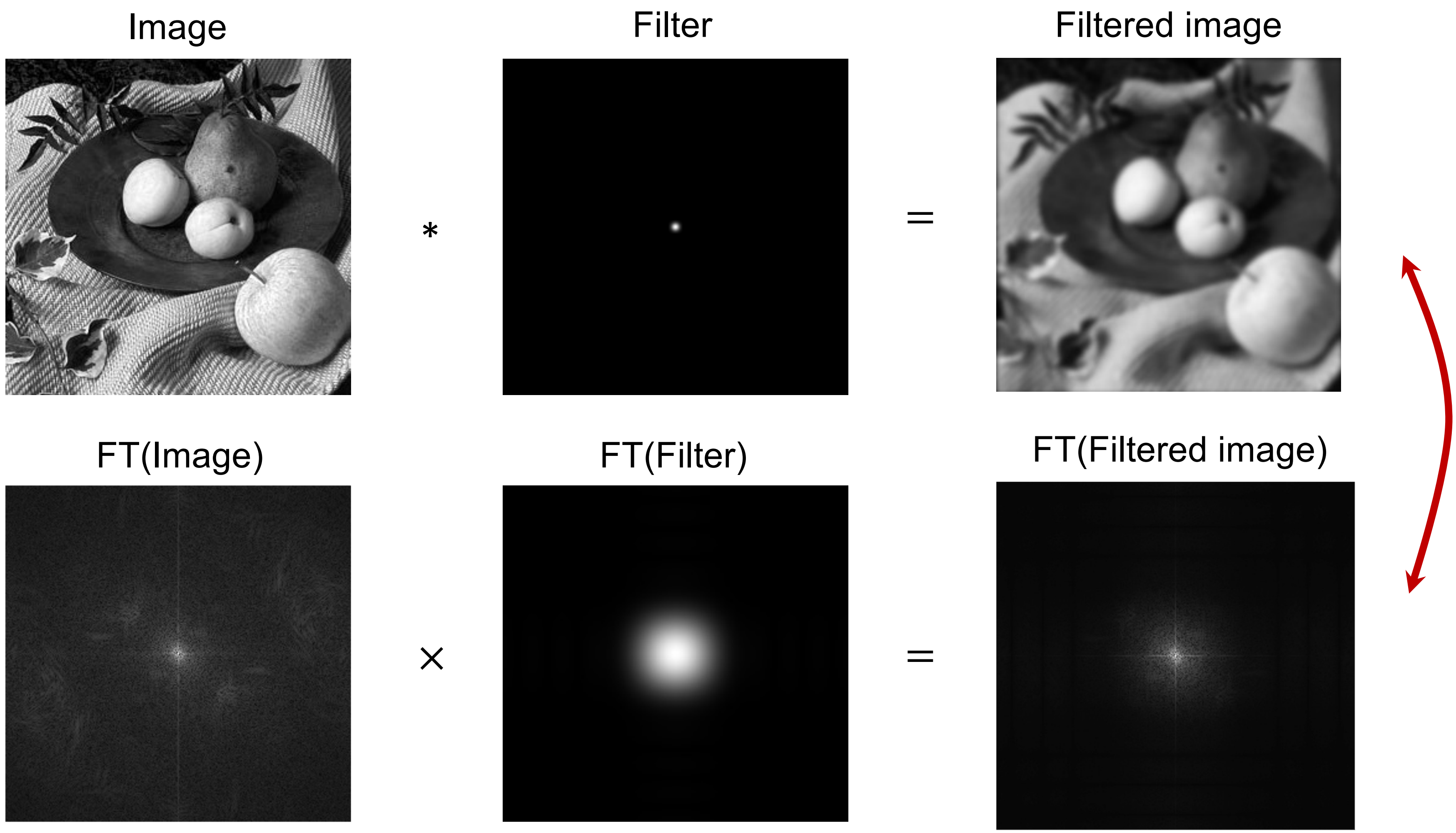
- The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{FG\} = \mathcal{F}^{-1}\{F\} * \mathcal{F}^{-1}\{G\}$$

2D convolution theorem example



Easy to prove

$$
\begin{aligned}
\mathcal{F}(f * g) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \right) \exp\left[-i2\pi u t\right] dt
\end{aligned}
$$

Swap integration bounds

$$
= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) g(\tau) \right) \exp\left[-i2\pi u t\right] dt d\tau
$$

Move g out of integral

$$
= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \exp\left[-i2\pi u t\right] dt g(\tau) \right) d\tau
$$

Shift result from before

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left[\mathcal{F}(f)\right] \exp\left[-i2\pi u \tau\right] g(\tau) d\tau \\
&= \left[\mathcal{F}(f)\right] \int_{-\infty}^{\infty} g(\tau) \exp\left[-i2\pi u \tau\right] d\tau \\
&= \left[\mathcal{F}(f)\right] \left[\mathcal{F}(g)\right].
\end{aligned}
$$

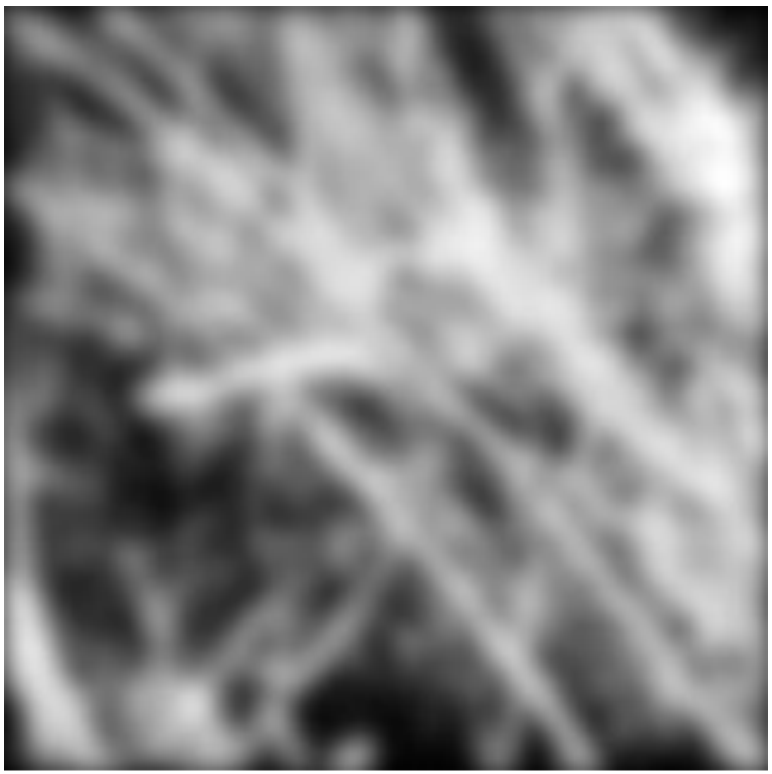
Convolution theorem

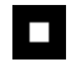
- Suppose f and g both consist of N pixels
- What is the complexity of computing $f * g$ in the spatial domain?
  - $O(N^2)$
- And what is the complexity of computing $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\mathcal{F}\{g\}\}$?
  - $O(N \log N)$ using FFT
- Thus, convolution of an image with a large filter can be more efficiently done in the frequency domain

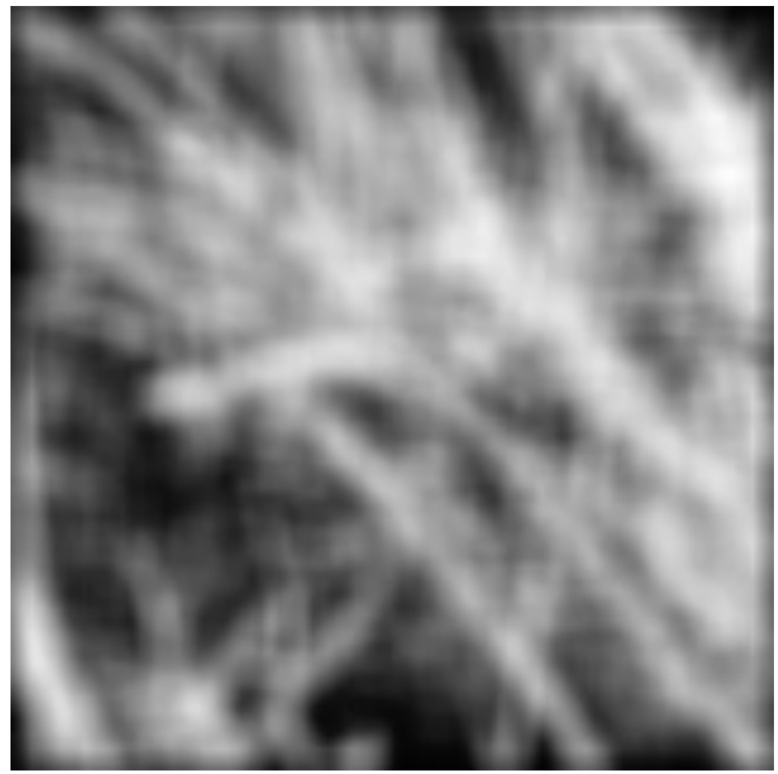
Mystery 1

- Why does filtering with a Gaussian give a nice smooth image, but filtering with a box filter gives artifacts?

Gaussian 



Box filter 



Consider this in FT space

Convolution theorem:

FT is
  FT(gaussian)*FT(image)

FT(gaussian)=another
  gaussian

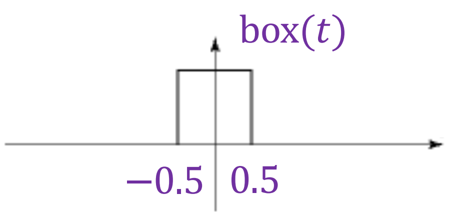
Weight falls off smoothly at
  higher frequencies

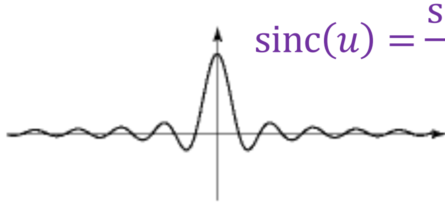
In FT space

Convolution theorem:

FT is
FT(box)*FT(image)

FT(box)=Sinc



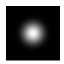


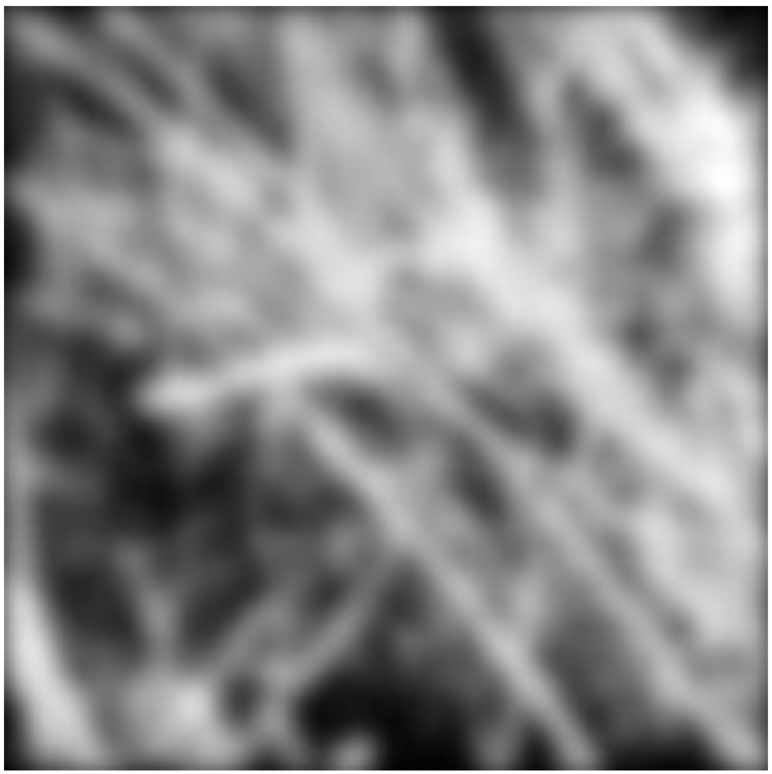
$$\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

Box filter

Mystery 1 SOLVED

- Why does filtering with a Gaussian give a nice smooth image, but filtering with a box filter gives artifacts?

Gaussian 



Box filter