A gentle introduction to Fourier analysis

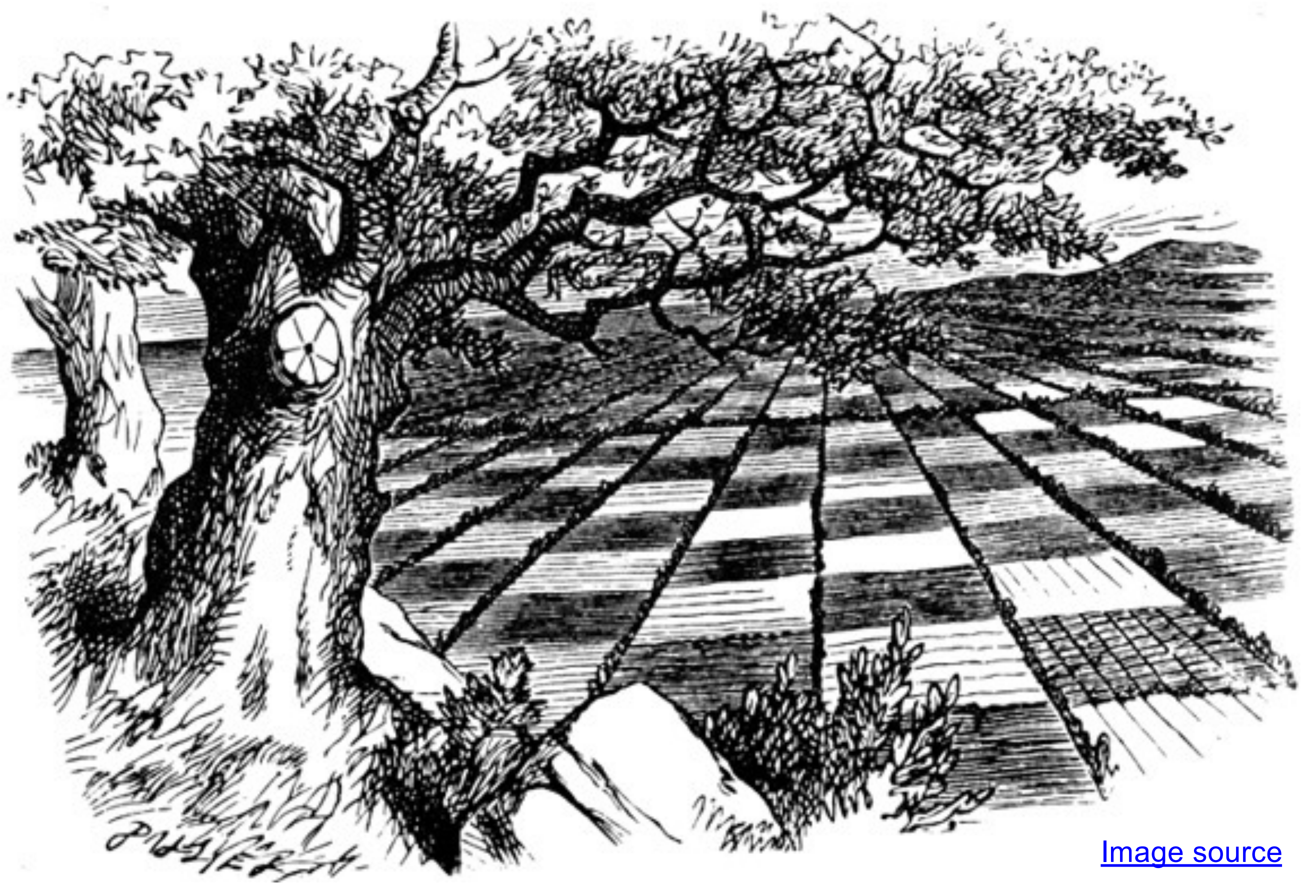


Image source

Many slides borrowed from S. Seitz, A. Efros, D. Hoiem, B. Freeman, A. Zisserman

1D Fourier transform

- Let's define an (overcomplete) set of basis functions:

$$\psi_u(t) = e^{i2\pi ut}, \qquad u \in (-\infty, \infty)$$

- Compare

Integer

$$e^{i2k\pi t} = \cos(2k\pi t) + i\sin(2k\pi t)$$

1D Fourier transform

- Let's define a (continuously parameterized) set of basis functions:

$$\psi_u(t) = e^{i2\pi ut}, \qquad u \in (-\infty, \infty)$$

- Inner product for complex functions is given by:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dt$$

Complex conjugate:
real part stays the same,
imaginary part is flipped

1D Fourier transform

- Let's define a (continuously parameterized) set of basis functions:

$$\psi_u(t) = e^{i2\pi ut}, \qquad u \in (-\infty, \infty)$$

- Inner product for complex functions is given by:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dt$$

- Orthonormality:

$$\langle \psi_{u_1}, \psi_{u_2} \rangle = \delta(u_1 - u_2) = \begin{cases} ? & \text{if} \quad u_1 = u_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t) dt = f(0)$$

1D Fourier transform

- Given a signal $f(t)$, we want to represent it as a weighted combination of the basis functions $\psi_u(t) = e^{i2\pi ut}$ with weights $F(u)$:

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ut} du$$

- Each weight $F(u)$ is given by the inner product of f and ψ_u:

$$F(u) = \langle f, \psi_u \rangle = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

1D Fourier transform

- Forward transform: $f(t) \xrightarrow{\mathcal{F}} F(u)$

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

- Note: for the FT to exist, the *energy* $\int_{-\infty}^{\infty} |f(t)|^2 \, dt$ has to be finite

1D Fourier transform

- Forward transform: $f(t) \xrightarrow{\mathcal{F}} F(u)$

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

- For each u, $F(u)$ is a *complex number* that encodes both the *amplitude* A and *phase* ϕ of the sinusoid $A\sin(2\pi ut + \phi)$ in the decomposition of $f(t)$:

$$F(u) = \mathrm{Re}(F(u)) + i\,\mathrm{Im}(F(u)),$$

$$A = \sqrt{\mathrm{Re}(F(u))^2 + \mathrm{Im}(F(u))^2}, \qquad \phi = \tan^{-1}\frac{\mathrm{Im}(F(u))}{\mathrm{Re}(F(u))}$$

- If $f(t)$ is real, then $\mathrm{Re}(F(u)) = \mathrm{Re}(F(-u))$,

$$\mathrm{Im}(F(u)) = -\mathrm{Im}(F(-u))$$

1D Fourier transform

- Forward transform: $f(t) \xrightarrow{\mathcal{F}} F(u)$

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

- Important properties:
  - Energy preservation:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(u)|^2 \, du$$

Parseval's Theorem!

  - Linearity: $\mathcal{F}\{af_1 + bf_2\} = a\mathcal{F}\{f_1\} + b\mathcal{F}\{f_2\}$

1D Fourier transform

- Forward transform: $f(t) \xrightarrow{\mathcal{F}} F(u)$

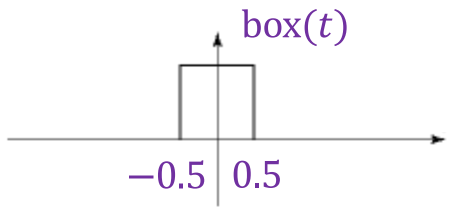
$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

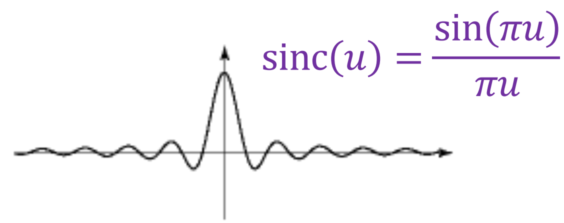
- Inverse transform: $F(u) \xrightarrow{\mathcal{F}^{-1}} f(t)$

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ut} du$$

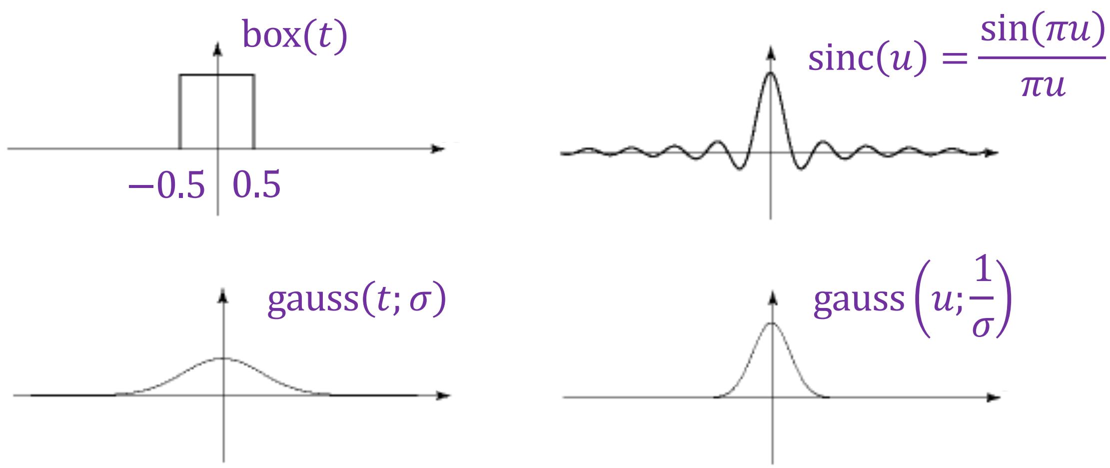
- Duality: if $f(t) \overset{\mathcal{F}}{\rightarrow} F(u)$, then $F(t) \overset{\mathcal{F}}{\rightarrow} f(-u)$
  - Thus, we can talk about *Fourier transform pairs* $f(t) \leftrightarrow F(u)$

Important Fourier transform pairs

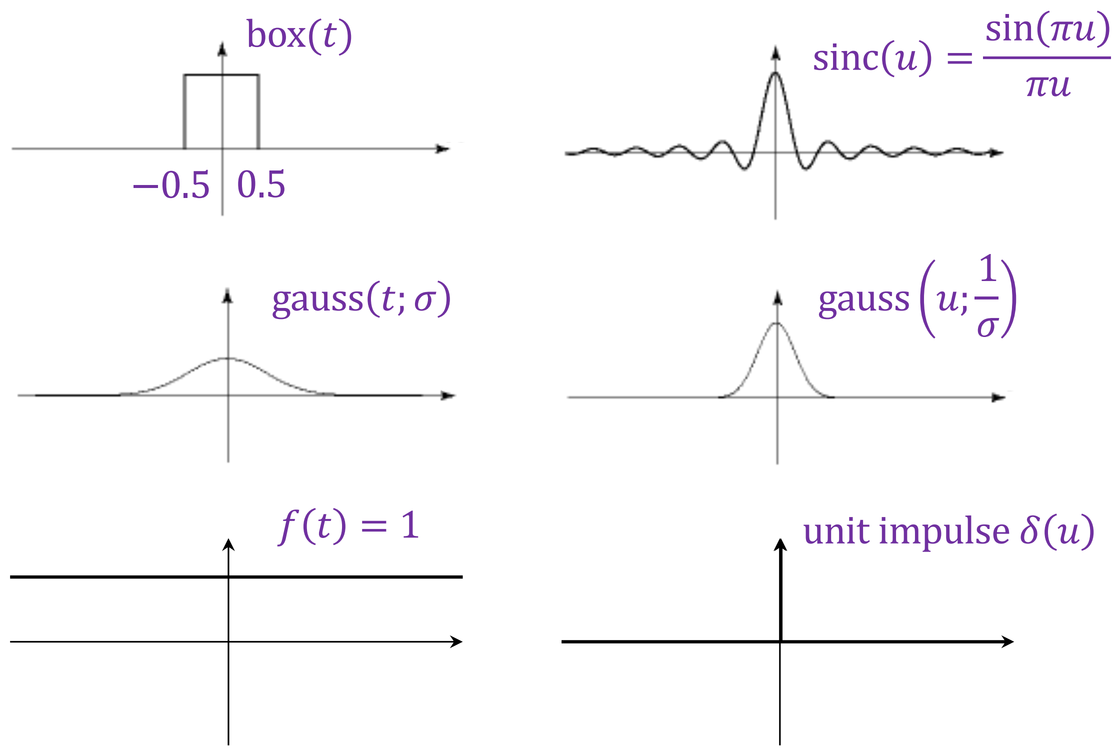




Important Fourier transform pairs



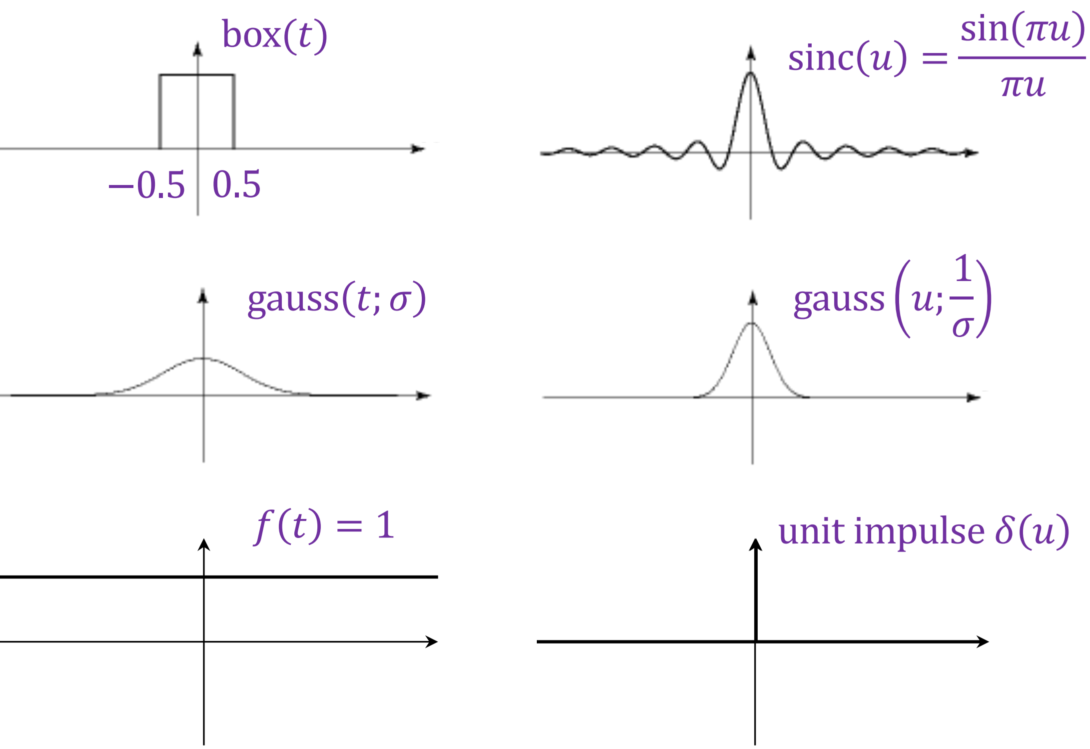
Important Fourier transform pairs



*The last one is formal since these functions don't meet the mathematical requirements for FT

Important Fourier transform pairs

Notice that when
f has narrower support,
FT(f) has broader, and
Vice versa!



*The last one is formal since these functions don’t meet the mathematical requirements for FT

Outline

- 1D Fourier transform
  - Definition and properties
  - Discrete Fourier transform

Discrete Fourier transform (DFT)

- Now suppose our signal consists of N samples $f(n)$,
$$n = 0, \dots, N-1$$
- We can also discretize frequencies to k/N, $k = 0, \dots, N-1$ (k cycles per N samples)

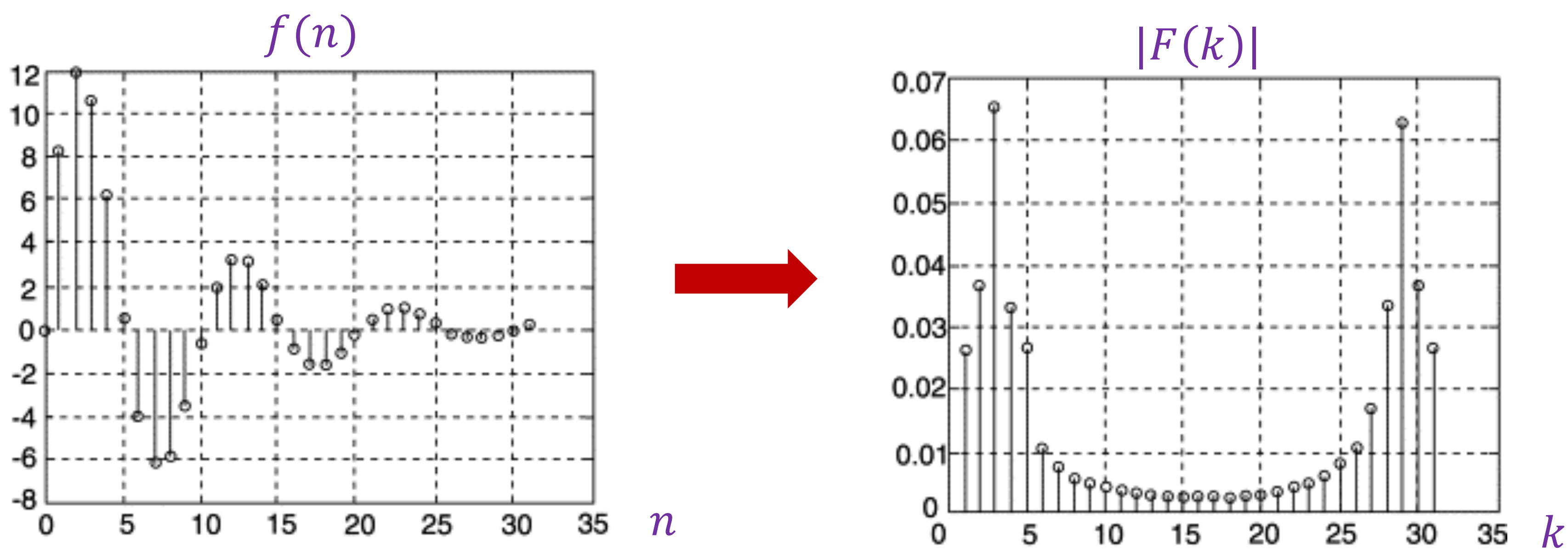


Image source

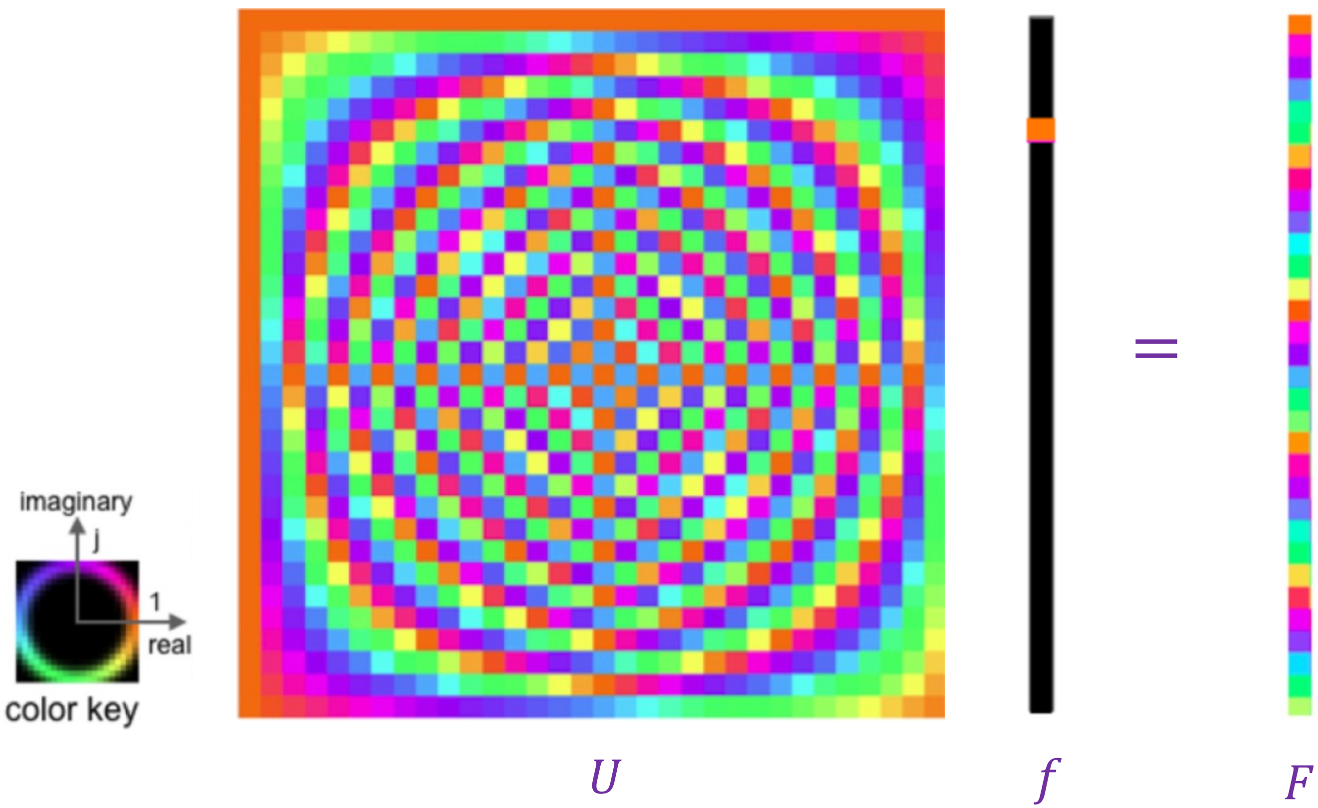
Discrete Fourier transform (DFT)

- Now suppose our signal consists of N samples $f(n)$,
$$n = 0, \ldots, N-1$$
- We can also discretize frequencies to k/N, $k = 0, \ldots, N-1$ (k cycles per N samples)
- DFT formula:
$$F(k) = \sum_{n=0}^{N-1} f(n) \exp\left(-i \frac{2\pi k}{N} n\right)$$
- We can pack the values $\exp\left(-i \frac{2\pi k}{N} n\right)$, $k, n = 0, \ldots, N-1$ into an $N \times N$ matrix U, and DFT becomes just a matrix-vector multiplication!
- Fast Fourier transform: only $N \log N$ complexity!

Image source

DFT: Just a change of basis!

$$U\,f\ =\ F$$



Source

Inverse DFT

- Forward DFT:

$$F(k) = \sum_{n=0}^{N-1} f(n) \exp\left(-i\frac{2\pi}{N}kn\right) \qquad \text{or } F = Uf$$

- Inverse DFT:

$$f(n) = \frac{1}{N}\sum_{n=0}^{N-1} F(k) \exp\left(i\frac{2\pi}{N}kn\right) \qquad \text{or } f = \frac{1}{N}U^{-1}F$$

where U^{-1} is the transpose of the *complex conjugate* of U

Periodicity of DFT and inverse DFT

- The result of DFT is periodic: because $F(k)$ is obtained as a sum of complex exponentials with a common period of N samples, $F(k + aN) = F(k)$ for any integer a:

$$F(k + aN) = \sum_{n=0}^{N-1} f(n)\exp\left(-i\frac{2\pi}{N}n(k + aN)\right)$$

$$= \sum_{n=0}^{N-1} f(n)\exp\left(-i\frac{2\pi n}{N}k\right)\exp(-i2\pi an) = F(k)$$

- Likewise, the result of the inverse DFT is a periodic signal: $f(t + aN) = f(t)$ for any integer a

Outline

- 1D Fourier transform
  - Definition and properties
  - Discrete Fourier transform
- 2D Fourier transform

2D Fourier transform

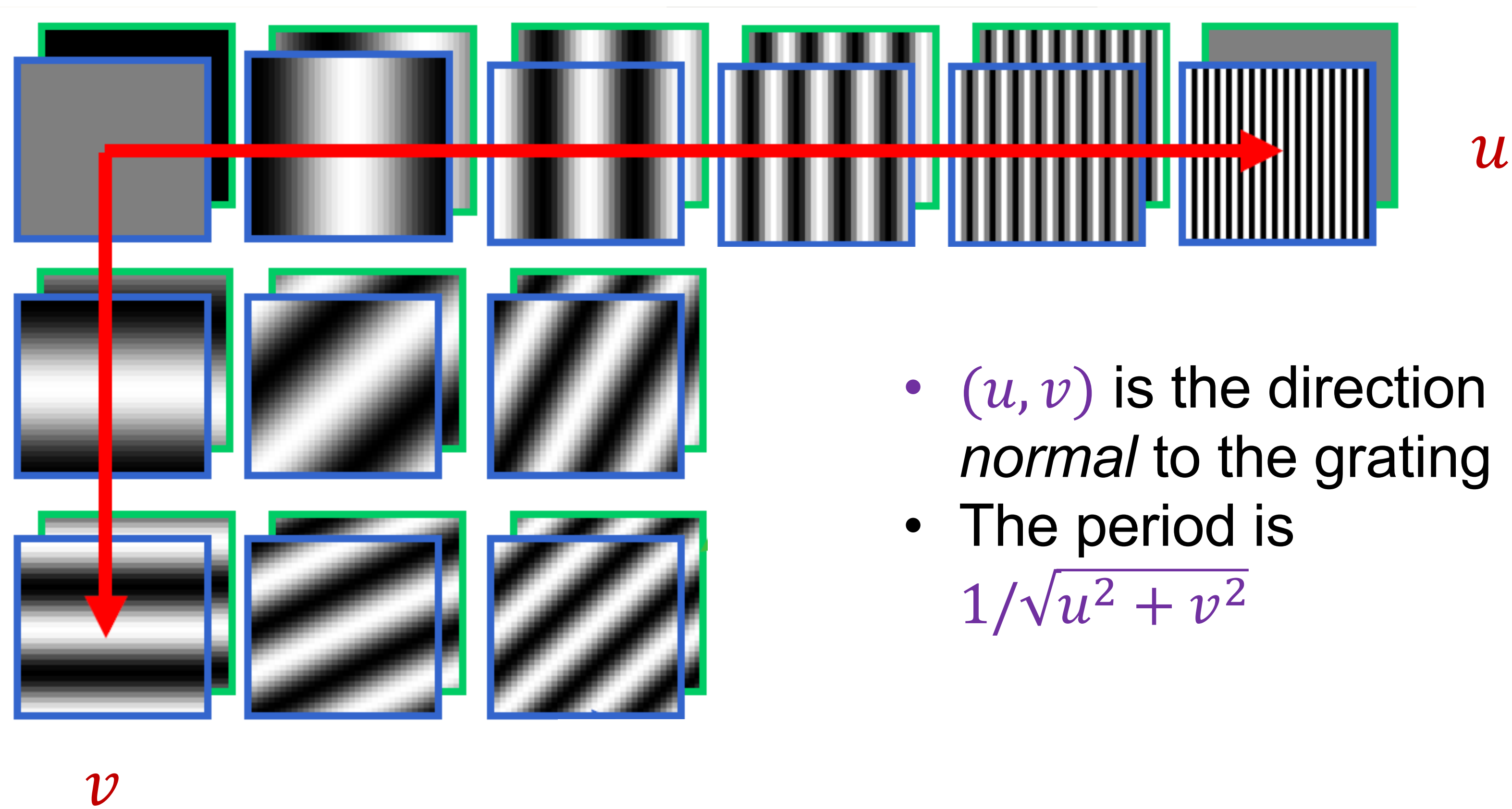
- To represent 2D signals $f(x, y)$, we need to extend our 1D basis functions $\psi_u(t) = e^{i2\pi ut}$ to two variables:

$$\psi_{u,v}(x, y) = e^{i2\pi ux} e^{i2\pi vy} = e^{i2\pi(ux+vy)}$$
$$= \cos 2\pi(ux + vy) + i \sin 2\pi(ux + vy)$$

- What does this look like?

2D Fourier transform

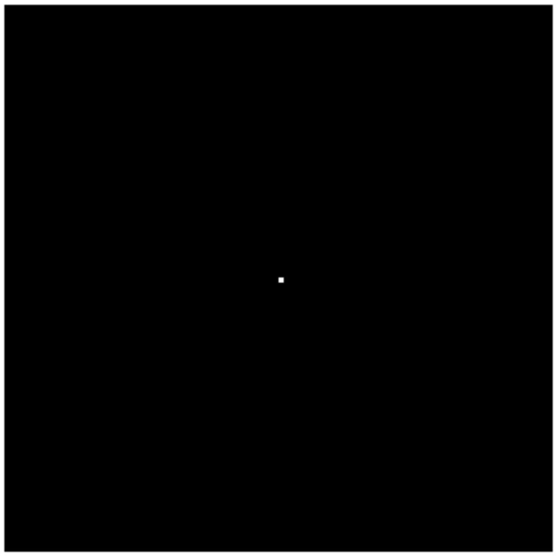
- 2D basis functions are oriented sinusoidal "gratings":



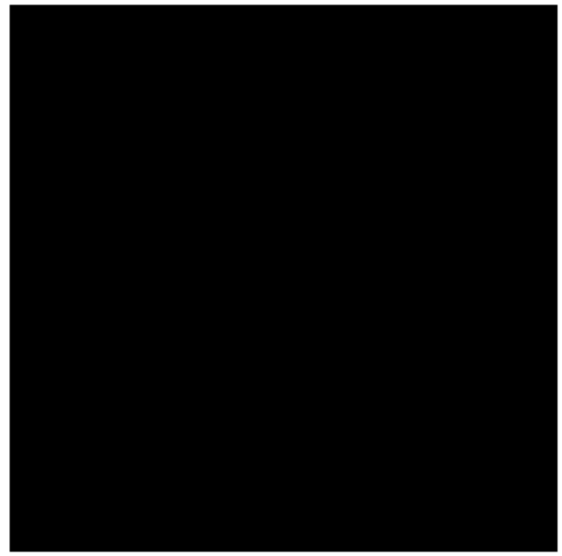
- (u, v) is the direction *normal* to the grating
- The period is $1/\sqrt{u^2 + v^2}$

Basis function examples

(u, v)

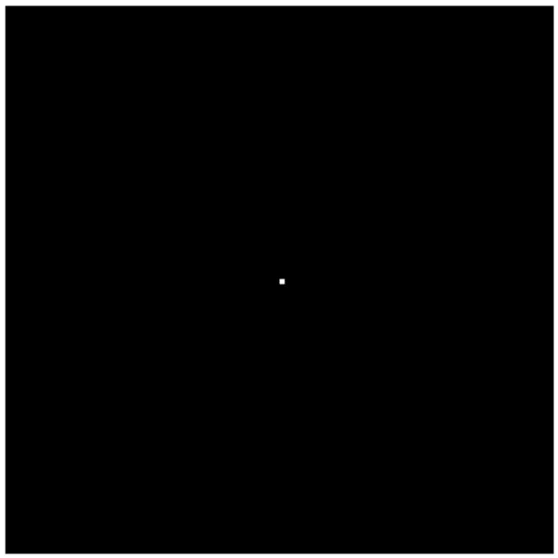


Real component

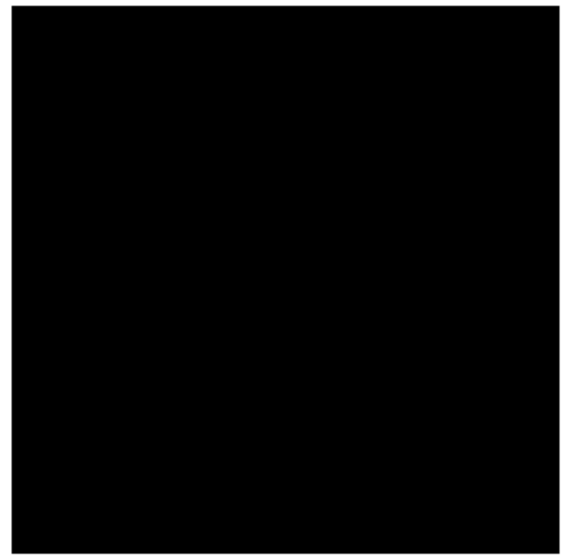


Basis function examples

(u, v)

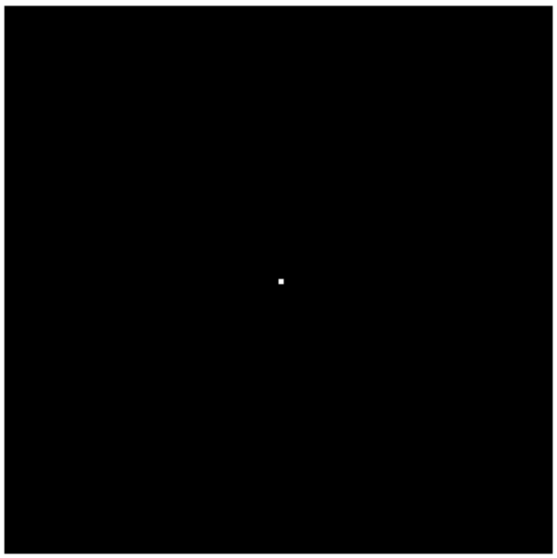


Real component

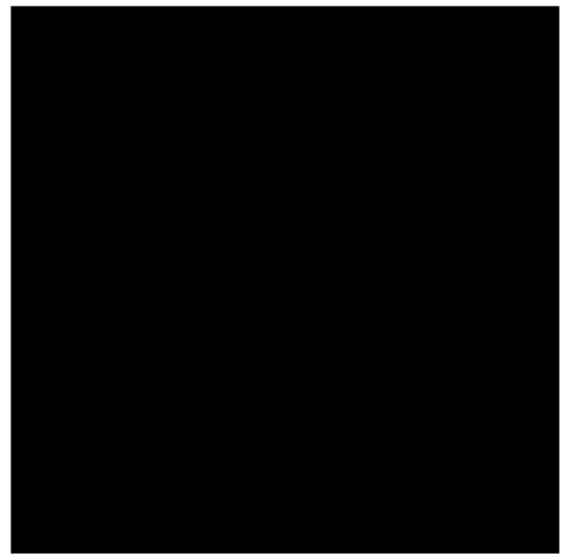


Basis function examples

(u, v)

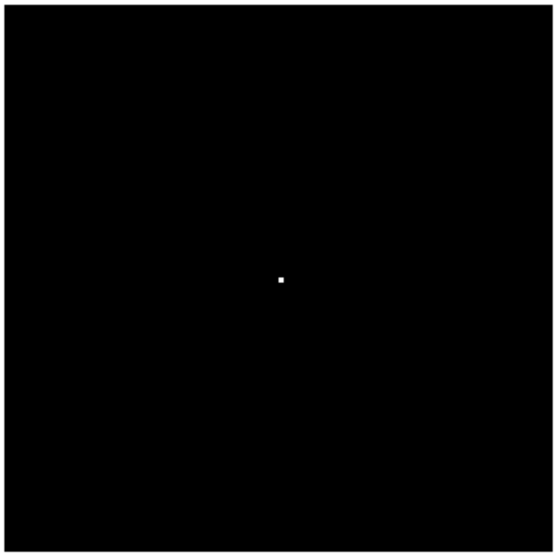


Real
component

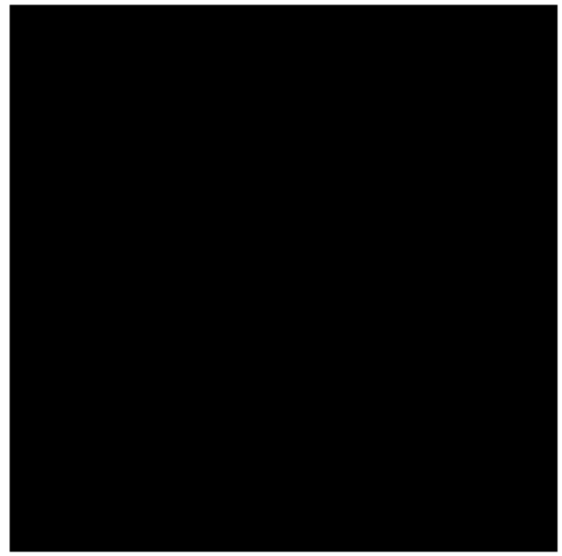


Linear combination of basis functions

(u, v)



Real component



2D Fourier transform

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+vy)}dx\,dy$$

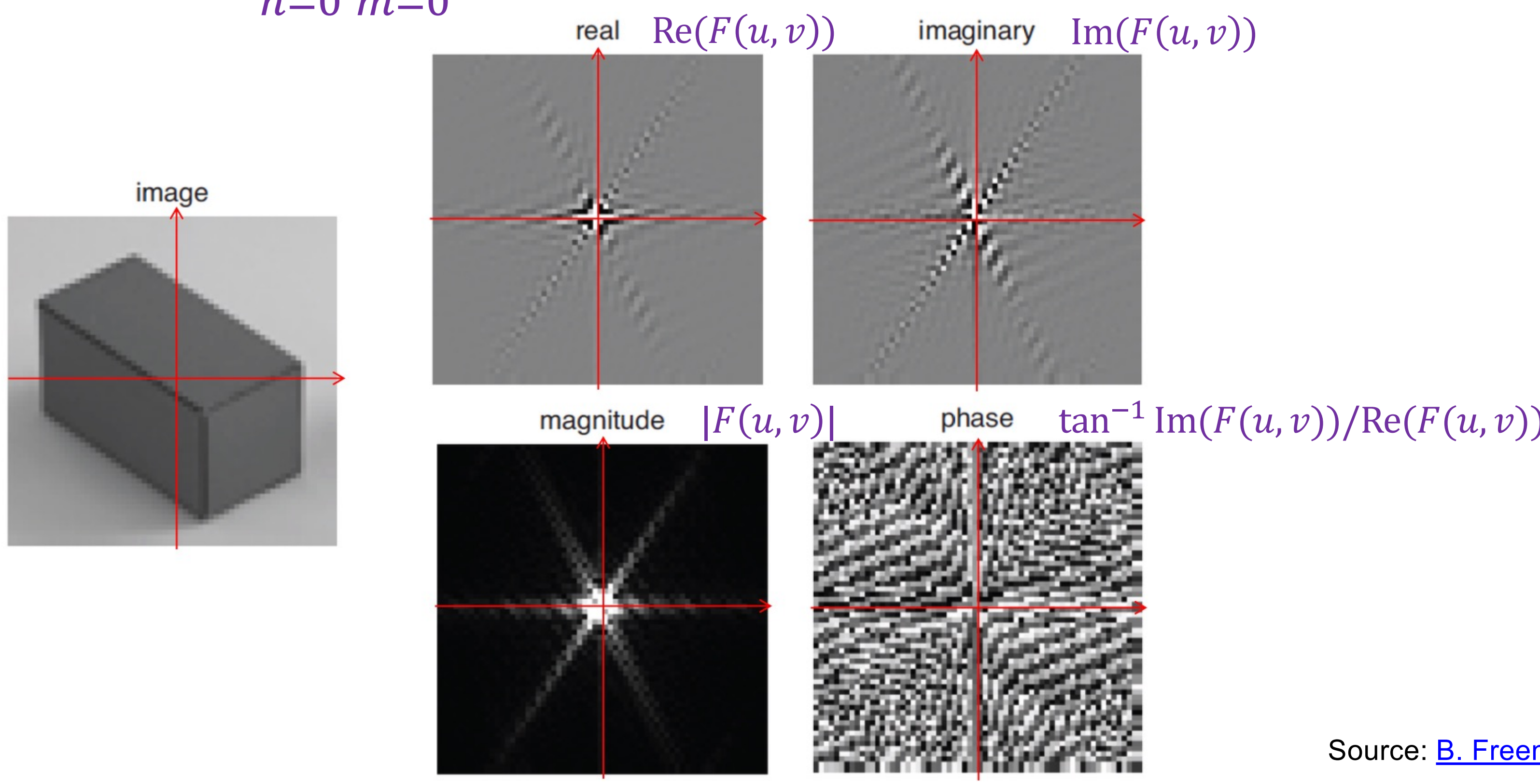
- Output is 2D and complex-valued:

$$F(u,v) = \mathrm{Re}(F(u,v)) + i\,\mathrm{Im}(F(u,v))$$

  - Magnitude spectrum: $|F(u,v)| = \sqrt{\mathrm{Re}(F(u,v))^2 + \mathrm{Im}(F(u,v))^2}$
  - Phase angle spectrum: $\tan^{-1}\frac{\mathrm{Im}(F(u,v))}{\mathrm{Re}(F(u,v))}$
- Symmetry: the Fourier transform of a real-valued image has coefficients that come in pairs, with $F(u,v)$ being the *complex conjugate* of $F(-u,-v)$
  - This means that the magnitude spectrum is symmetric about the origin

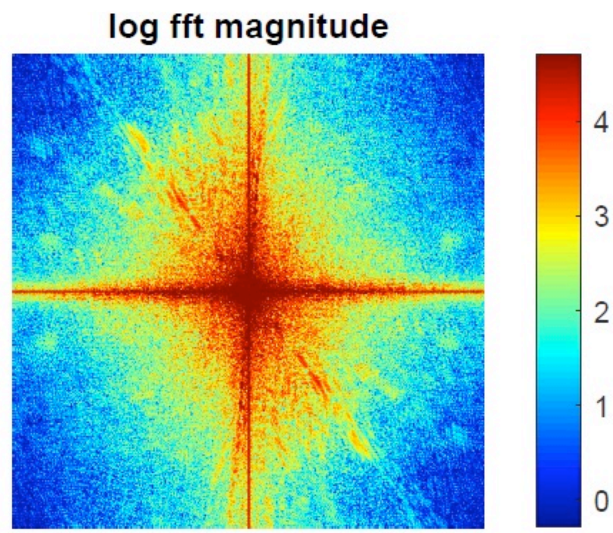
2D discrete Fourier transform

$$F(u,v) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(n,m) \exp\left(-i2\pi\left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$



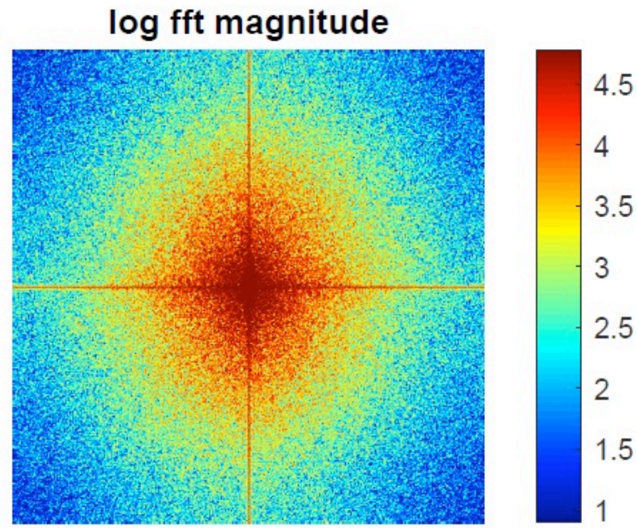
Source: [B. Freeman]

Real image examples

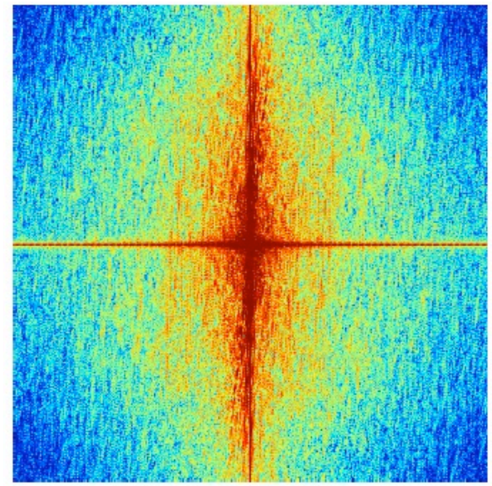


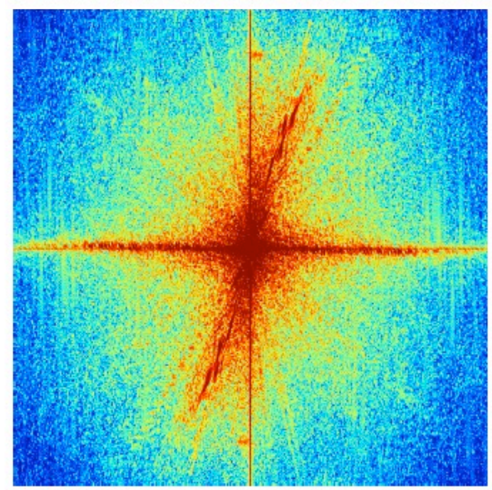
Real image examples

intensity image



Which image goes with which spectrum?





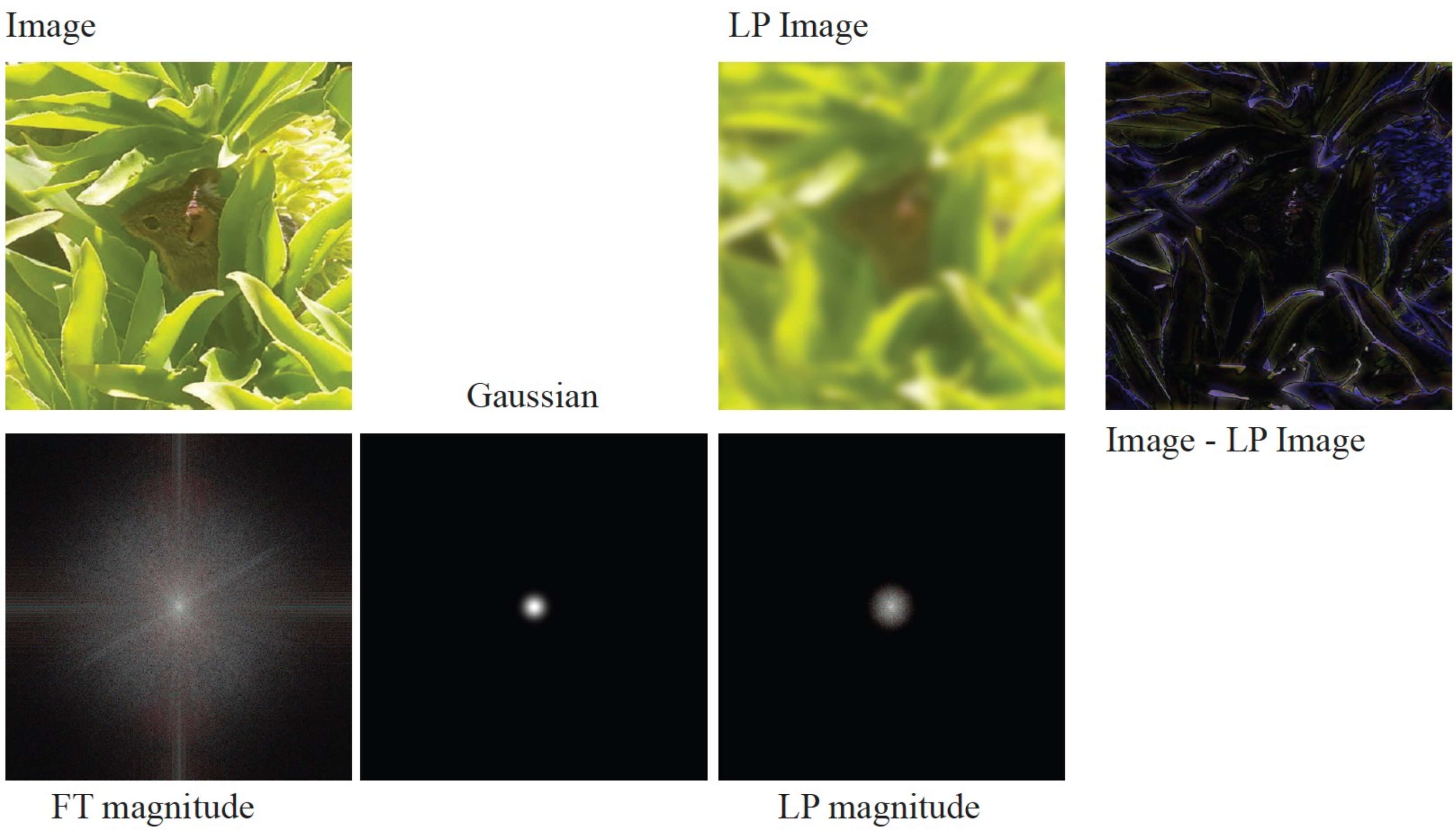
Trick – low pass filter

Multiply FT magnitude by Gaussian

Inverse FT

High frequencies are suppressed

Smoothing by FT



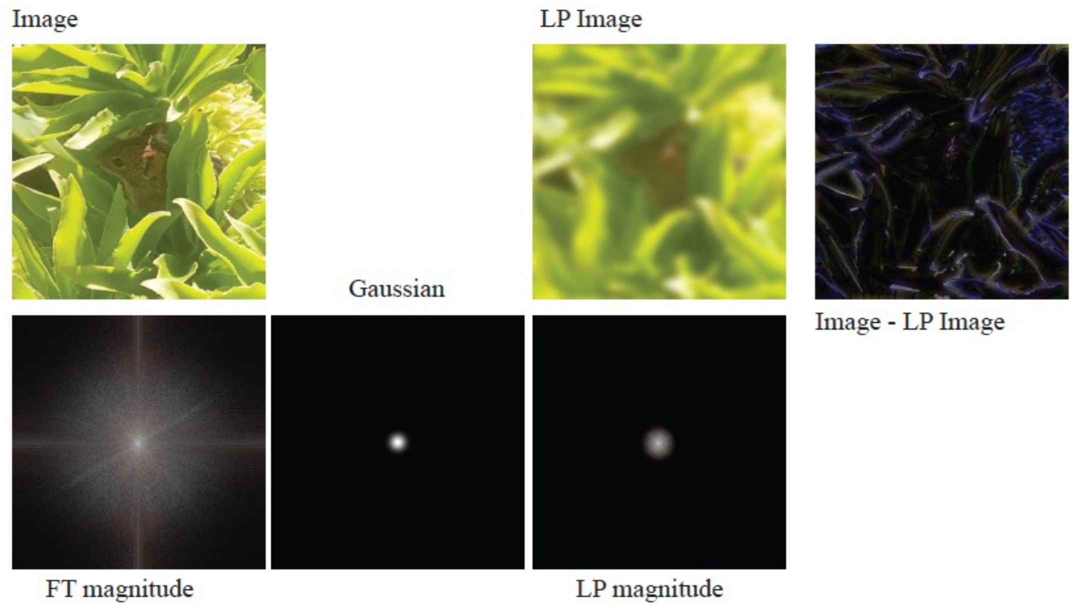
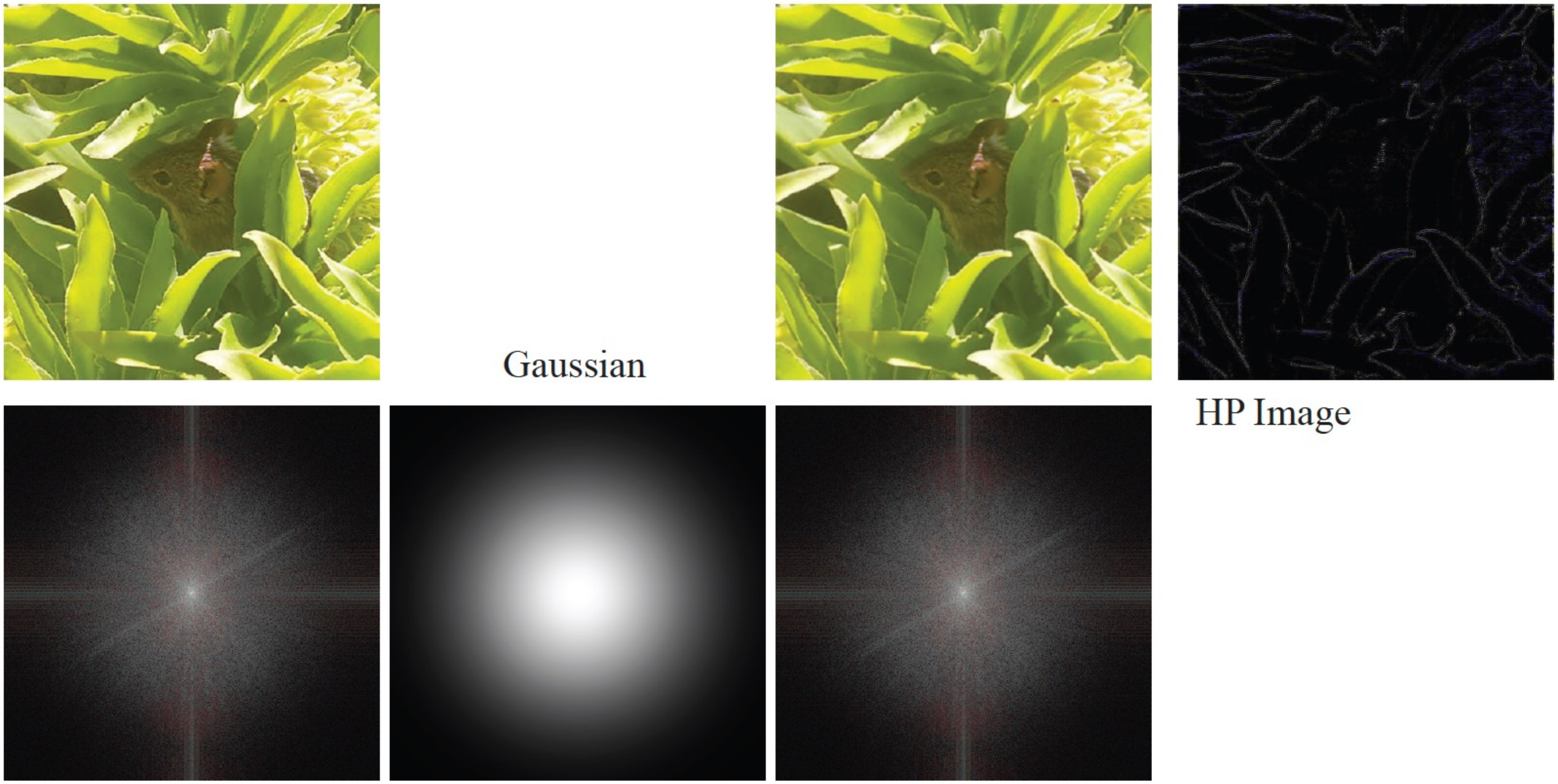


FIGURE 6.2: *On the* **top left**, *the image of a four striped grass mouse with the log magnitude of its Fourier transform on the* **bottom left**. **Center left**, *the gaussian with* $\sigma = 10$ *in* u, v *space. This is multiplied by the weights, and the log magnitude of the result appears* **center right**. **Above** *this is the image obtained by inverting the Fourier transform – equivalently, the low pass filtered image.* **Far left** *shows the high pass filtered image, obtained by subtracting the low pass filtered image from the original. I have* not *shown the log magnitude of the high pass filtered image, because scaling makes the result quite difficult to interpret (it doesn't look filtered). The low pass filtered version is heavily blurred, because only the lowest spatial frequencies appear in the result. Note the high pass filtered version contains what is missing from the low pass version, so has few large values which appear at edges.* Image credit: *Figure shows my photograph, taken at Kirstenbosch and Long Beach respectively.*

Smoothing with FT



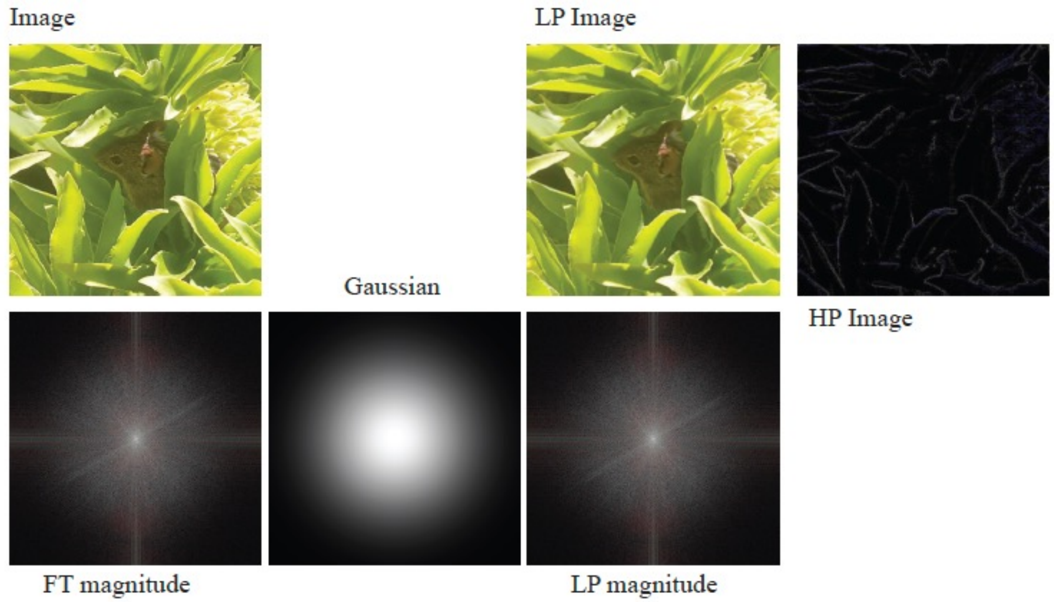


FIGURE 6.3: *On the* **top left**, *the image of a four striped grass mouse with the log magnitude of its Fourier transform on the* **bottom left**. **Center left**, *the gaussian with* $\sigma = 100$ *in* u, v *space. This is multiplied by the weights, and the log magnitude of the result appears* **center right**. **Above** *this is the image obtained by inverting the Fourier transform – equivalently, the low pass filtered image.* **Far left** *shows the high pass filtered image, obtained by subtracting the low pass filtered image from the original. I have* not *shown the log magnitude of the high pass filtered image, because scaling makes the result quite difficult to interpret (it doesn't look filtered). The low pass filtered version is less heavily blurred than that in Figure 6.2, because only the lowest spatial frequencies appear in the result. Note the high pass filtered version contains what is missing from the low pass version, so has very few large values which appear at edges.* Image credit: *Figure shows my photograph, taken at Kirstenbosch and Long Beach respectively.*

Phase vs. magnitude

- Which has more information, the phase or the magnitude?
- Let’s take the phase from one image and combine it with the magnitude from another image

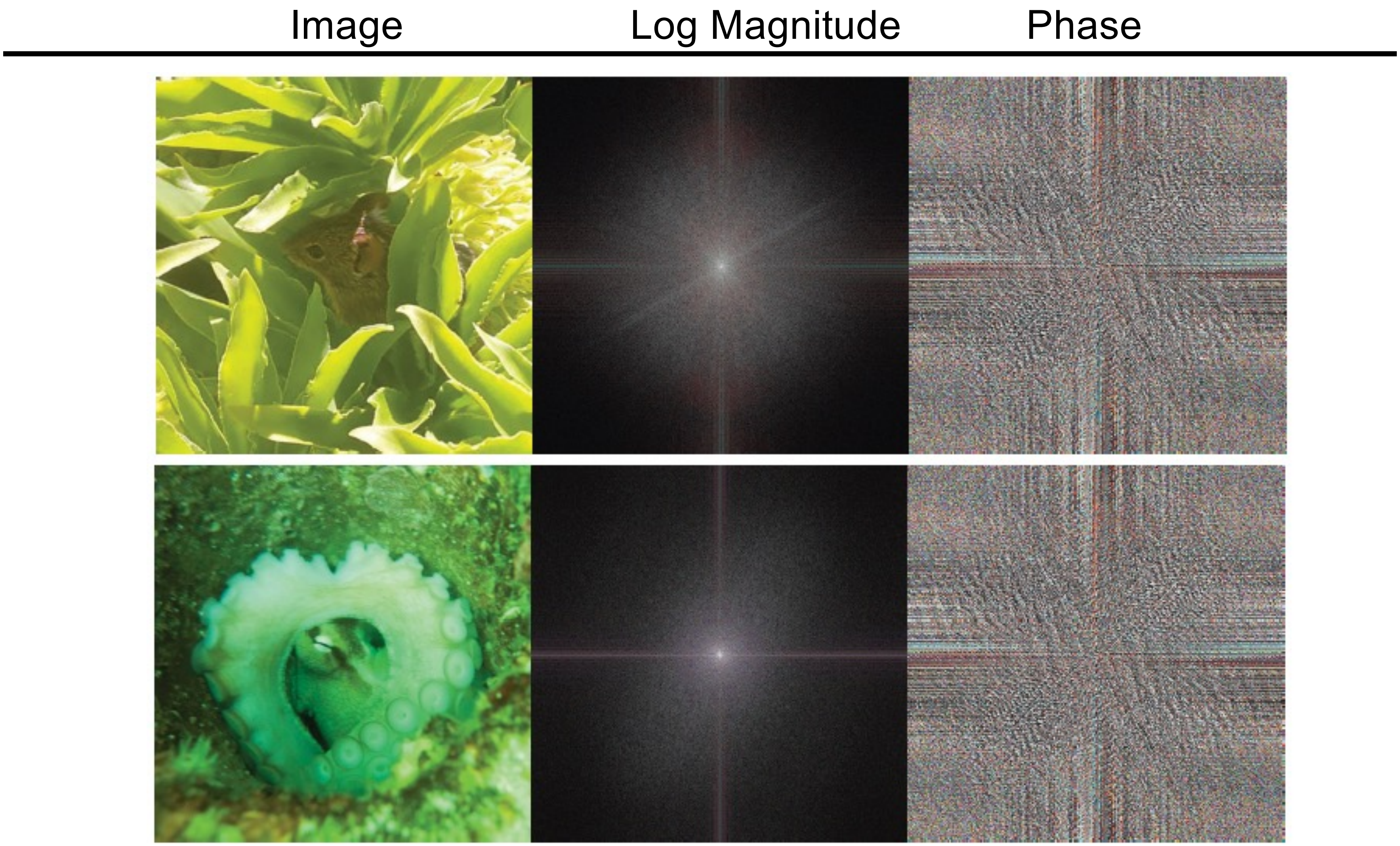
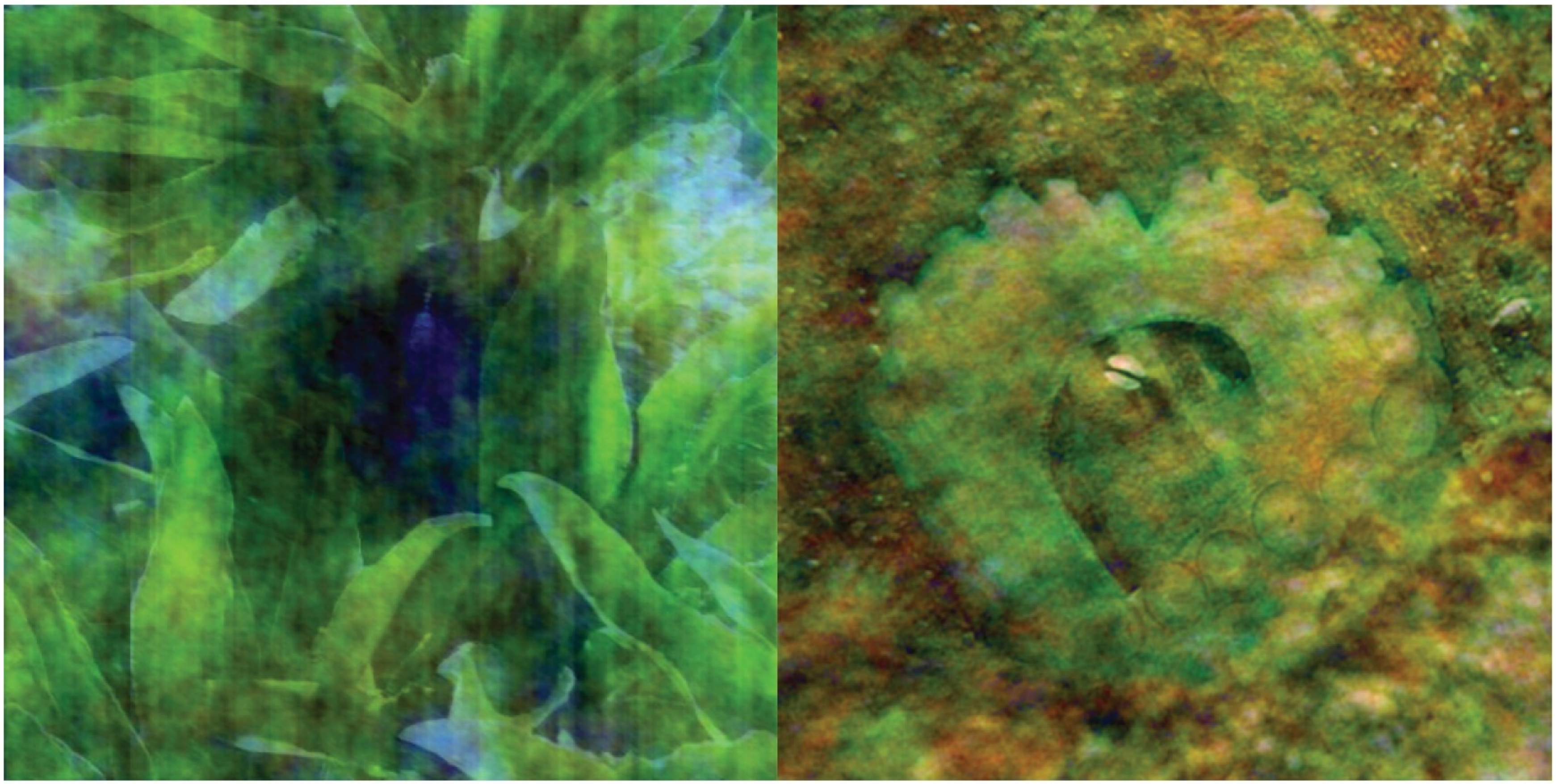


Image
Log Magnitude
Phase

Mouse phase, octopus mag

Octopus phase, mouse mag



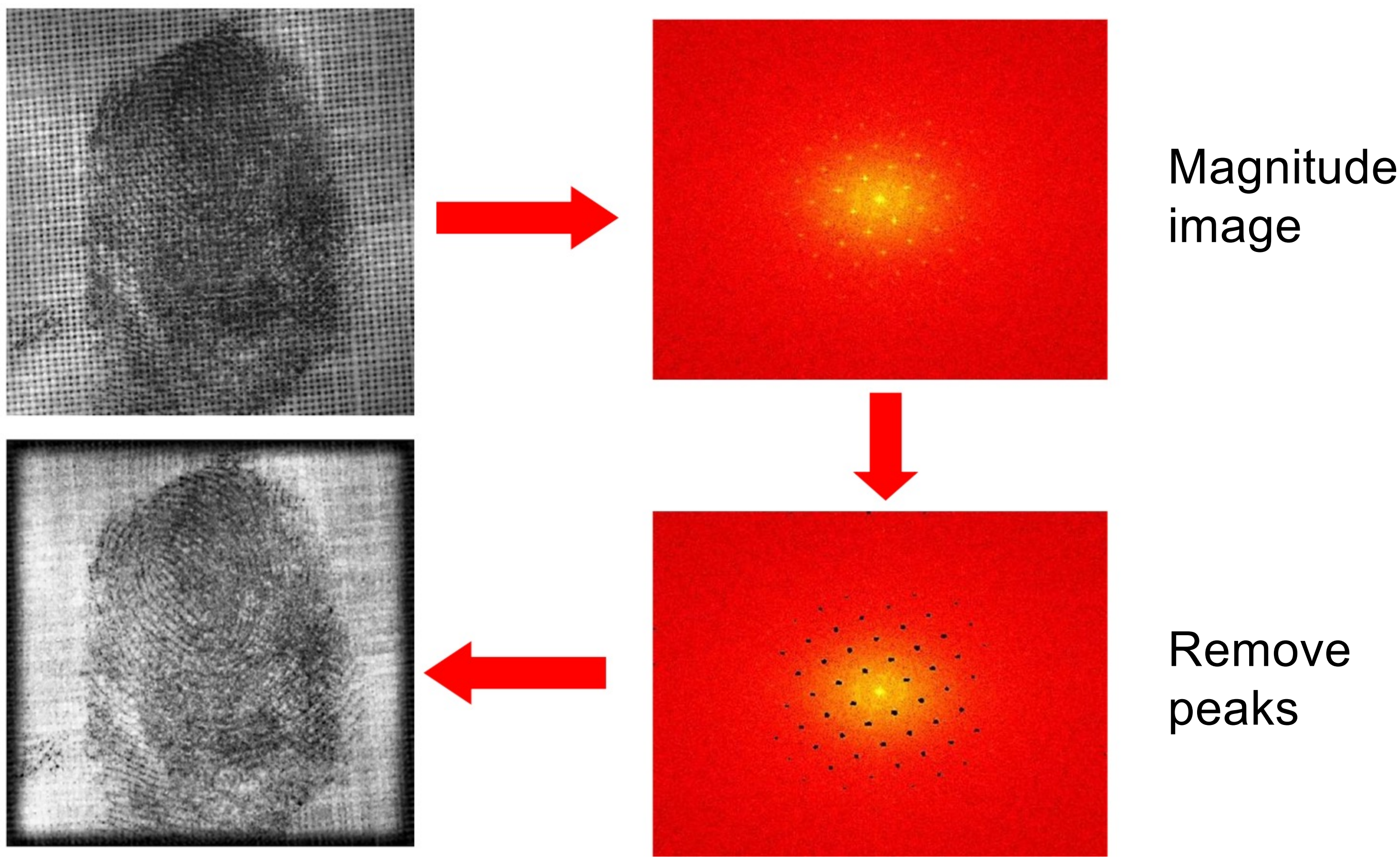
Images with periodic patterns

- The magnitude image has peaks corresponding to the frequencies of repetition

Image

Source: A. Zisserman

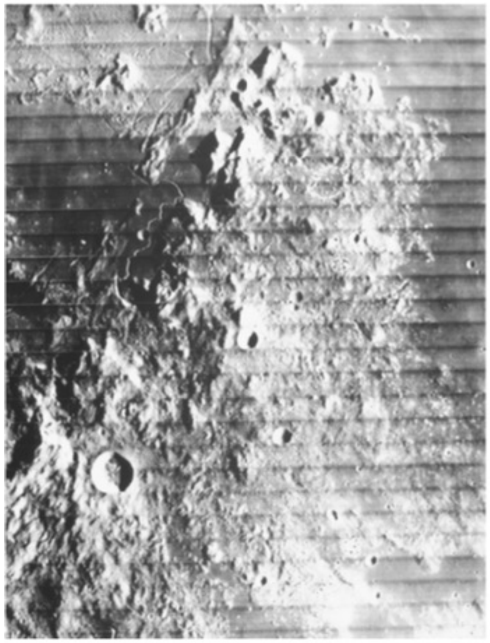
Application: Removing periodic patterns



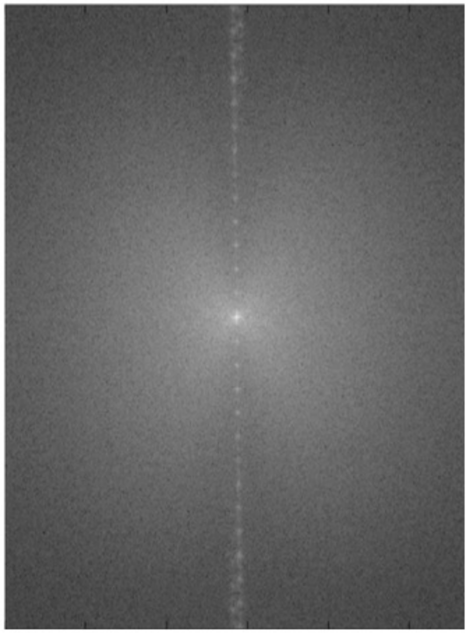
Source: [A. Zisserman]

Periodic patterns

Lunar orbital image (1966)



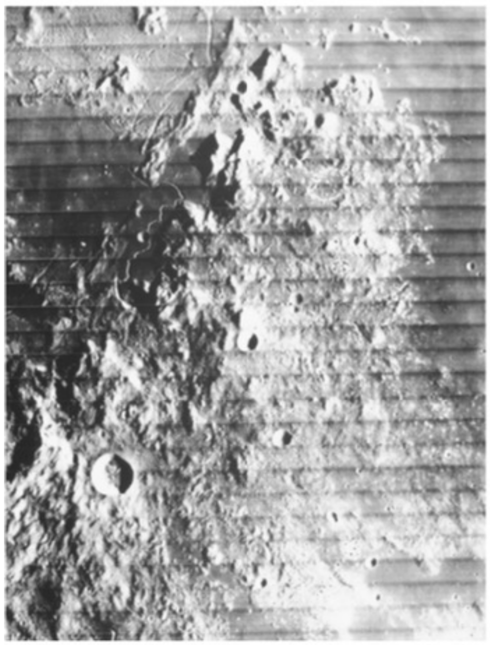
Magnitude image



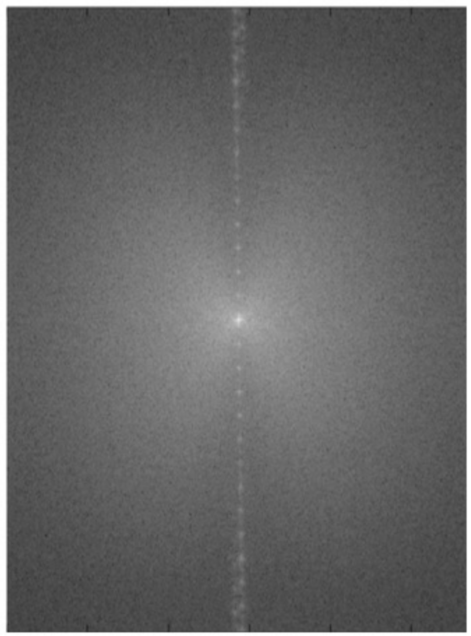
Why are there multiple peaks in the magnitude image?

Application: Removing periodic patterns

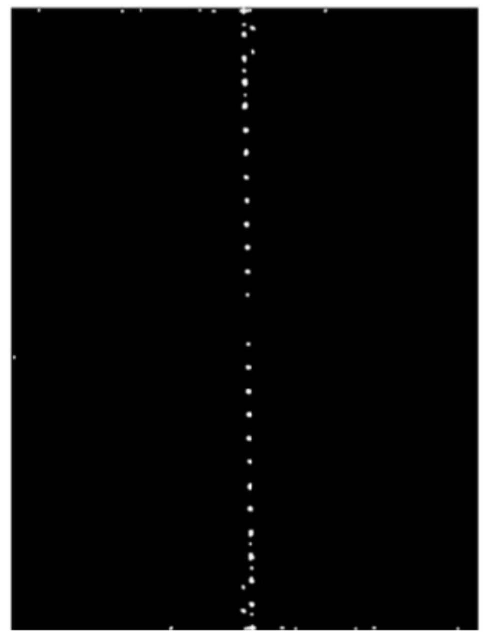
Lunar orbital image (1966)



Magnitude image



Remove peaks



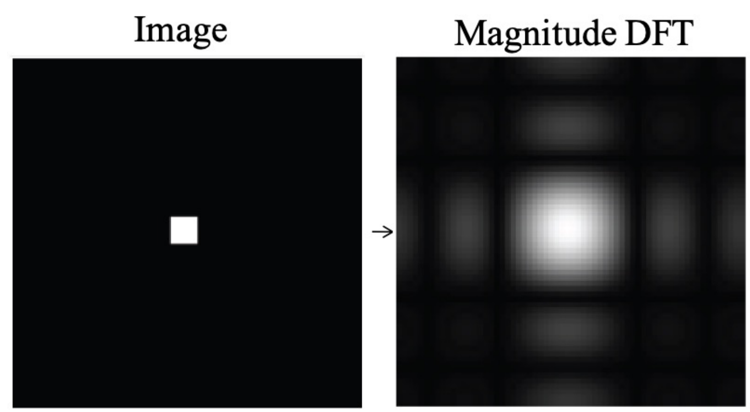
Join lines removed

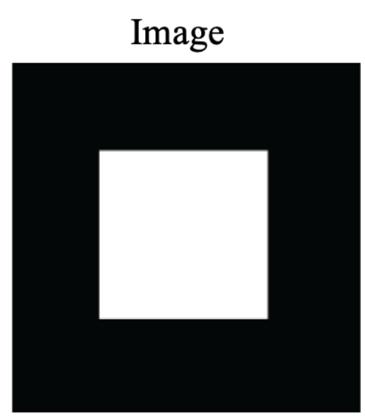
You should think of this as a kind of local smoothing
But in the Fourier domain!

Source: A. Zisserman

Image transformations

- How does the FT change when the image is scaled?





Scaled by the inverse factor!

In 1D

2D is easy, follows this form

$$\begin{aligned}\mathcal{F}(f(at)) &= \int_{-\infty}^{\infty} f(at)\exp\left[-i2\pi ut\right]dt \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(s)\exp\left[-i2\pi u/as\right]dt \\ &= \frac{1}{a}\mathcal{F}(f)(u/a).\end{aligned}$$

Important effect

"wider" function
has
"narrower"
Fourier transform

"narrower" function
has
"wider"
Fourier transform

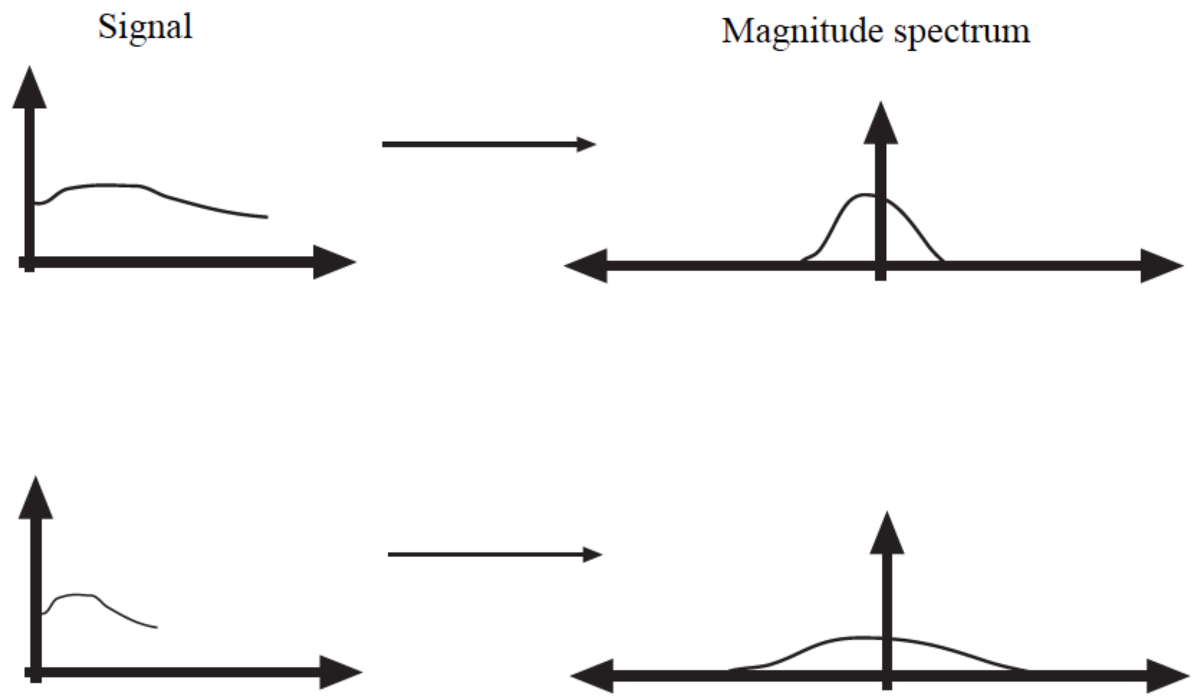


FIGURE 7.1: **Top** *shows* $f(t)$ *and its magnitude spectrum, and* **bottom** $f(2t)$ *and its magnitude spectrum. Notice how narrowing the function broadens the Fourier transform (from top to bottom); or broadening it narrows the Fourier transform (from bottom to top).*

Reference table in notes

Function	Fourier transform	Tag
$f(x,y)$	$\int\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+vy)}dxdy = \mathcal{F}(f)(u,v)$	1
$\int\int_{-\infty}^{\infty} \mathcal{F}(f)(u,v)e^{i2\pi(ux+vy)}dudv = f(x,y)$	$\mathcal{F}(f)(u,v)$	2
$\frac{\partial f}{\partial x}(x,y)$	$u\mathcal{F}(f)(u,v)$	3
$0.5\delta(x+a,y) + 0.5\delta(x-a,y)$	$\cos 2\pi au$	4
$\cos 2\pi ax$	$0.5\delta(u+a,v) + 0.5\delta(u-a,v)$	5
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$	6
$box_1(x,y)$	$\frac{\sin u}{u}\frac{\sin v}{v}$	7
$f(ax,by)$	$\frac{\mathcal{F}(f)(u/a,v/b)}{ab}$	8
$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(x-i,y-j)$	$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\delta(u-i,v-j)$	9
$f(x-a,y-b)$	$e^{-i2\pi(au+bv)}\mathcal{F}(f)$	10
$f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$	$\mathcal{F}(f)(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$	11
$(f * g)(x,y)$	$\mathcal{F}(f)\mathcal{F}(g)(u,v)$	12