A gentle introduction to Fourier analysis

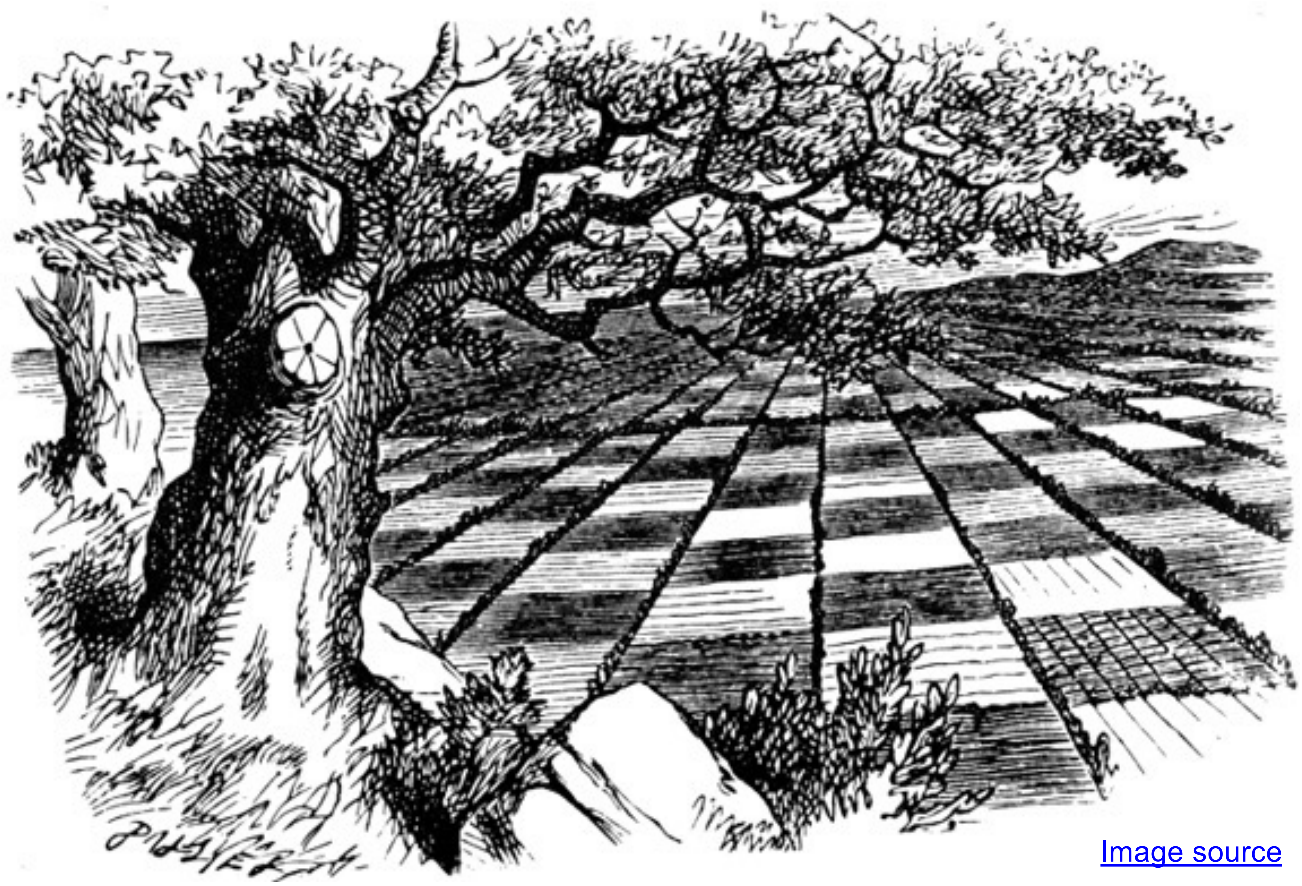
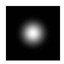


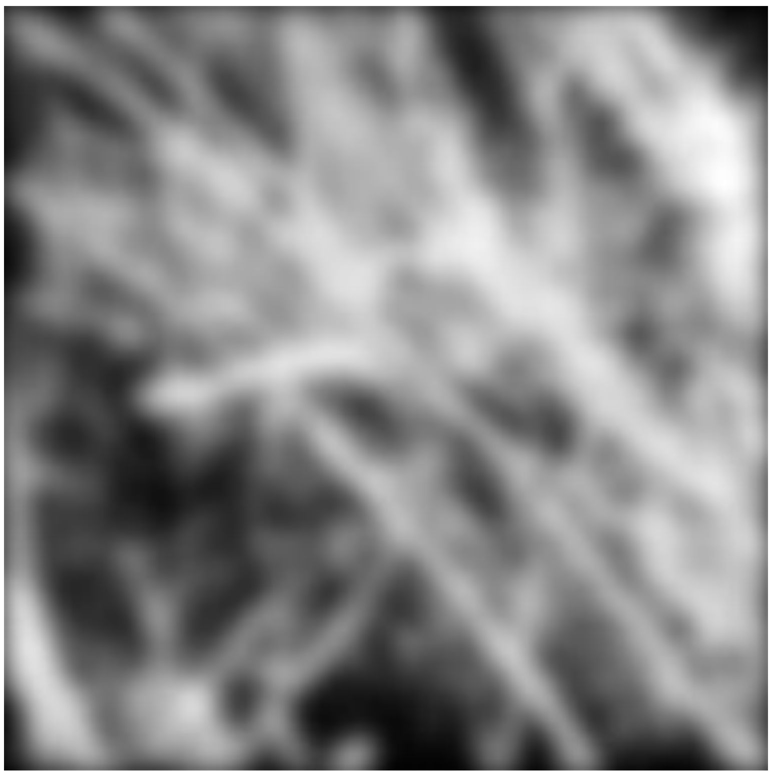
Image source

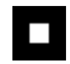
Many slides borrowed from S. Seitz, A. Efros, D. Hoiem, B. Freeman, A. Zisserman

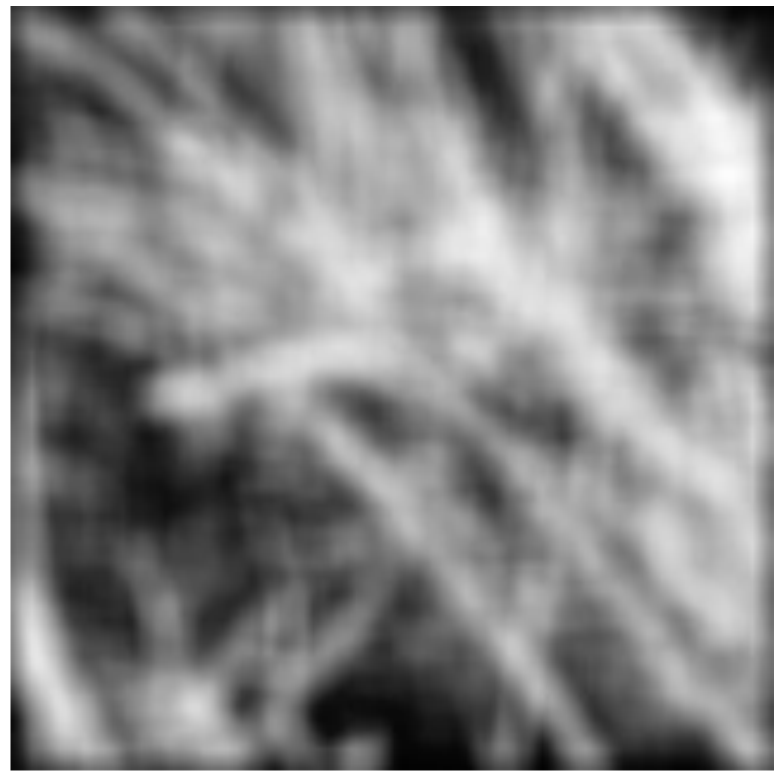
Mystery 1

- Why does filtering with a Gaussian give a nice smooth image, but filtering with a box filter gives artifacts?

Gaussian 

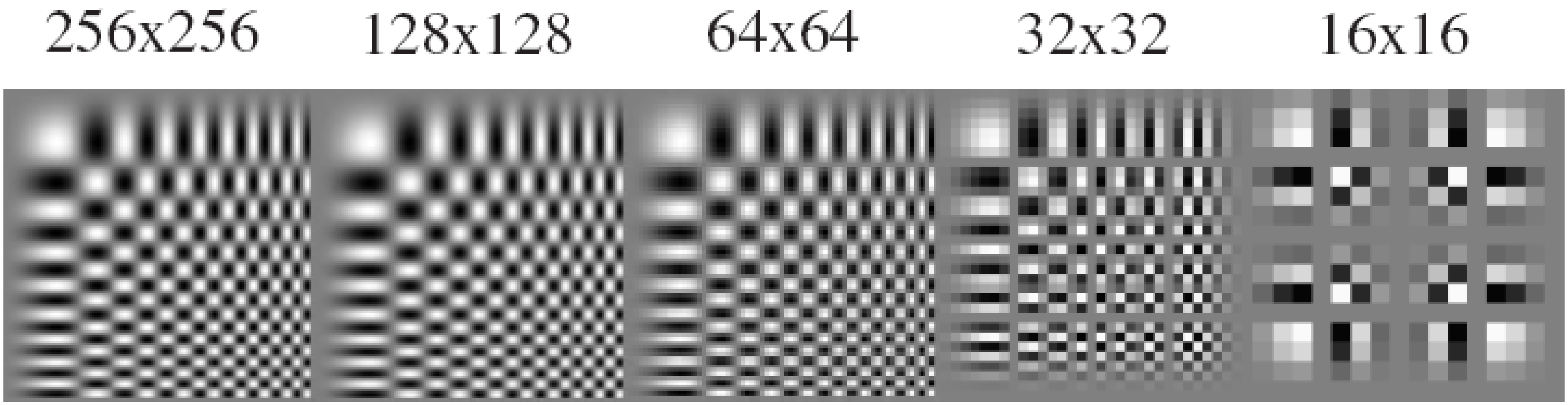


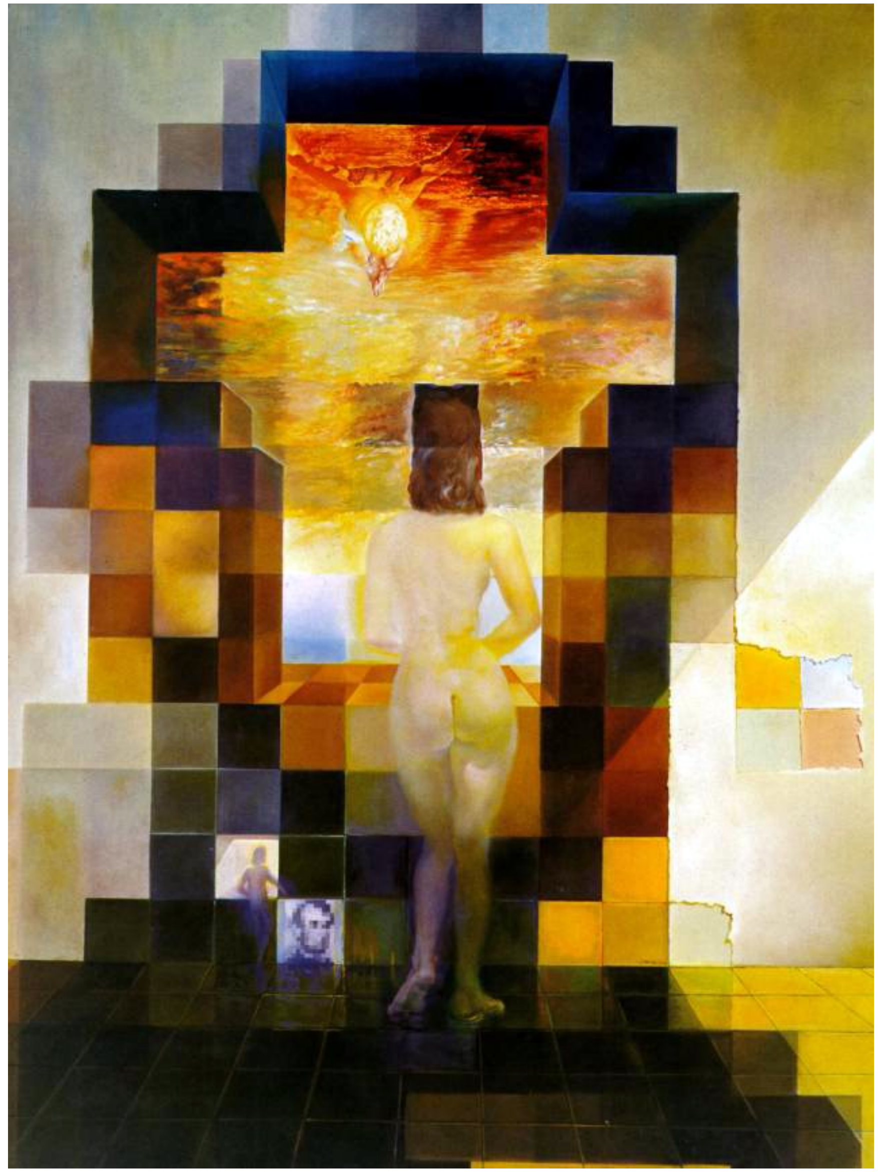
Box filter 



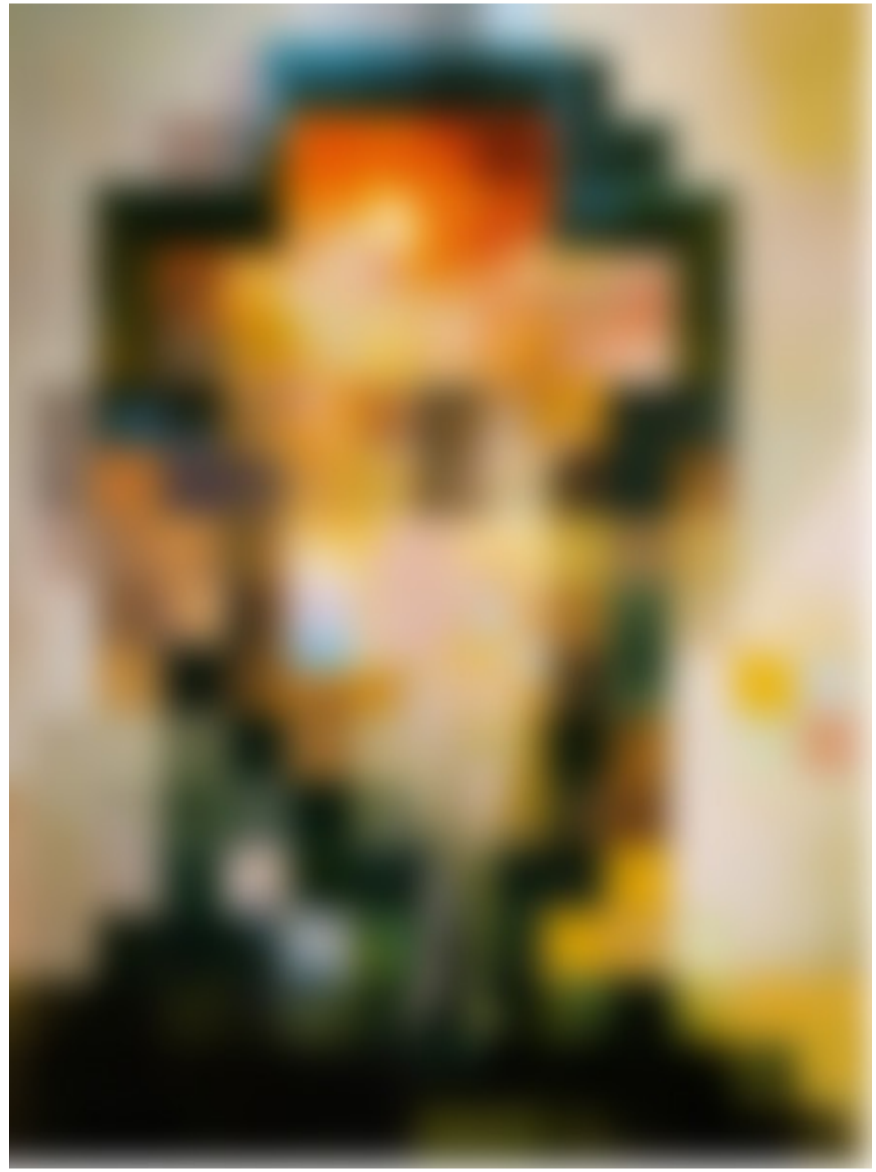
Mystery 2

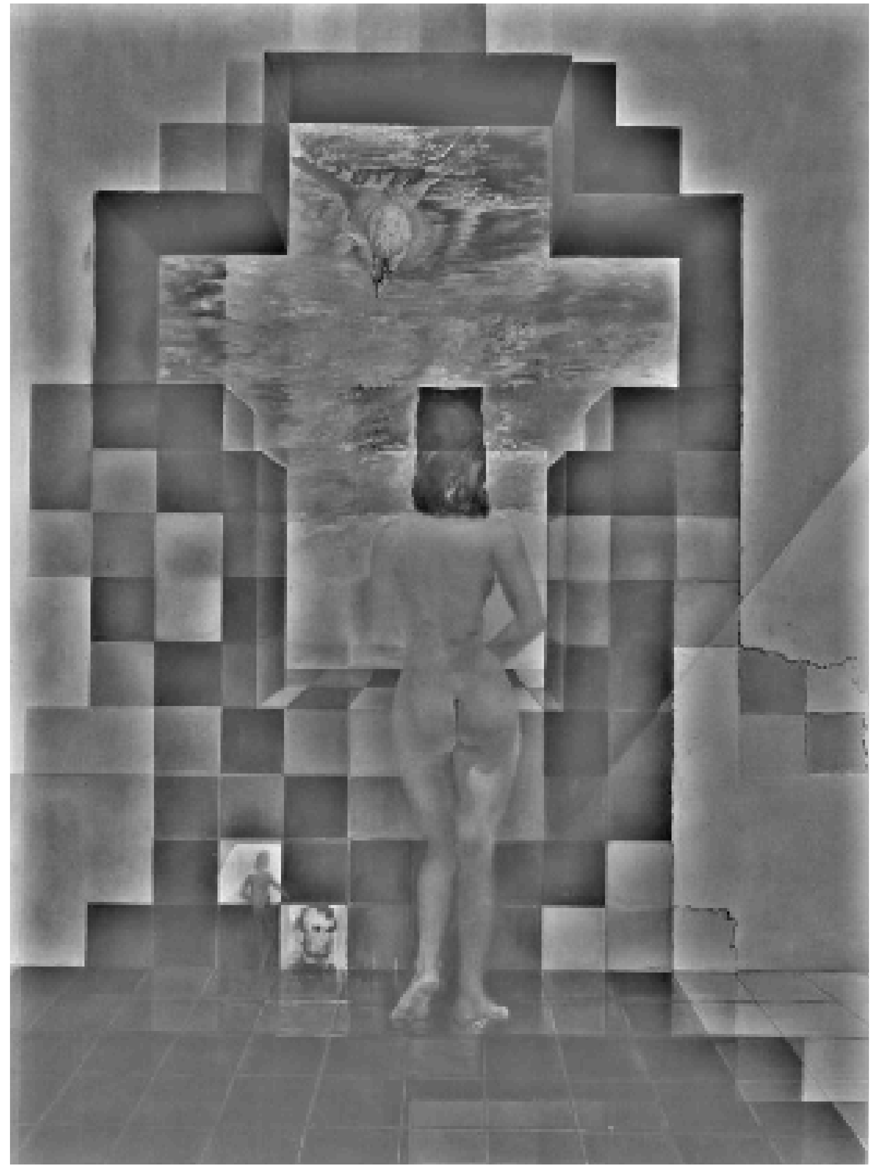
- Why can downsampling sometimes lead to aliasing?





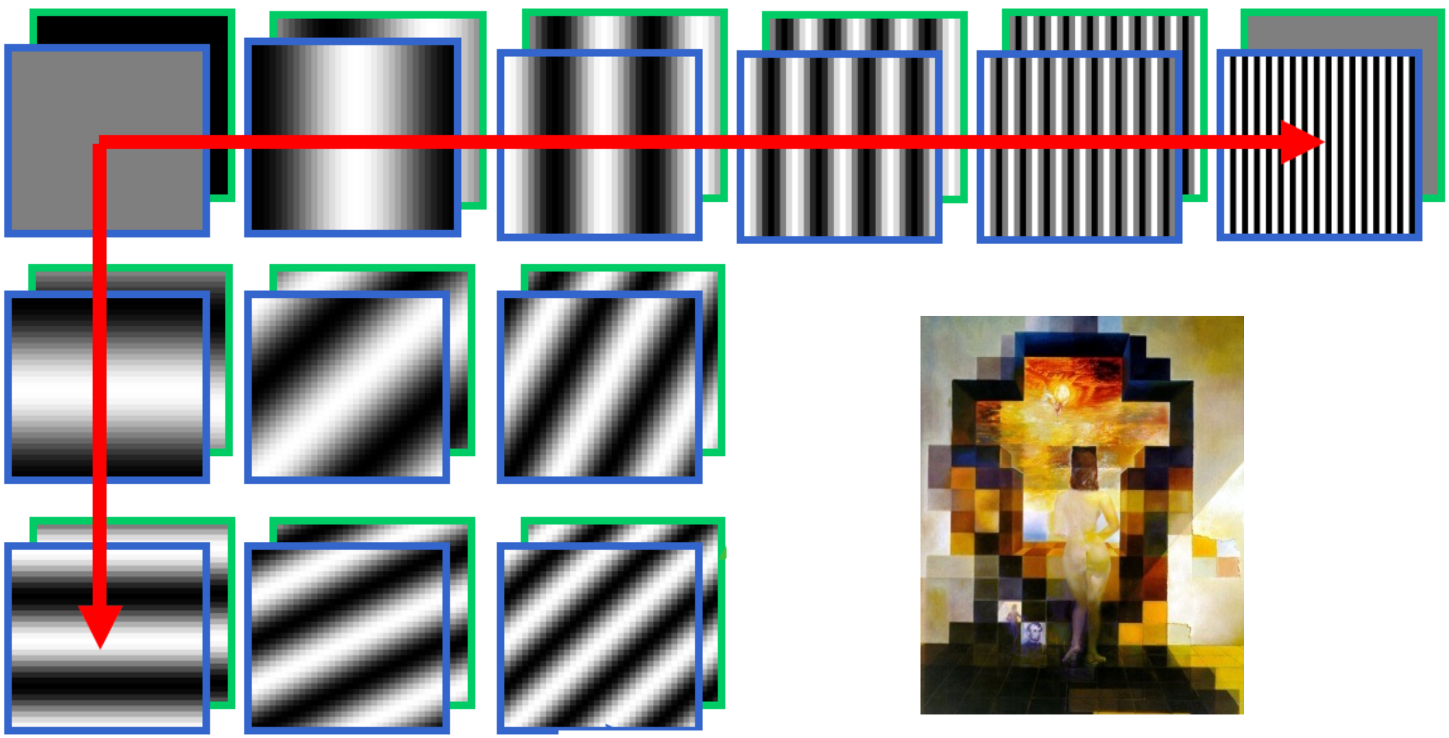
Salvador Dali
“Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln”, 1976





Fourier analysis

- To understand such phenomena, we need a representation of images that allows us to tease apart slow and fast changes



Outline

- Fourier series
- 1D Fourier transform
  - Definition and properties
  - Discrete Fourier transform
- 2D Fourier transform
  - Definition
  - Examples and properties
- Convolution theorem
- Understanding the sampling theorem

Fourier series

- Any(**) periodic function on [0, 1] can be expressed as a weighted sum of sinusoids of different frequencies (1807)

**=bunch of important details here

Jean-Baptiste Joseph Fourier (1768-1830)

Example: series for a square wave

$$\sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k}\sin(kt)$$

Periodic means f(0)=f(1)

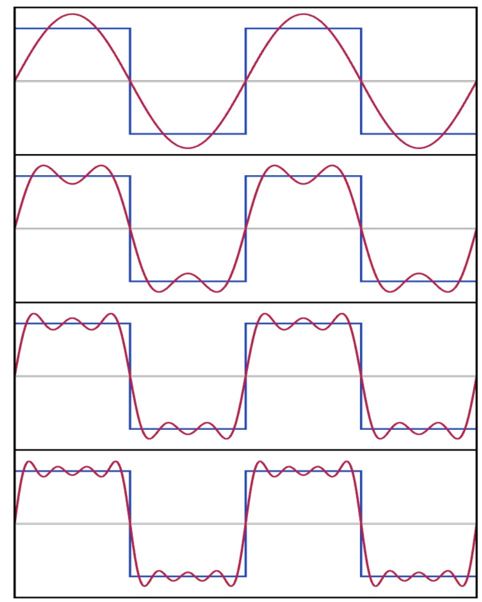


Image: Wikipedia

Fourier series

Generally, we have for a (reasonable) periodic f(t)

$$f(t) \sim A_0 + \sum_{i=1}^{\infty} \left[A_i \cos\left(i2\pi t\right) + B_i \sin\left(i2\pi t\right)\right]$$

And we need to figure out the weights for a given f(t).

Fourier series: useful facts

$$\int_0^1 \cos(i2\pi t)\,dt = \int_0^1 \sin(i2\pi t)\,dt = 0 \text{ for } i \text{ integer, } i > 0$$

Fact 1

$$\int_0^1 \cos(i2\pi t)\sin(j2\pi t)\,dt = 0 \text{ for } i,j \text{ integer, } i \neq j,\ i > 0,\ j > 0$$

$$\int_0^1 \cos(i2\pi t)\cos(j2\pi t)\,dt = 0 \text{ for } i,j \text{ integer, } i \neq j,\ i > 0,\ j > 0$$

Fact 2

$$\int_0^1 \sin(i2\pi t)\sin(j2\pi t)\,dt = 0 \text{ for } i,j \text{ integer, } i \neq j,\ i > 0,\ j > 0$$

$$\int_0^1 \sin^2(i2\pi t)\,dt = 1/2 \text{ for } i \text{ integer}$$

$$\int_0^1 \cos^2(i2\pi t)\,dt = 1/2 \text{ for } i \text{ integer}$$

Fact 3

Fourier series: using facts

If:

$$f(t) \sim A_0 + \sum_{i=1}^{\infty} \left[A_i \cos\left(i2\pi t\right) + B_i \sin\left(i2\pi t\right) \right]$$

$$\int_0^1 f(t)dt = A_0$$

(fact 1 makes all the cosine/sine terms go away!)

$$\int_0^1 f(t)\sin(i2\pi t)dt = \frac{A_i}{2}$$

$$\int_0^1 f(t)\cos(i2\pi t)dt = \frac{B_i}{2}$$

(fact 2 makes all the other terms go away!
And fact 3 sets the scale)

Fourier series: issues

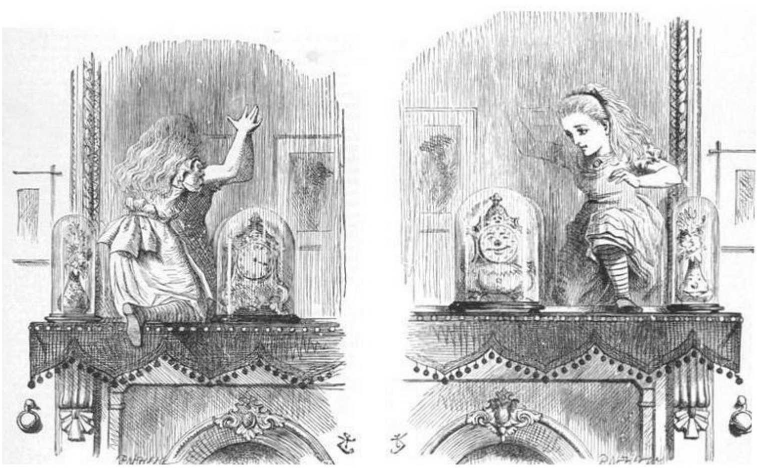
- A’s and B’s are inelegant -> complex exponentials
- Did NOT show that the series converges to the function
  - Read Korner’s wonderful book Fourier Analysis
  - We’re OK for anything we care about
- In principle, we can go forward
  - Function -> A’s, B’s
- Or backward
  - A’s, B’s -> Function
- Is this right? (mostly yes, but details…)

Complex exponentials

This i is the square root of -1 !!!

$$e^{i2k\pi t} = \cos(2k\pi t) + i \sin(2k\pi t)$$

$$f(t) \sim \sum_{k=0}^{\infty} c_k e^{i2k\pi t}$$



Advantage:

if the function is complex, can represent cleanly
don't need to remember which is A, which B

Complex exponentials: compact facts

$$\int_0^1 e^{i2k\pi t} e^{-i2n\pi t} dt = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

k, n integers

This minus sign matters!

Fourier series with complex exponentials: using fact

If:

$$f(t) \sim \sum_{k=0}^{\infty} c_k e^{i2k\pi t}$$

$$c_k = \int_0^1 f(t) e^{-i2k\pi t} dt$$

Using the fact! (this is analogous to an orthonormal basis in linear algebra)

Fourier series with complex exponentials: issues

- But this is just for a periodic function on [0, 1]
  - Easy to extend to other intervals
  - Easy to extend to the circle

- But what about functions on [-infinity, infinity]?
  - These could wiggle often in numerous places
  - IDEA: use “more” basis elements

- The Fourier transform