Coordinate geometry: The essential matrix

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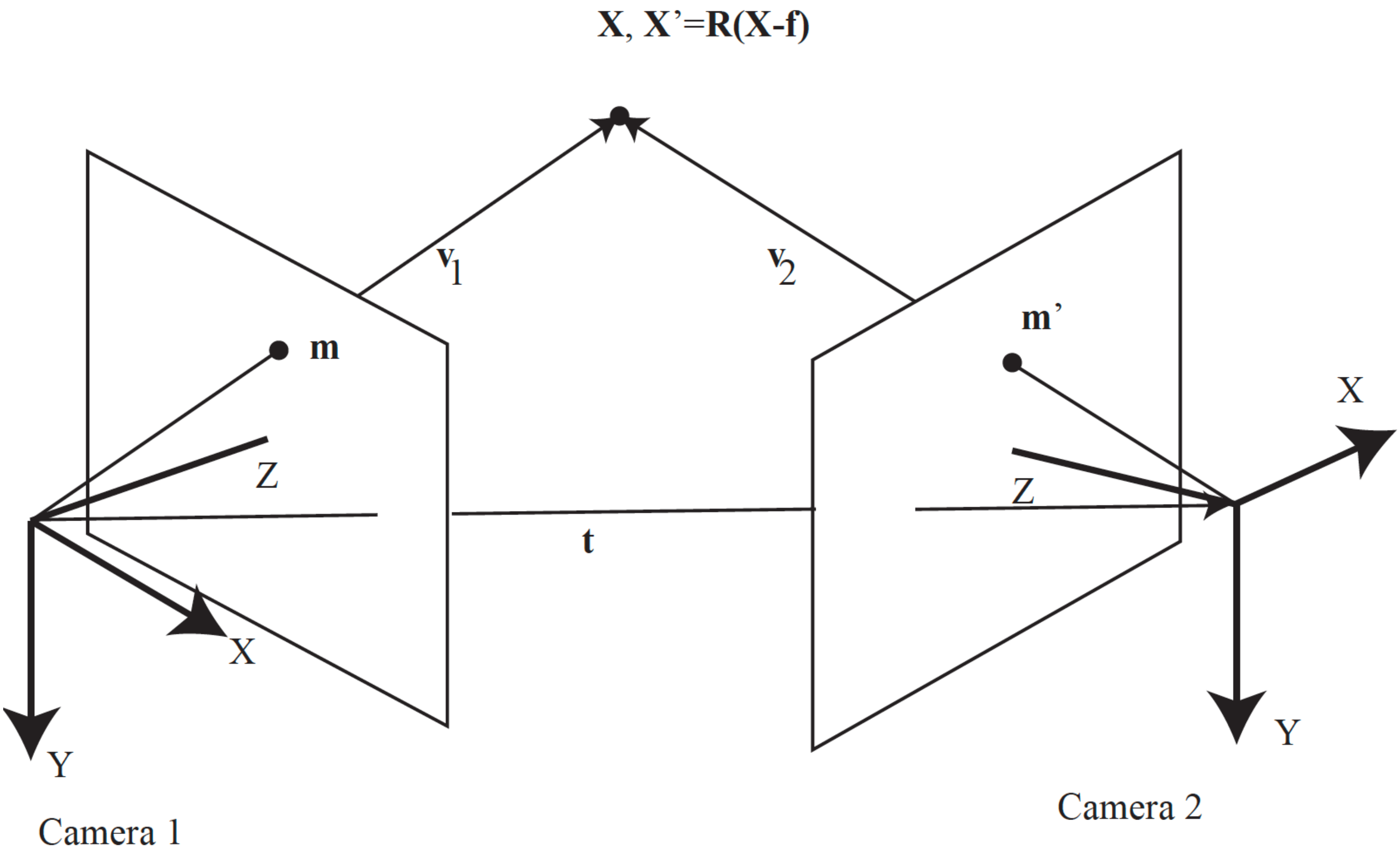
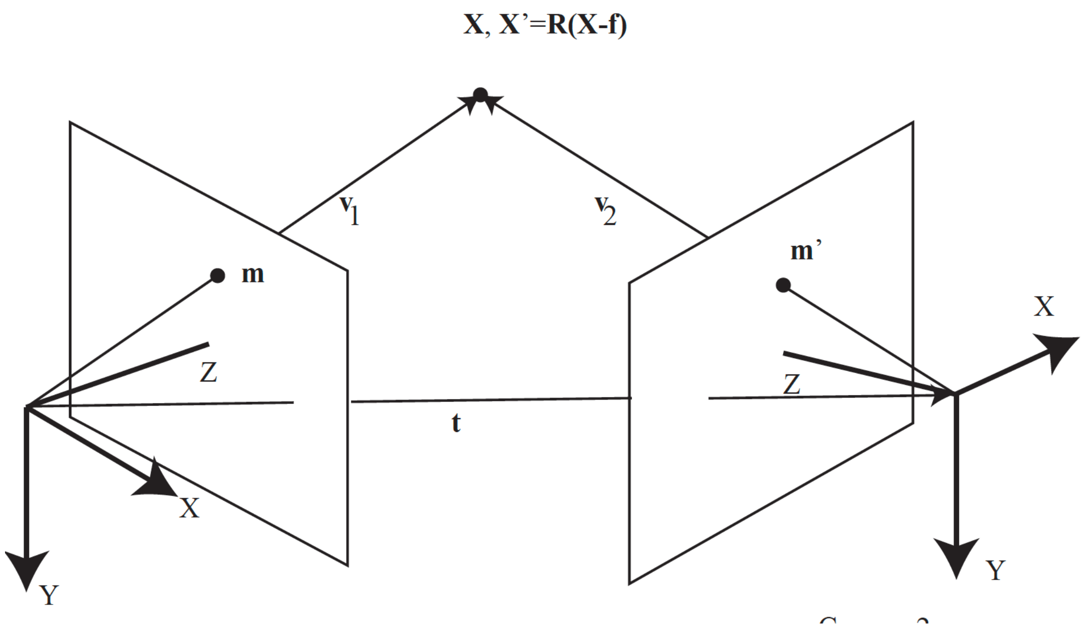
Fundamental vs essential matrix

- Fundamental matrix is a geometric concept
  - true whatever the coordinate system
- Essential matrix
  - what happens in coordinates


X, X'=R(X-f)
v1
v2
m
m'
Z
Z
t
X
X
Y
Y
Camera 1
Camera 2



The first camera has focal point at the origin, so $\mathbf{v}_1 = \mathbf{X}$ is the vector from $\mathbf{f}_1$ to $\mathbf{X}$. The vector from $\mathbf{f}_2$ to $\mathbf{X}$ is $\mathbf{v}_2 = \mathbf{X} - \mathbf{t}$ *in the first camera's coordinate system.* From Figure 32.3 $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{t}$ must be coplanar. This means that

$$[\mathbf{t} \times \mathbf{v}_1]^T \mathbf{v}_2 = 0$$

A convenient trick from linear algebra helps here. For a vector $\mathbf{a} = [a_1, a_2, a_3]^T$, write

$$[\mathbf{a}]_X = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

and notice $\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_X \mathbf{b}$. Further, you can write $\mathbf{v}_2$ in the second camera's coordinate system, yielding $\mathbf{v}_2' = \mathcal{R}^T \mathbf{v}_2$. This means that

$$\begin{aligned} [[\mathbf{t}]_X \mathbf{v}_1]^T \mathbf{v}_2 &= 0 \\ &= \mathbf{v}_1^T [\mathbf{t}]_X{}^T \mathbf{v}_2 \\ &= \mathbf{v}_1^T [\mathbf{t}]_X{}^T \mathcal{R} \mathbf{v}_2' \end{aligned}$$

In homogenous coords,

Camera 1's image plane is at $z = 1$ in that camera's coodinate system, so the point $\mathbf{x}_1$ is at $(x_{1,1}, x_{2,1}, 1)^T$ in 3D and in camera 1's coordinate system. Alternatively, $(x_{1,1}, x_{2,1}, 1)^T$ is the location of $\mathbf{x}_1$ on camera 1's image plane in homogeneous coordinates. Further, $\mathbf{v}_1 \equiv \mathbf{x}_1$. Express both $\mathbf{x}_1$ and $\mathbf{x}'_2$ in homogeneous coordinates. Then

$$\begin{aligned} \mathbf{x}_1^T [\mathbf{t}]_X{}^T \mathcal{R} \mathbf{x}'_2 &= \mathbf{v}_1^T [\mathbf{t}]_X{}^T \mathcal{R} \mathbf{v}'_2 \\ &= 0 \\ \text{and so} & \\ \mathbf{x}_1^T \mathcal{E} \mathbf{x}'_2 &= 0 \end{aligned}$$

The essential matrix

Remember this: *Given two cameras related by a rotation* $\mathcal{R}$ *and a translation* $\mathbf{t}$, *the essential matrix is*

$$\mathcal{E} = {[\mathbf{t}]_X}^T \mathcal{R}.$$

A point $\mathbf{X}$ *in 3D projects to* $\mathbf{x}_1$ *on camera 1's image plane in camera 1's coordinate system. The same point projects to* $\mathbf{x}_2'$ *on camera 2's image plane in camera 2's coordinate system. For any* $\mathbf{X}$ *not on the baseline,*

$$\mathbf{x}_1^T \mathcal{E} \mathbf{x}_2' = 0$$

- it follows from fundamental matrix that something like this should be there; but here we have an expression relating it to camera transformation

Recall: singular value decomposition

Orthonormal

$$\mathcal{M} = \mathcal{U}\Sigma\mathcal{V}^T$$

Diagonal matrix
Of non-negative singular
values

The essential matrix: Properties

$$\mathcal{E} = [\mathbf{t}_\times]\,\mathcal{R}$$

$$\mathrm{svd}(\mathcal{E}) = \mathcal{U}\Sigma\mathcal{V}^T$$

- But $\mathcal{V}^T$ is orthonormal, so

$$\mathrm{svd}(\mathcal{E}) = \mathrm{svd}([\mathbf{t}_\times])$$

The essential matrix: Properties

$$\mathrm{svd}([\mathbf{t}_\times]) = \mathcal{U}\Sigma\mathcal{V}^T$$

$$\| \mathbf{t} \times \mathbf{v} \| = \| \mathbf{t} \| \| \mathbf{v} \| \sin\phi$$

From this deduce

$$\Sigma = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The essential matrix: Properties

- The essential matrix has one singular value zero
  - the other two are equal
- (slightly less easy)
  - any such matrix is an essential matrix

Estimating the essential matrix

Procedure: 32.5 *Estimating the essential matrix*

Use procedure 32.3 to estimate the fundamental matrix, then use the camera intrinsics to compute

$$\mathcal{M} = \mathcal{K}_1^T \mathcal{F} \mathcal{K}_2.$$

From $\mathcal{M}$ compute the SVD to obtain $\mathcal{U}$, Σ and $\mathcal{V}$. Write $\Sigma_e = \text{diag}(1, 1, 0)$. Then the estimate of the essential matrix is

$$\hat{\mathcal{E}} = \mathcal{U} \Sigma_e \mathcal{V}^T.$$