Epipolar Geometry and the Fundamental Matrix

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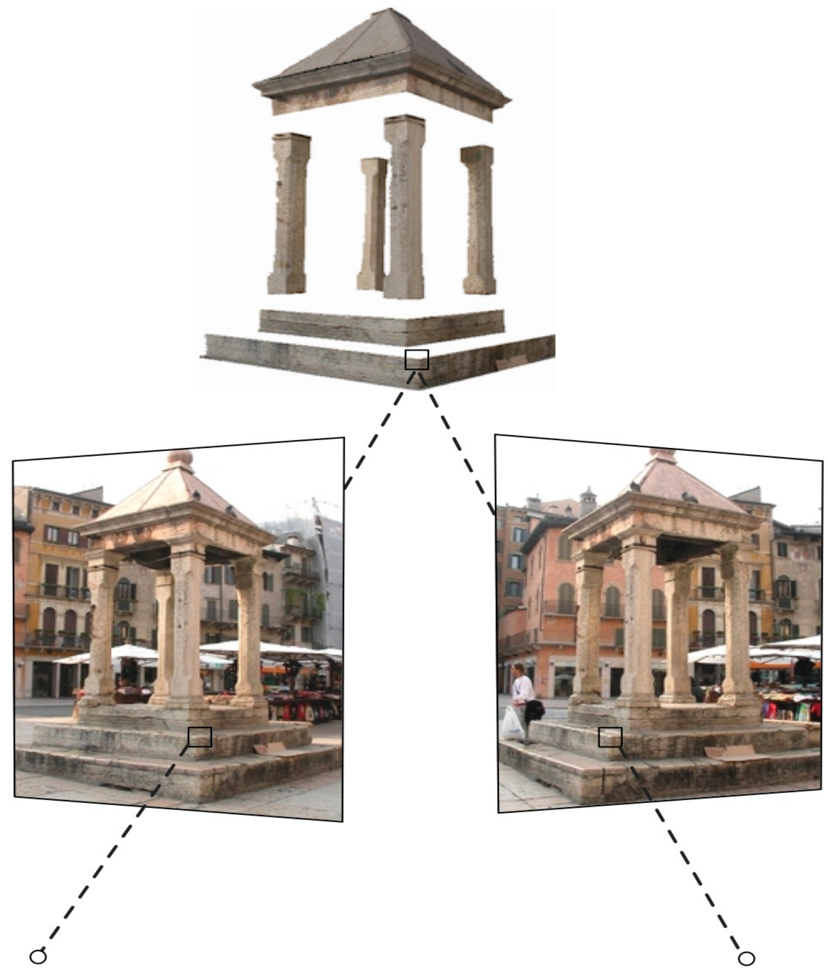
Two views of the same 3D scene

- Because:
    - You have two
        - Eyes
        - Cameras
    - OR the camera moves



The Pinhole Camera

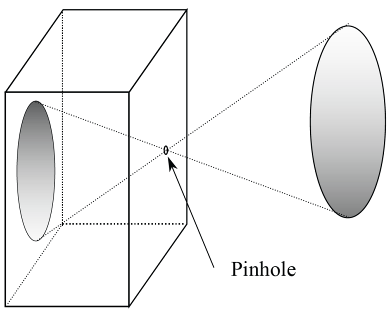


FIGURE 2.3: *In the pinhole imaging model, a light-tight box with a pinhole in it views an object. The only light that a point on the back of the box sees comes through the very small pinhole, so that an inverted image is formed on the back face of the box.*

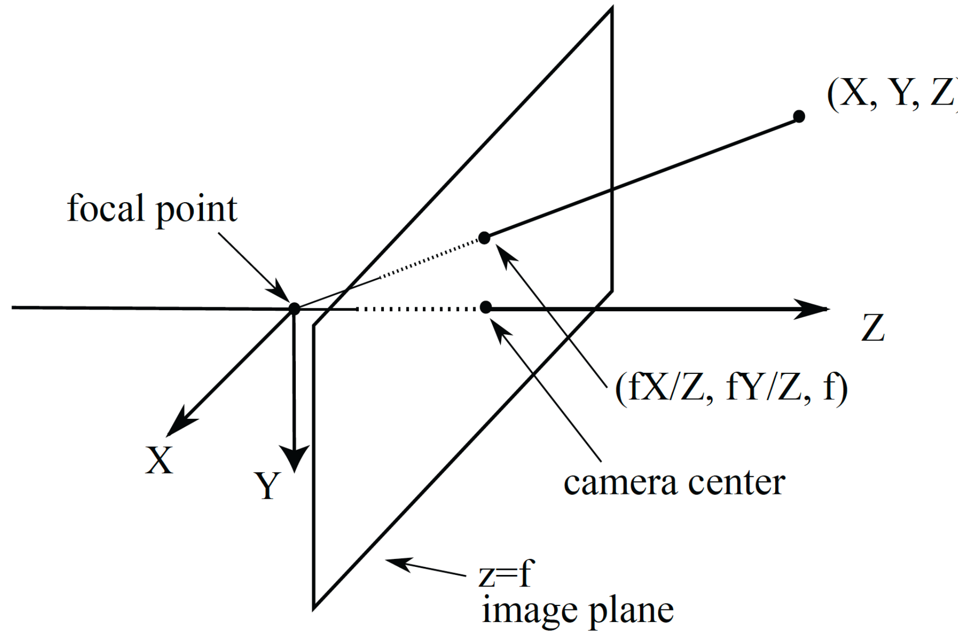
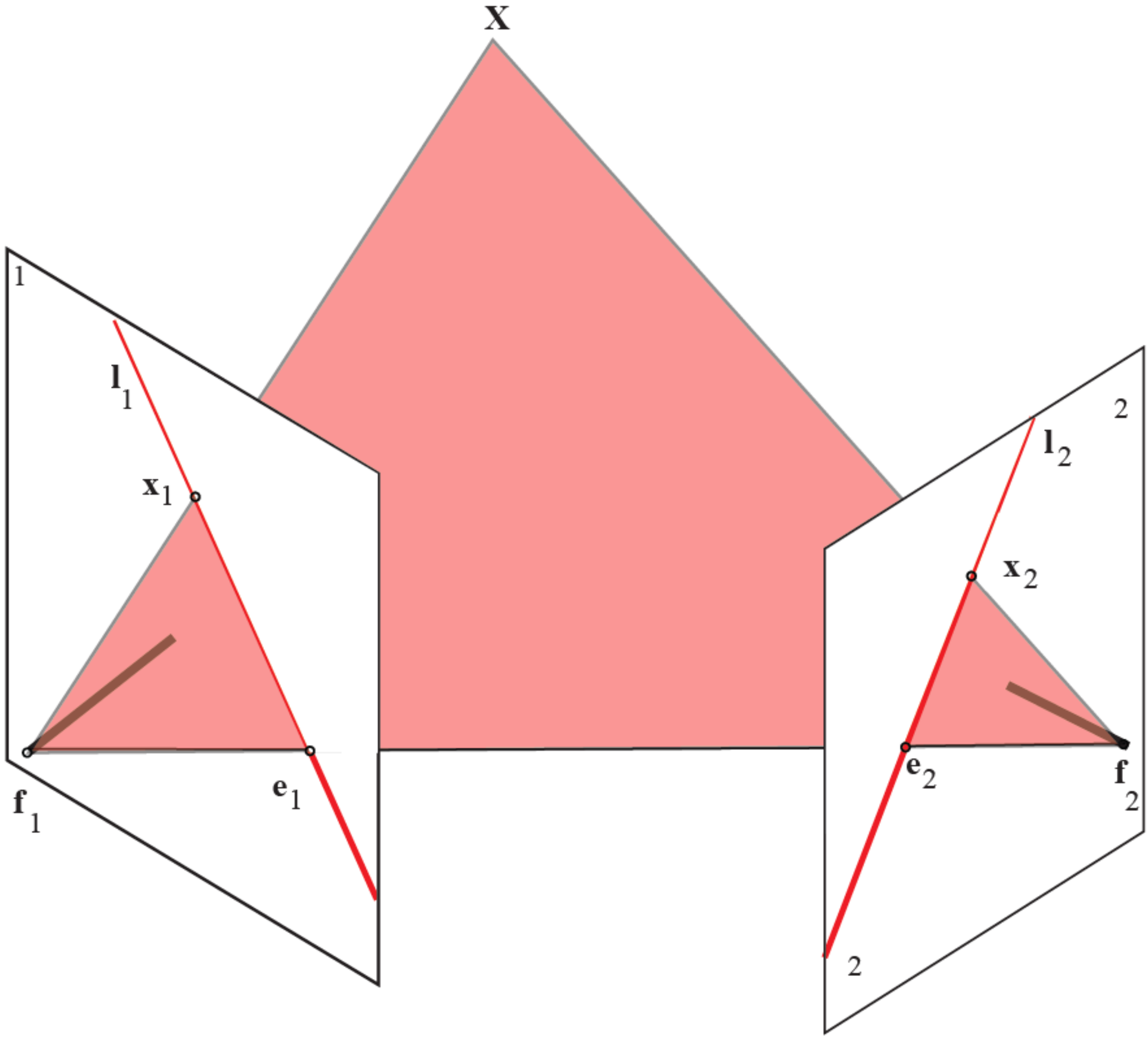
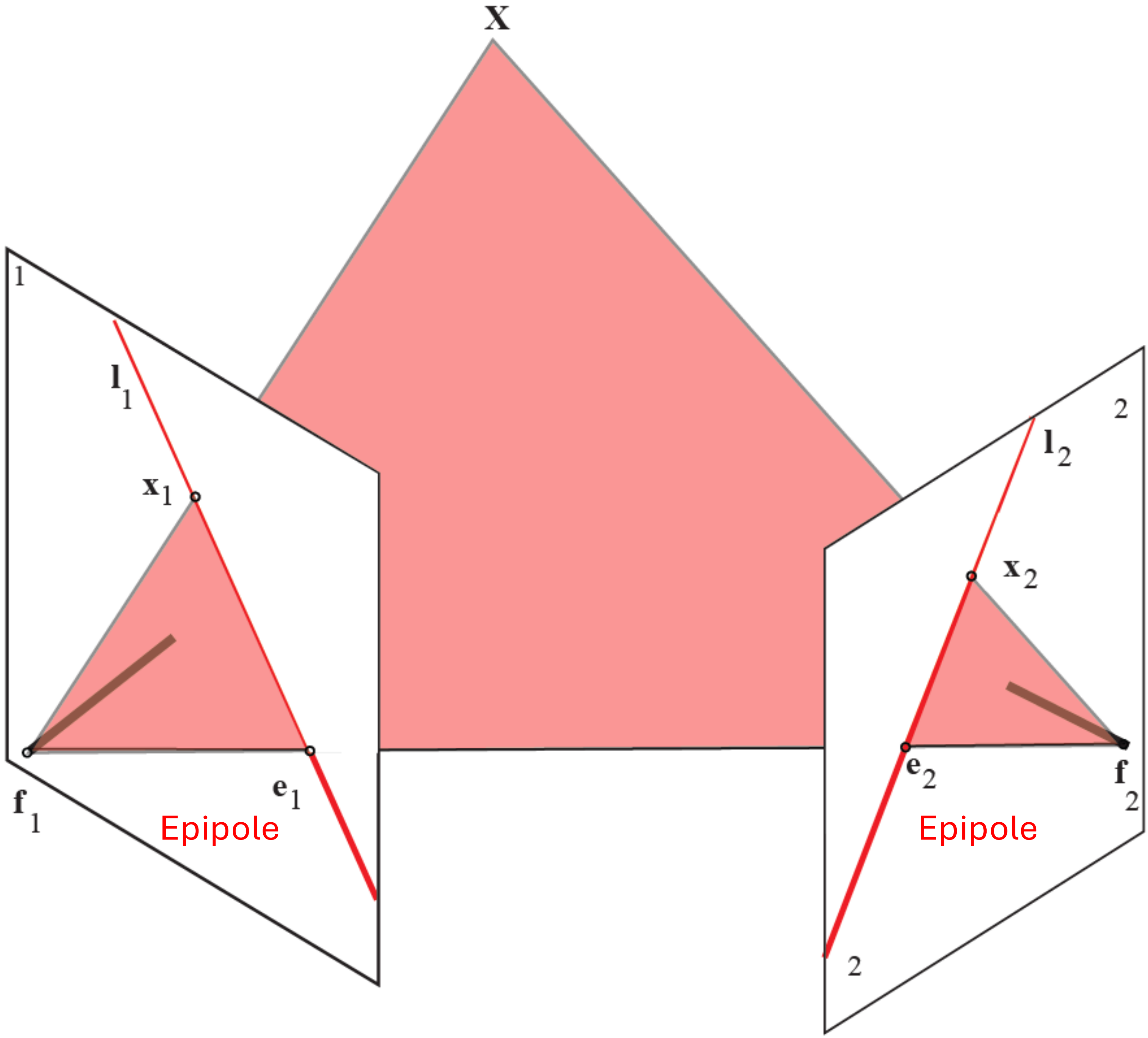
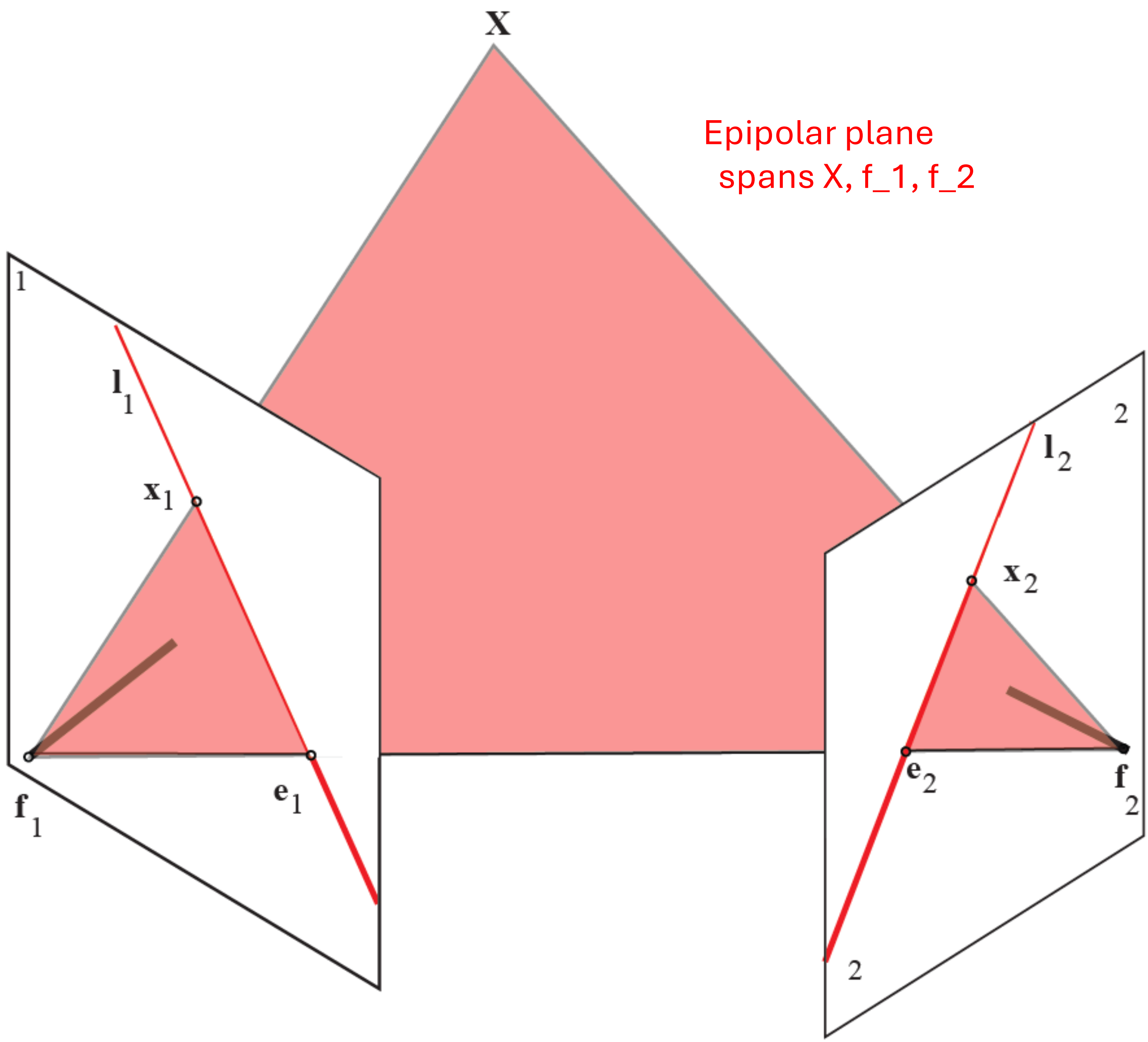


FIGURE 2.4: *The usual geometric abstraction of the pinhole model. The box doesn't affect the geometry, and is omitted. The pinhole has been moved to the back of the box, so that the image is no longer inverted. The image is formed on the plane* $z = f$, *by convention. Notice the y-axis goes down in the image. This allows me to use a* right handed *coordinate system and also have* z increase *as one moves into the image.*

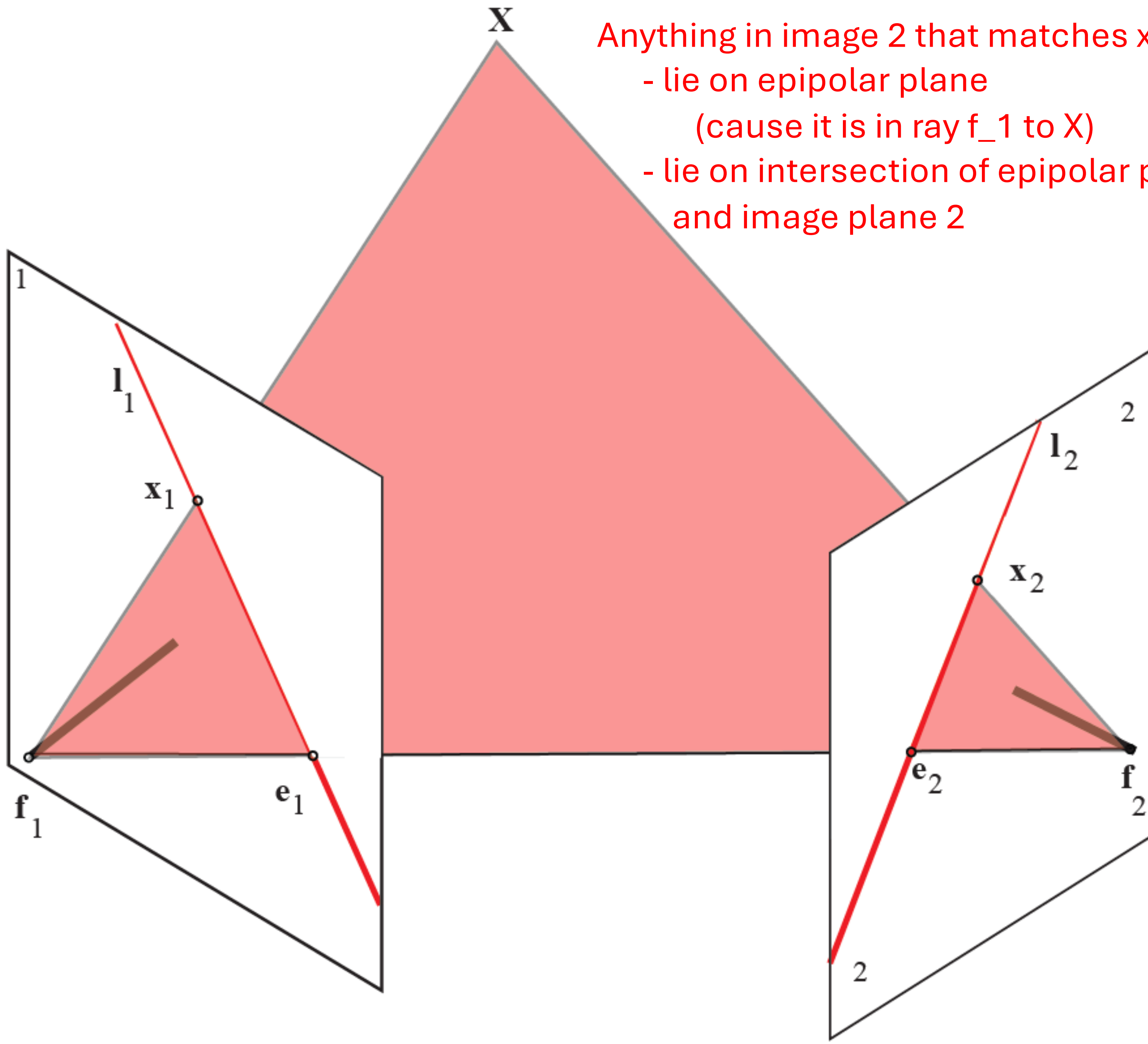

X
1
l1
x1
e1
f1
2
l2
x2
e2
f2
2

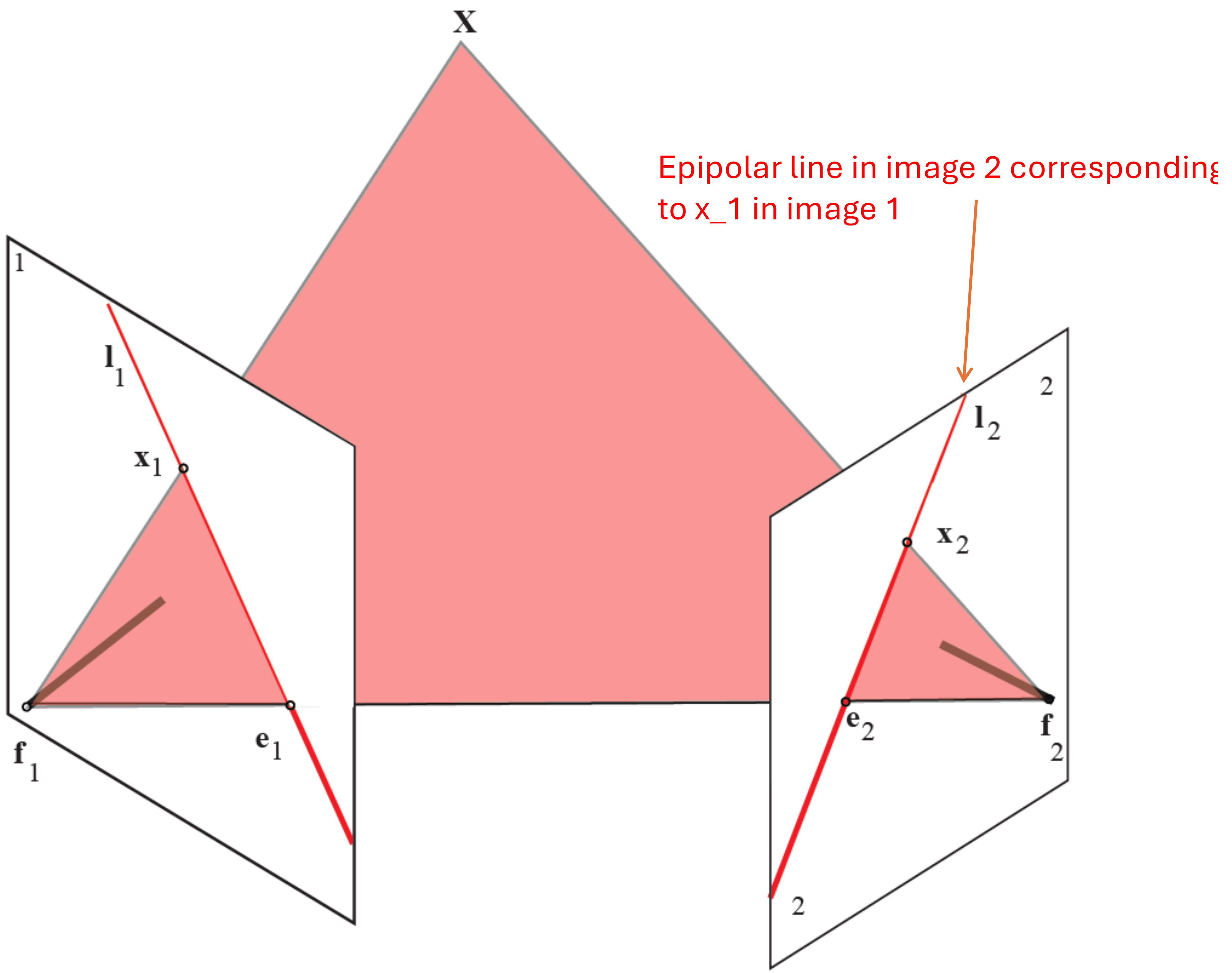

X
1
l_1
x_1
f_1
e_1
Epipole
2
l_2
x_2
e_2
f_2
Epipole
2

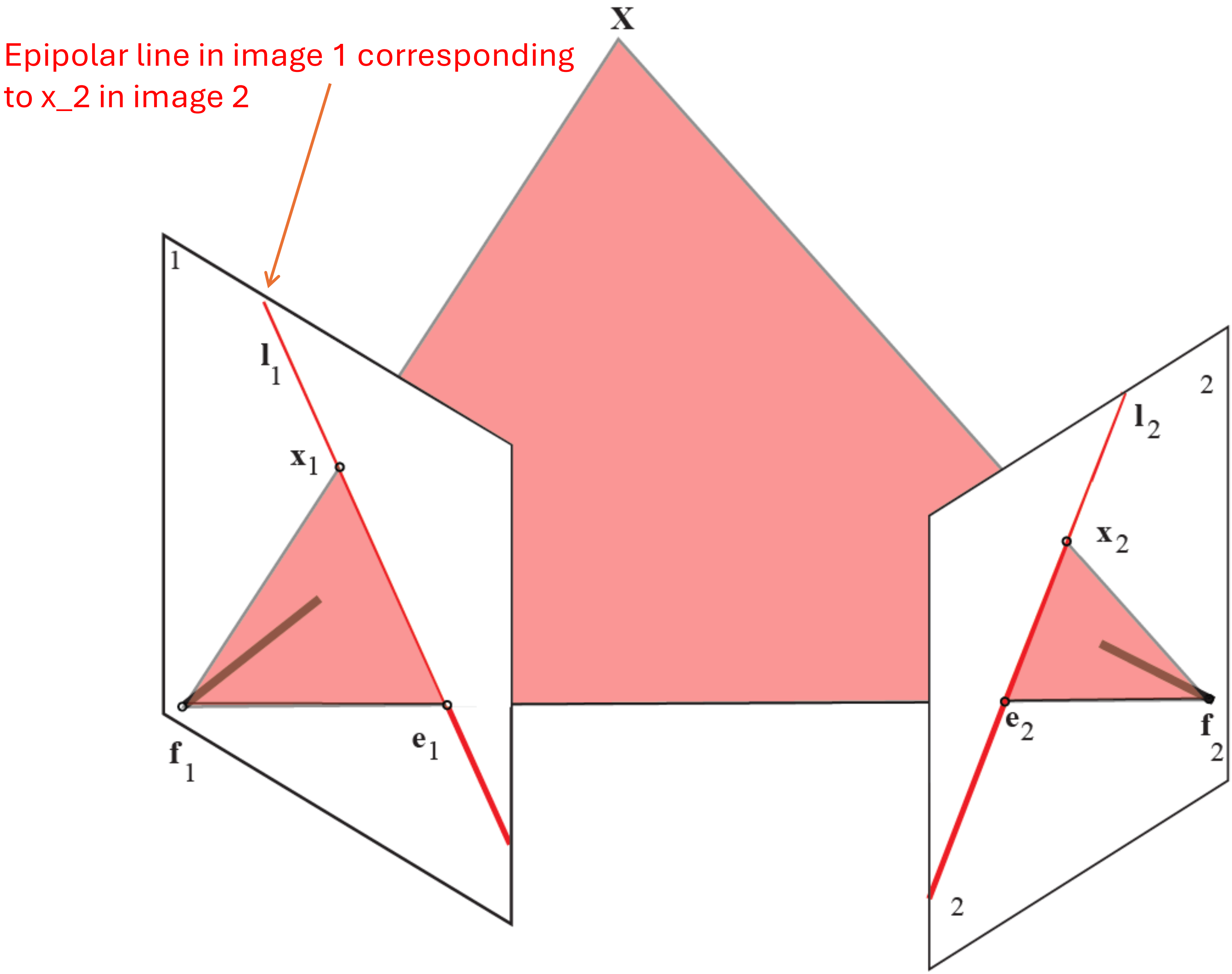

X
Epipolar plane
spans X, f_1, f_2
1
l1
x1
e1
f1
2
l2
x2
e2
f2
2

Anything in image 2 that matches x_1 MUST

- lie on epipolar plane
  (cause it is in ray f_1 to X)
- lie on intersection of epipolar plane
  and image plane 2




X
Epipolar line in image 2 corresponding
to x_1 in image 1
1
l_1
x_1
f_1
e_1
2
l_2
x_2
e_2
f_2
2



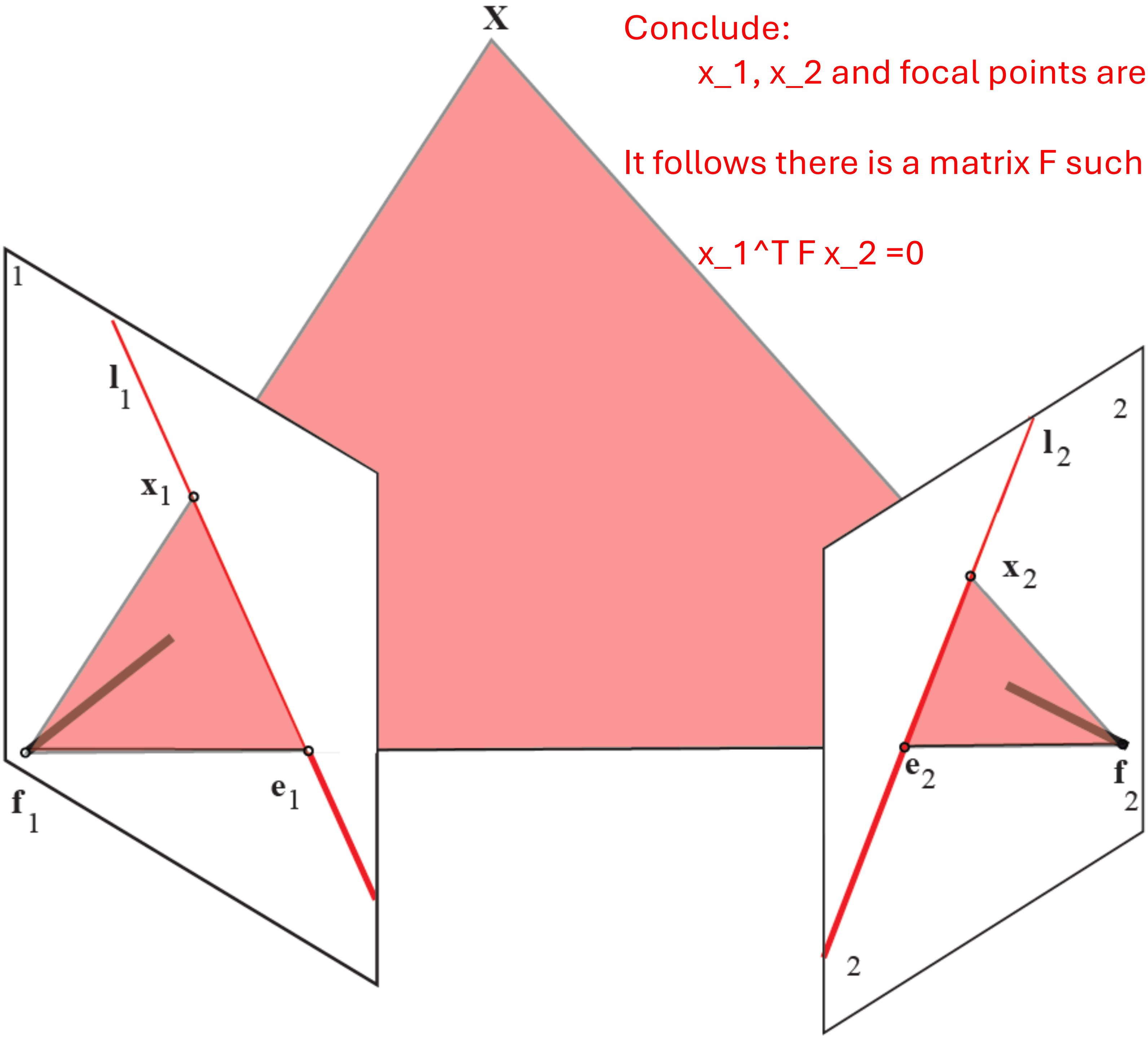
Epipolar line in image 1 corresponding to x_2 in image 2
X
1
l
1
x
1
f
1
e
1
2
l
2
x
2
e
2
f
2
2

Conclude:

x_1, x_2 and focal points are coplanar

It follows there is a matrix F such that

$x_1^T F x_2 = 0$



It follows…

This construction exposes an extremely important relationship between $\mathbf{x}_1$ and $\mathbf{x}_2$. Each of these points lies on its image plane, and so has three homogeneous coordinates. However, you can express these points in 3D. Write $\mathbf{X}_1$ for the point $\mathbf{x}_1$ written in four homogeneous coordinates, and notice that there is some 4×3 matrix $\mathcal{P}_1$ so that $\mathbf{X}_1 = \mathcal{P}_1\mathbf{x}_1$ (**exercises**).

The four points $\mathbf{X}_1$, $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{X}_2$ lie on the same plane. Each is a point in 3D. If you write each point in homogenous coordinates, you must have

$$\text{determinant}([\mathbf{X}_1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{X}_2]) = 0.$$

(**exercises**). In turn, this means that there is some 4×4 matrix $\mathcal{G}$ such that

$$\mathbf{X}_1^T \mathcal{G} \mathbf{X}_2 = 0.$$

But $\mathbf{X}_1 = \mathcal{P}_1\mathbf{x}_1$, etc. So there is some 3×3 matrix $\mathcal{F}$ such that

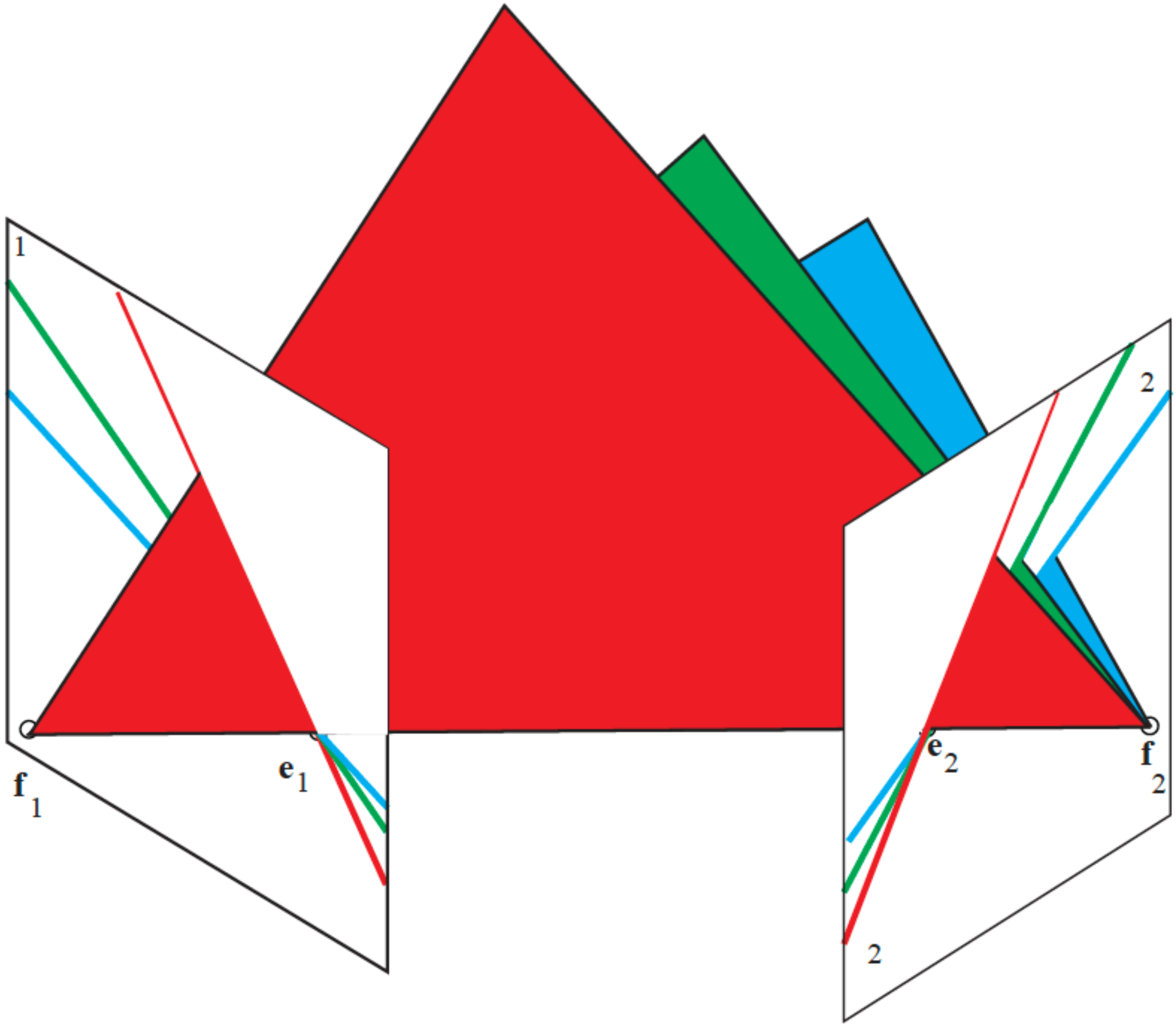
$$\mathbf{x}_1^T \mathcal{F} \mathbf{x}_2 = 0.$$

This matrix is known as the *fundamental matrix.*

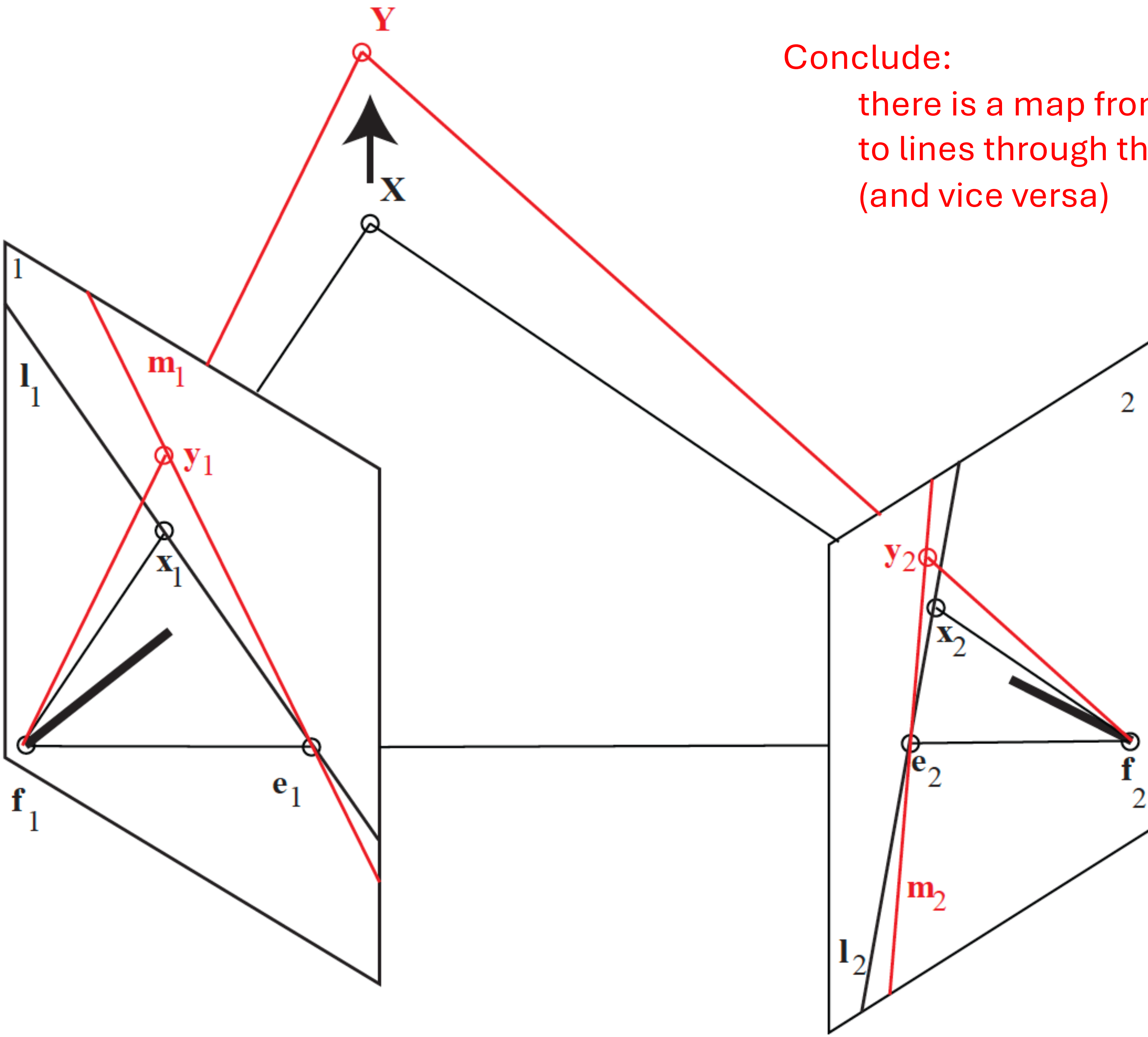
The fundamental matrix

Remember this: *For any pair of cameras which do not share a focal point, there is a fundamental matrix* $\mathcal{F}$ *with the property that for any pair* $\mathbf{x}_1$, $\mathbf{x}_2'$, *where* $\mathbf{x}_1$ *is the image of a 3D point in the first camera and* $\mathbf{x}_2'$ *is the image of that point in the second camera,*

$$\mathbf{x}_1^T \mathcal{F} \mathbf{x}_2' = 0$$



1
f 1
e 1
2
e 2
f 2
2



Conclude:

there is a map from points in 1
to lines through the epipole in 2
(and vice versa)

The fundamental matrix

The fundamental matrix for a pair of cameras reveals both epipoles and epipolar lines for both cameras. Choose some point $\mathbf{x}_1$ in the first camera. Now for *every* point $\mathbf{x}'$ in camera 2 that could match $\mathbf{x}_1$,

$$\mathbf{x}_1^T \mathcal{F} \mathbf{x}' = \left(\mathbf{x}_1^T \mathcal{F}\right) \mathbf{x}' = 0$$

and you can think of $\mathcal{F}^T \mathbf{x}_1$ as a vector containing the coefficients of a line. This line is the epipolar line corresponding to $\mathbf{x}_1$. You can think of the fundamental matrix as a map from points in one camera to lines in the other camera.

Procedure: 32.1 *Obtaining an epipolar line from a fundamental matrix*

The epipolar line in camera 2 corresponding to $\mathbf{x}_1$ in camera 1 consists of the set of points $\mathbf{x}'$ in camera 2 which satisfy the equation

$$\left(\mathbf{x}_1^T \mathcal{F}\right) \mathbf{x}' = 0$$

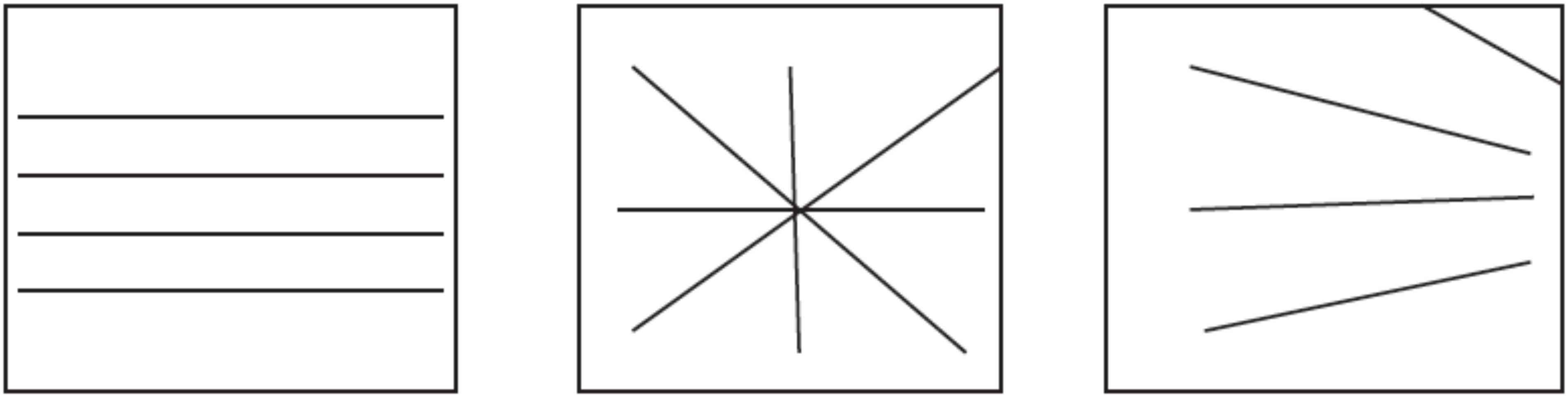
and the coefficients of the line are

$$\left(\mathcal{F}^T \mathbf{x}_1\right).$$

It follows that the coefficients of the epipolar line in camera 1 corresponding to $\mathbf{x}_2'$ in camera 2 are

$$\left(\mathcal{F} \mathbf{x}_2'\right).$$

Interpreting epipoles



The fundamental matrix

Every such line passes through the epipole, so the epipole must be the point in camera 2, $\mathbf{e}_2'$, such that for *any* choice of $\mathbf{x}_1$, $\left(\mathbf{x}_1^T \mathcal{F}\right) \mathbf{e}_2' = 0$. The only way to achieve this is if

$$\mathcal{F}\mathbf{e}_2' = \mathbf{0}$$

so $\mathcal{F}$ cannot have full rank.

The fundamental matrix

Procedure: 32.2 *Obtaining epipoles from a fundamental matrix*

The epipole in camera 2 is the point $\mathbf{e}'_2$ such that

$$\mathcal{F}\mathbf{e}'_2 = \mathbf{0}.$$

It follows that the epipole in camera 1 is the point $\mathbf{e}_1$ such that

$$\mathcal{F}^T\mathbf{e}_1 = \mathbf{0}.$$