Homogenous Coordinates

D.A. Forsyth

University of Illinois at Urbana Champaign

Homogeneous coordinates

- Add an extra coordinate and use an equivalence relation
- for 2D
  - equivalence relation
    k*(X,Y,Z) is the same as (X,Y,Z)
- for 3D
  - equivalence relation
    k*(X,Y,Z,T) is the same as (X,Y,Z,T)
- “Ordinary” or “non-homogeneous” coordinates
  - properly called affine coordinates
  - in 3D, affine -> homogeneous $(x, y, z) \to k * (x, y, z, 1)$
  - in 3D, homogeneous to affine $(X, Y, Z, T) \to \left(\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}\right)$

Homogeneous coordinates

- Notice (0, 0, 0, 0) is meaningless (HC’s for 3D)
  - also (0, 0, 0) in 2D
- Basic notion
  - Possible to represent points “at infinity” by careful use of zero
  - Where parallel lines intersect
    - eg

$$(tX, tY, tZ, 1) \quad \text{and} \quad (tX + a, tY + b, tZ + c, 1)$$

$$\text{intersect at} \quad (X, Y, Z, 0)$$

  - Where parallel planes intersect (etc)
- Can write the action of a perspective camera as a matrix

Homogeneous coordinates

Example: 23.1 *Lines on the affine plane*

Lines on the affine plane form one important example of homogeneous coordinates. A line is the set of points (x, y) where $ax + by + c = 0$. We can use the coordinates (a, b, c) to represent a line. If $(d, e, f) = \lambda(a, b, c)$ for $\lambda \neq 0$ (which is the same as $(d, e, f) \equiv (a, b, c)$), then (d, e, f) and (a, b, c) represent the same line. This means the coordinates we are using for lines are homogeneous coordinates, and the family of lines in the affine plane is a projective plane. Notice that encoding lines using affine coordinates must leave out some lines. For example, if we insist on using $(u, v, 1) = (a/c, b/c, 1)$ to represent lines, the corresponding equation of the line would be $ux + vy + 1 = 0$. But no such line can pass through the origin – our representation has left out every line through the origin.

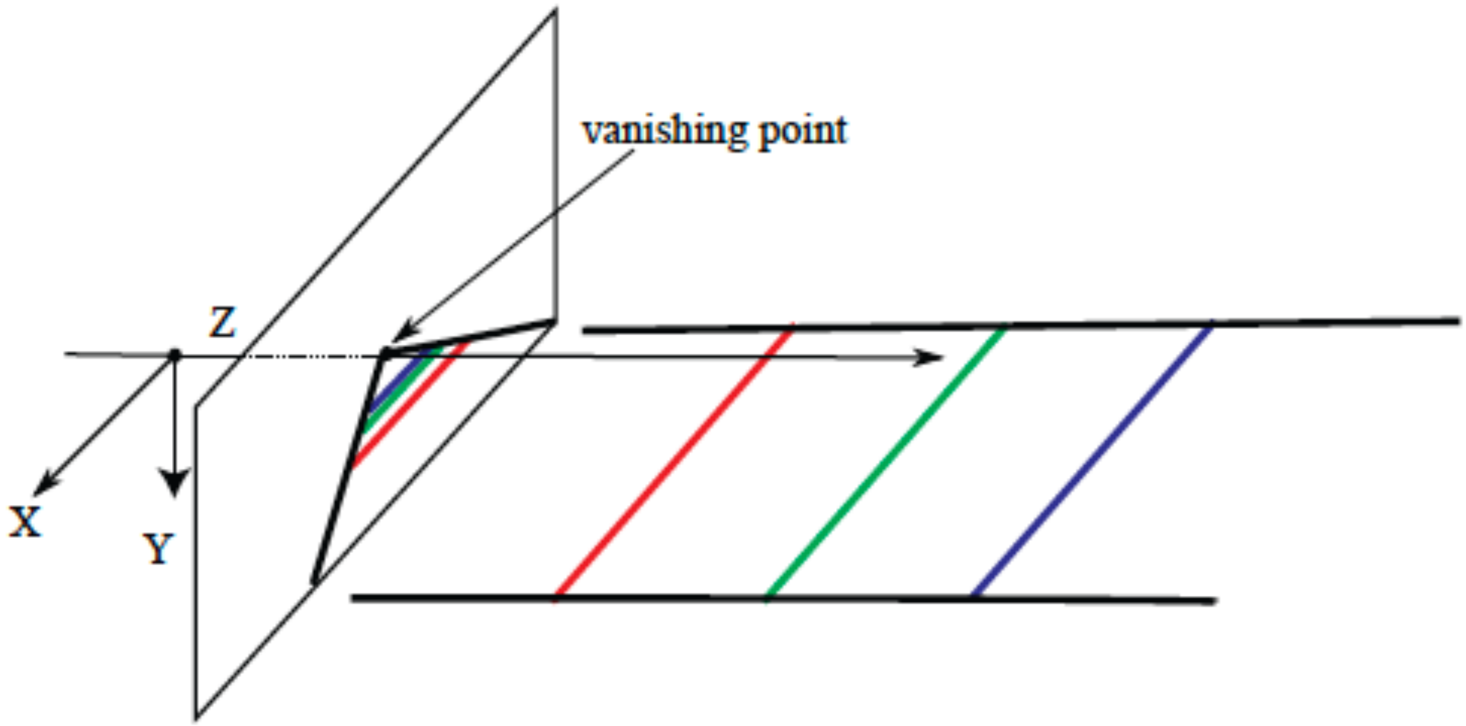
Homogeneous coordinates for a line

23.1.2 The projective line

In homogenous coordinates, we represent a point on a 1D space with two coordinates, so (X_1, X_2) (by convention, homogeneous coordinates are written with capital letters). Two sets of homogeneous coordinates (U_1, U_2) and (V_1, V_2) represent different points if there is no $\lambda \neq 0$ such that $\lambda(U_1, U_2) = (V_1, V_2)$. The set of all distinct points is known as a *projective line*. You should think of the projective line as an ordinary line (an *affine line*) with an "extra point". Every point on an affine line has a corresponding point on a projective line. A point on an affine line is given by a single coordinate x. This point can be identified with the point on a projective line given by $(X_1, X_2) = \lambda(x, 1)$ (for $\lambda \neq 0$) in homogeneous coordinates. The extra point has coordinates $(X_1, 0)$. These are the homogeneous coordinates of a single point (check this), but this point would be "at infinity" on the affine line.

There isn't anything special about the point on the projective line given by $(X_1, 0)$. You can see this by identifying the point x on the affine line with $(X_1, X_2) = \lambda(1, x)$ (for $\lambda \neq 0$). Now $(X_1, 0)$ is a point like any other, and $(0, X_2)$ is "at infinity". A little work establishes that there is a 1-1 mapping between the projective line and a circle (exercises).

You can see the point at infinity

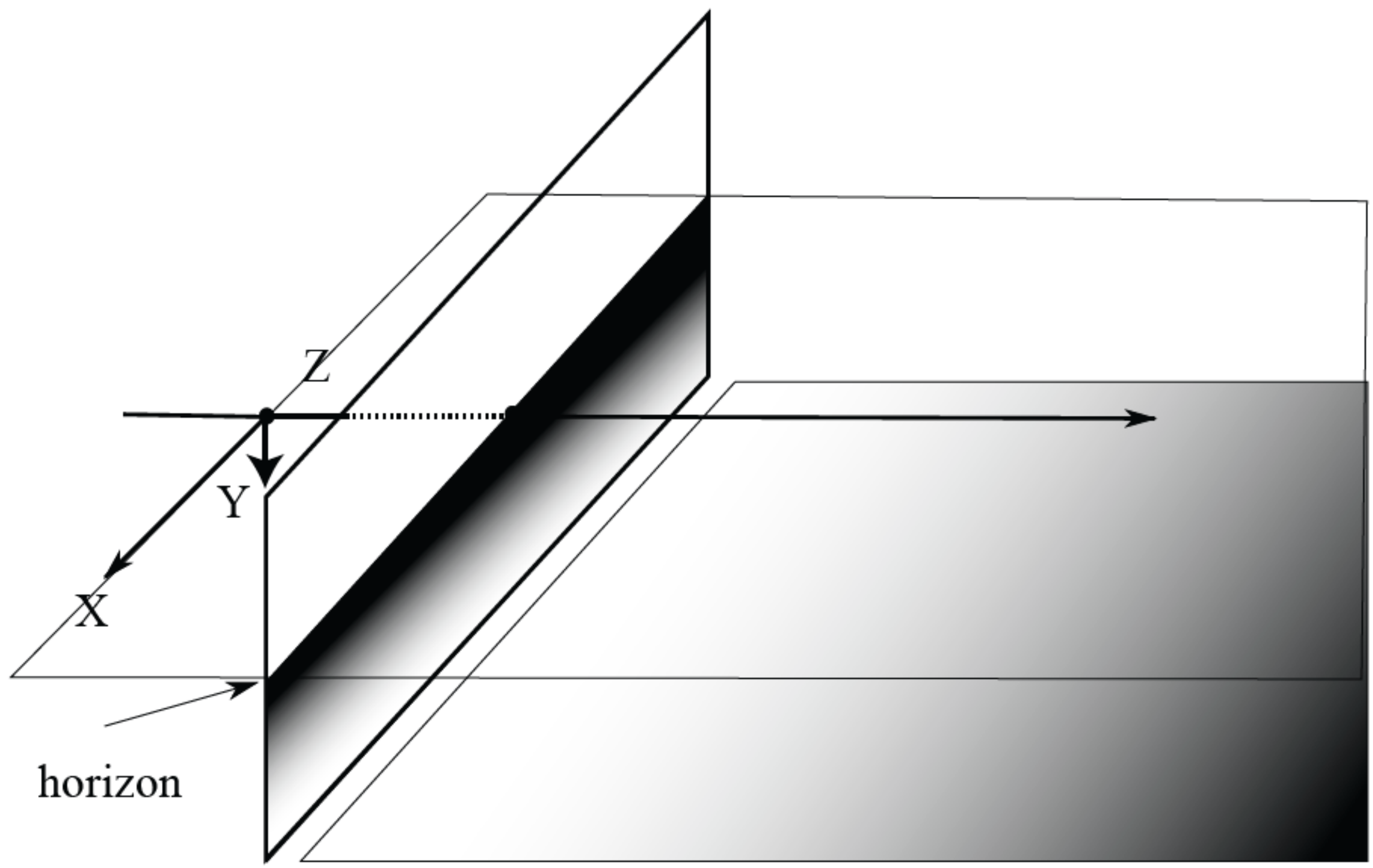


Homogeneous coordinates for the plane

23.1.3 The projective plane

The space represented by three homogeneous coordinates is known as a *projective plane*. You can map an *affine plane* (the usual plane, with coordinates x, y) to a projective plane by writing $(X_1, X_2, X_3) = (x, y, 1)$. Notice that there are points on the projective plane — the points where $X_3 = 0$ — that are missing. These points form a projective line (check this!). This line is often referred to as the *line at infinity*.

You can see the line at infinity, too!



Main issues

- Keep track of coordinates you're working in
  - or else...
- Transforming is easy
  - in 3D, affine -> homogeneous

$$(x, y, z) \rightarrow k * (x, y, z, 1)$$

  - in 3D, homogeneous to affine

$$(X, Y, Z, T) \rightarrow \left(\frac{X}{T}, \frac{Y}{T}, \frac{Z}{T}\right)$$

- HC's account for observable phenomena
  - point at infinity
  - line at infinity, etc.
- Remember dividing by zero annoys computers