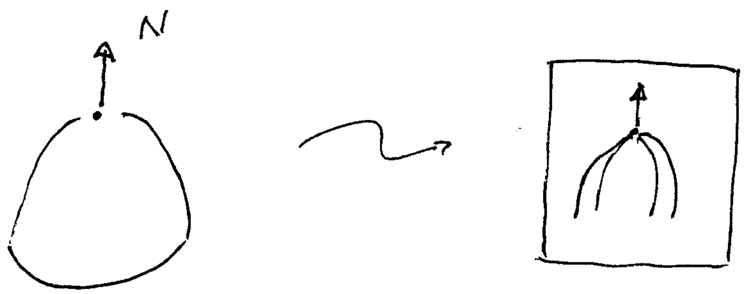
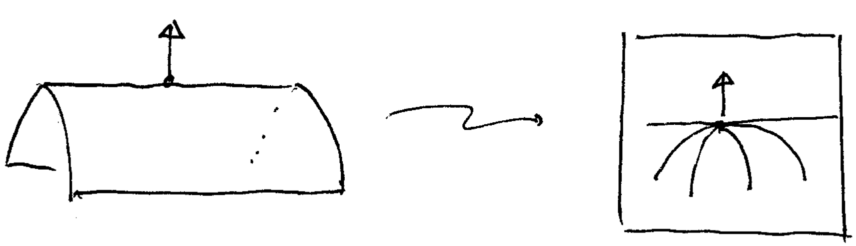
Local (Differential) Geometry of surfaces:

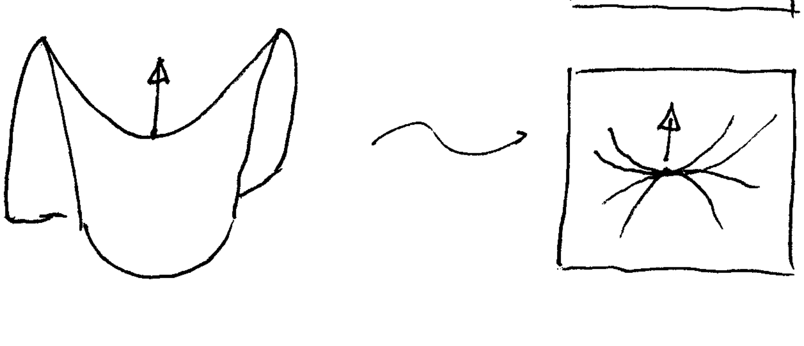
Choose a point on a surf.
- compute normal
- Build family of planes thru pt, normal
- consider these X-sections of surf
- 3 cases



Elliptic



Parabolic



Hyperbolic

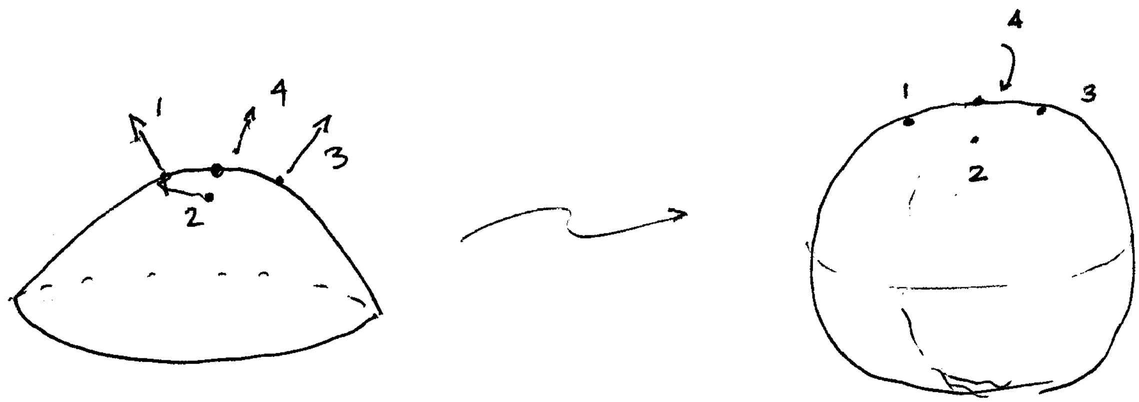
A finer classification would be helpful

Both Elliptic

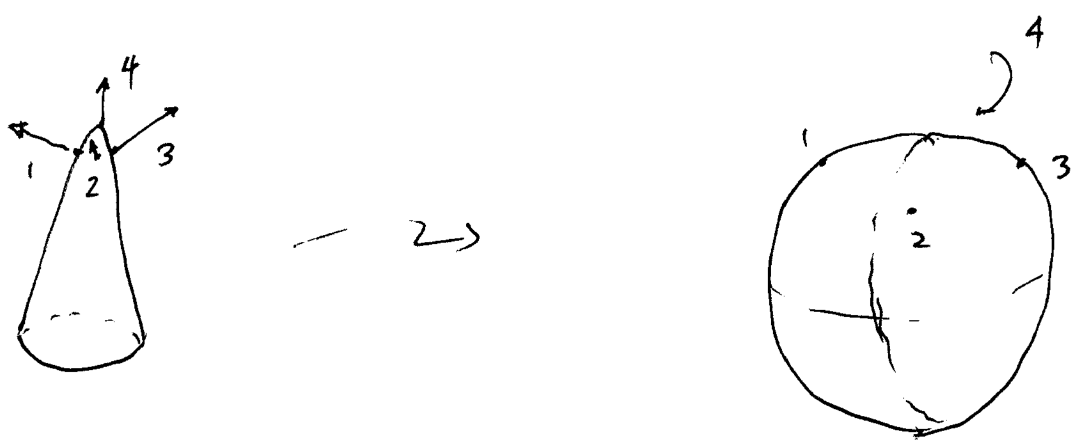
We get this from the <u>Gauss map</u>.

$\underset{\sim}{x}$ pt. on surface $\longrightarrow$ $N(\underset{\sim}{x})$ pt on sphere given by normal

I

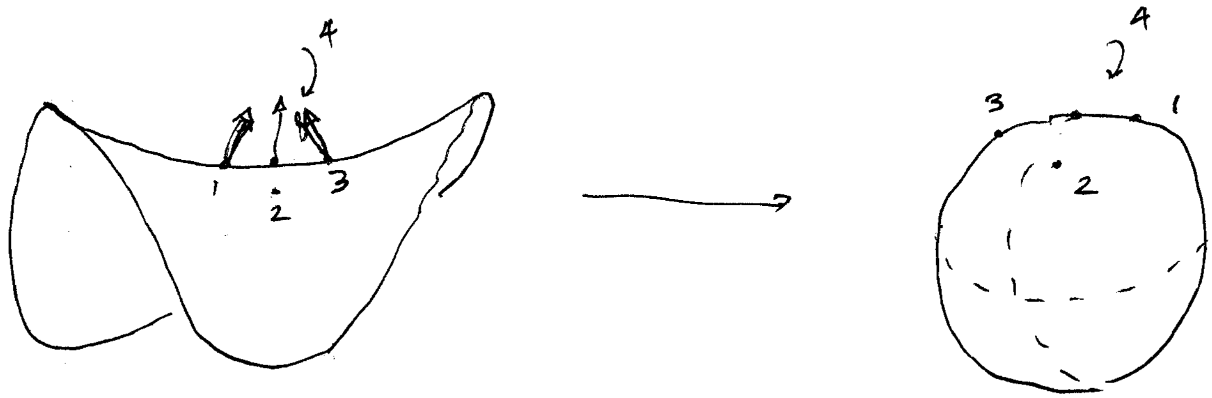


II

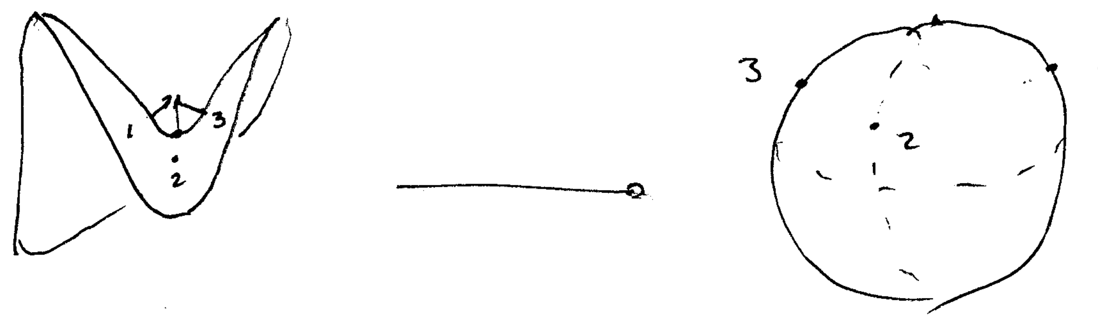


- map a small circle round p to sphere
- case I : small circle $\downarrow$ small "
- case II : small $\longrightarrow$ big.

I



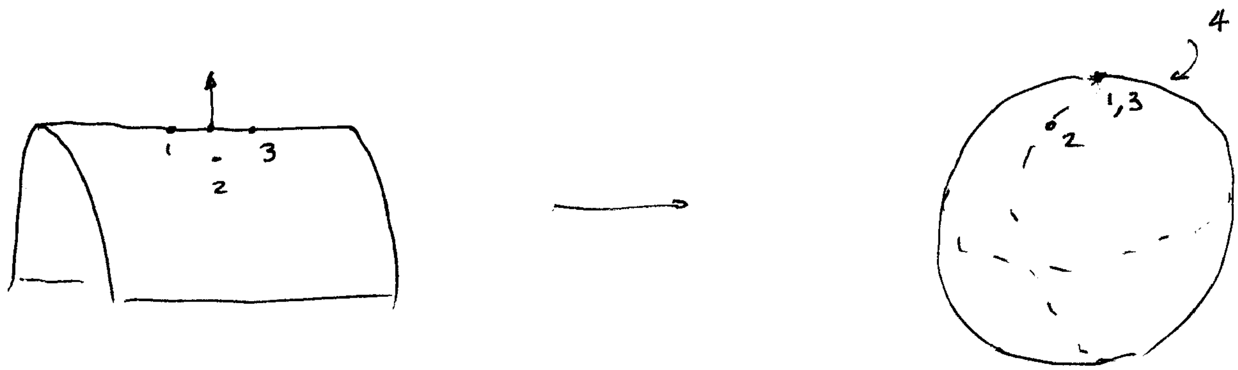
II



- Notice <u>direction reverses</u>

I : small $\longrightarrow$ small

II : small $\longrightarrow$ big



· small $\longrightarrow$ area zero.

Defn

K = Gaussian curvature

$$= \lim_{\text{radius} \to 0} \left\{ \frac{\text{Area on Gauss map}}{\text{Area on Surf}} \right\}$$

$$K = \begin{cases} < 0 & \text{Hyperbolic} \\ 0 & \text{Parabolic} \\ > 0 & \text{Elliptic} \end{cases}$$

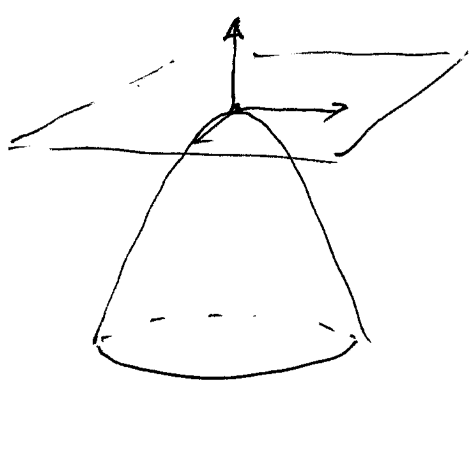
<u>Bending</u> does not change K

– You must $\left\{\begin{matrix}\text{add} \\ \text{subtract}\end{matrix}\right\}$ area.

(So there must be another Description to add Detail.)

- Take a point on a surface.
- Construct a coord system ~~in~~

(x,y) in tangent plane, with z normal



- IN THIS COORD SYSTEM, near this pt, write Taylor Series.

Surface is

$$(x, y, z(x,y))$$

$$\approx \left(x, y, z_0 + (\nabla z)\cdot(x,y) + \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix} + O(x,y)^3\right)$$

but
$$z_0 = 0$$
$$\nabla z = 0$$

so $$(x, y, z(x,y)) = \left(x, y, \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix} + O(3)\right)$$

- this is a quadratic form
- symmetric

$\therefore$ rotate coord sys

$$\big(u, v, z(u,v)\big) = \left(u, v, \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right) + O(3)\right)$$

Now recall a curve

$(u, \frac{1}{2} a u^2)$ has curvature a at $u = 0$

- So the curvature of the u section is K_1

v " is K_2

~~$s = u\cos\theta + v\sin\theta$~~ "

$s = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ " is $K_1 \cos^2\theta + K_2 \sin^2\theta$

$\therefore$ The directional curvature has

maximum $\max(K_1, K_2)$

min $\min(K_1, K_2)$

∴ at each point on surf, there are 2 dir'ns, orthonormal, where dir' curvature is max, resp. min

$$\boxed{\text{Principal} \begin{cases} \text{directions} \\ \text{curvatures} \end{cases}}$$

- With easy extra work

$$\boxed{K = k_1 k_2}$$

- Normal curvature is

$$\boxed{H = k_1 + k_2}$$

this gives bending.

we have $H = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$ as a local

Description. We can see this as

a quadratic form on tangent vecs

$X = \begin{pmatrix} x_u \\ x_v \end{pmatrix}$ in u, v coords

$Y = \begin{pmatrix} y_u \\ y_v \end{pmatrix}$ in " coords

$X^T H Y$ is quadratic form

Important cases

$$X^T H Y = 0$$

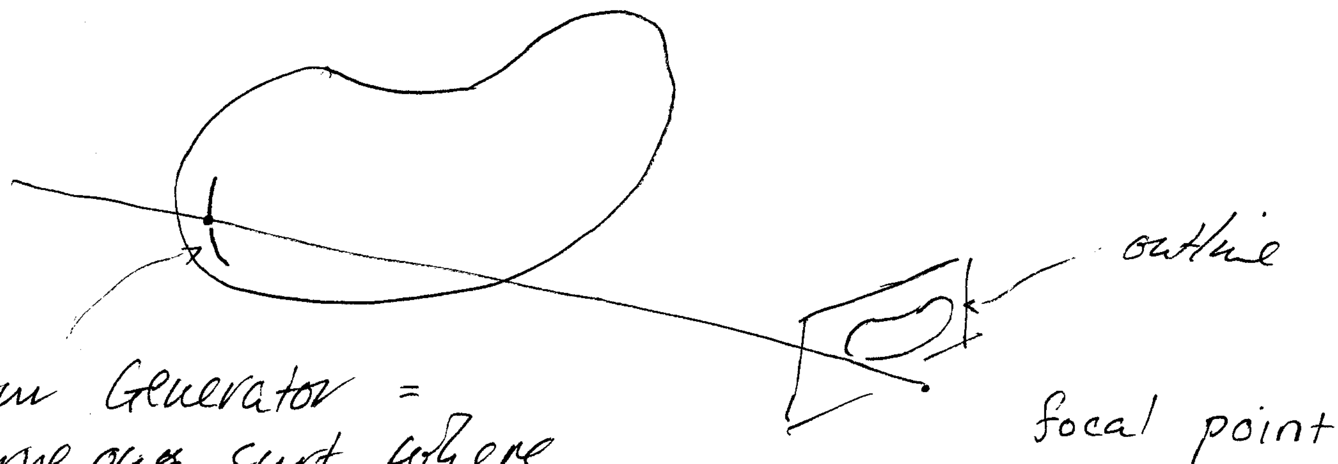
X, Y are conjugate

$$X^T H X = 0$$

X is <u>assymptotic</u>

Conjagate dirs have an important application

§ Important phenomenon.



Contour Generator =
Curve on surf where
view is tangent

· C.G., outline move when view moves.

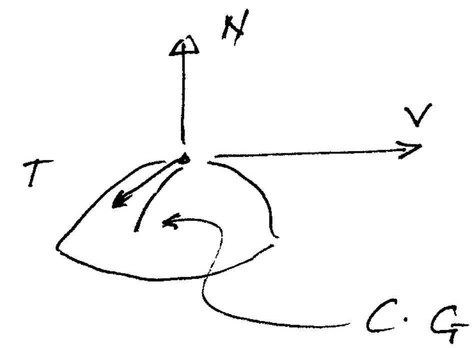
at C.G.

$$V \cdot N = 0$$

Q: what is tangent to C.G.?

~~at our~~

- Assume our point is ~~surface~~ C.G. pt



$N \cdot V = 0$

- now, we want the C.G. tangent T
    - T tangent to surface
    - directional Derivative of $(N \cdot V)$ in T Direction is 0

$$\left(D_T (N \cdot V) = 0 \right)$$

$T = (a, b)$

$$D_T = a \frac{\partial}{\partial u} + b \frac{\partial}{\partial v}$$

compute N

$$\left(u,\ v,\ \tfrac{1}{2}k_1 u^2 + \tfrac{1}{2}k_2 v^2\right)$$

$$N = \frac{-k_1 u\ ,\ -k_2 v\ ,\ 1}{\sqrt{1 + k_1^2 u^2 + k_2^2 v^2}}$$

notice <u>magnitude</u> of n has nothing to do with problem.

$$V = (V_0\ ,\ V_1\ ,\ V_2) \qquad [\text{orthographic case}]$$

$$N \cdot V = 0 \quad \propto \quad -K_1 u V_0 - K_2 v V_1 + V_2$$

$$a\left[-K_1 V_0\right] + b\left[-K_2 V_1\right] = 0$$

$$\equiv \boxed{V \text{ and } T \text{ are conjugate}}$$

Useful ideas

- it is often difficult to construct the coordinate system we have used.

eg parametric surface

$\underline{X}(s,t) = \big(x(s,t), y(s,t), z(s,t) \big)$

at s, t there are 2 tangents X_s, X_t

$$\left(X_s = \frac{\partial X}{\partial s}, \quad X_t = \frac{\partial X}{\partial t} \right)$$

if V is a tangent vector, how long is it?

$$V = a X_s + b X_t$$

$$V \cdot V = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X_s \cdot X_s & X_s \cdot X_t \\ X_s \cdot X_t & X_t \cdot X_t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

this matrix $\begin{bmatrix} X_s \cdot X_s & X_t \cdot X_s \\ X_t \cdot X_s & X_t \cdot X_t \end{bmatrix}$ is the

$$\boxed{\text{FIRST FUNDAMENTAL FORM}}$$

<u>Notice</u>

$$N \cdot N = 1$$

Now consider a tangent vector V

- the directional derivative in the V direction ~~is~~ is

$$a \frac{\partial}{\partial s} + b \frac{\partial}{\partial t}$$

if $V = a X_s + b X_t$

- write D_V

- $D_V (N \cdot N) = 0$

so
$$(D_v N) \cdot N = 0$$

so $D_v N$ is a tangent vector

for u a tangent vector,

$$(D_v \cdot N) \cdot u = \mathrm{II}(u, v)$$

$\mathrm{II}(u,v)$ is a quadratic form in u, v

SECOND FUNDAMENTAL FORM

$$\boxed{K = \frac{\det \mathrm{II}}{\det \mathrm{I}}}$$

Our form II is <u>second fundamental form</u>

$$\boxed{\text{Conjugate dirs if } \quad \mathrm{II}(u,v) = 0}$$

u,v are