Local (Differential) Geometry of Surfaces:

Choose a point on a surf.
- Compute normal
- Build family of planes through normal
- Consider these x-sections of surf

3 cases

Elliptic
Parabolic
Hyperbolic
A finer classification would be

Both Elliptic

We get this from the Gauss map.

\[ x \]

pt. on surface \[ \rightarrow \]

\[ N(x) \]

pt. on sphere given by normal
map a small circle round \( p \) to sphere

- case I: small circle
  - small \( \rightarrow \) small

- case II: small \( \rightarrow \) big
I: \[ \text{small} \rightarrow \text{small} \]

II: \[ \text{small} \rightarrow \text{big} \]

Notice, direction reverses
small \rightarrow \text{area zero.}

\textbf{Defn}

\[ K = \text{Gaussian curvature} \]

\[ = \lim_{\text{radius} \to 0} \left\{ \begin{array}{l}
\text{Area of Gauss map} \\
\text{Area on surf}
\end{array} \right\} \]

\[ K = \begin{cases} 
< 0 & \text{Hyperbolic} \\
0 & \text{Parabolic} \\
> 0 & \text{Elliptic}
\end{cases} \]
Bending does not change $K$ - you must \{add $\}$ area
(\textit{so there must be another description to add detail.})

\begin{itemize}
  \item Take a point on a surface
  \item Construct a coord system $(x,y)$ in tangent plane, with $z$ normal
\end{itemize}

\textit{In this coord system, near this pt, write Taylor Series.}
Surface is

\[(x, y, z(x, y))\]

\[= (x, y, z_0 + (\nabla z) \cdot (x, y) + \frac{1}{2} (x^T H(x)) y + O(x^3))\]

\[= C(z_0 = 0, \nabla z = 0)\]

\[= z(x, y) = (x, y, \frac{1}{2} (y^T H(x) y + O(x^3)))\]

This is a quadratic form

Symmetric

\[= \text{rotate coord sys}\]

\[(u, v, z(u, v)) = (u, v, \frac{1}{2} (k_1 u^2 + k_2 v^2) + O(x^3))\]
Now recall a curve

\[(u, \frac{1}{2} au^2)\] has curvature a at \(u = 0\).

So the curvature of the \(u\) section is \(k_1\),

\[v \quad \text{is} \quad k_2\]

\[s = u \cos \theta + v \sin \theta\]

\[s = \frac{\cos \theta}{\sin \theta}\]

is \(k_1 \cos^2 \theta + k_2 \sin^2 \theta\)

The directional curvature has maximum \(\max (k_1, k_2)\)

minimum \(\min (k_1, k_2)\)
At each point on a surf, there are 2 dirs., orthonormal, where dir' curvature is max, resp. min.

Principal directions

\[ K = k_1, k_2 \]

With easy extra work

Normal curvature is

\[ H = k_1 + k_2 \]

This gives bending.
we have \( H = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \) as a local description. We can see this as a quadratic form on tangent vectors.

\[ X^t = \begin{pmatrix} x_u \\ x_v \end{pmatrix} \] in \( u,v \) coords

\[ Y = \begin{pmatrix} y_u \\ y_v \end{pmatrix} \] in \( u,v \) coords

\[ X^t H Y \] is quadratic form

**Important cases**

\[ X^t H Y = 0 \]

\( X, Y \) are conjugate
\[ X^T H X = 0 \]

\[ X \text{ is asymptotic} \]

Conjugate dirs have an important application
Important phenomenon.

Contour Generator = Curve on surf where view is tangent

at C.G.

\[ \mathbf{V} \cdot \mathbf{N} = 0 \]

Q: what is tangent to C.G.?
. Assume our point is C.C. pt

\[ \mathbf{N} \cdot \mathbf{V} = 0 \]

. now we want the bi-tangent T

. \( \mathbf{T} \) tangent to surface

. directional derivative of \((\mathbf{N} \cdot \mathbf{V})\)

\[ \nabla \mathbf{T} \] direction is 0

\[ ( \frac{D}{\mathbf{T}} (\mathbf{N} \cdot \mathbf{V}) = 0 ) \]

\( \mathbf{T} = (a, b) \)

\[ D_{\mathbf{T}} = a \frac{\partial}{\partial u} + b \frac{\partial}{\partial v} \]
compute $N$

\[(u, v, \frac{1}{2} k_1 u^2 + \frac{1}{2} k_2 v^2)\]

\[N = \frac{-k_1 u, -k_2 v, 1}{\sqrt{1 + k_1^2 u^2 + k_2^2 v^2}}\]

notice magnitude of $n$ has nothing to do with problem.

\[V = (V_0, v, V_2)\]  \[\text{orthographic case}\]

\[N \cdot V = 0 \quad \alpha = -k_1 u V_0 - k_2 V v_1 + V_2\]

\[
\begin{bmatrix}
-a \\
-b
\end{bmatrix} + b \begin{bmatrix}
-k_1 V_0 \\
-k_2 V v_1
\end{bmatrix} = 0
\]

\[\Rightarrow V \text{ and } T \text{ are conjugate}\]
Useful ideas

- it is often difficult to construct the coordinate system we have used.

*eg parametric surface*

\[ X(s,t) = (x(s,t), y(s,t), z(s,t)) \]

at \( s, t \) there are 2 tangents \( X_s, X_t \)

\[
\begin{align*}
X_s &= \frac{\partial X}{\partial s}, \\
X_t &= \frac{\partial X}{\partial t}
\end{align*}
\]

if \( \mathbf{V} \) is a tangent vector, how long is it?

\[
\mathbf{V} = a \mathbf{X}_s + b \mathbf{X}_t
\]

\[
\mathbf{V} \cdot \mathbf{V} = \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X_s \cdot X_s & X_s \cdot X_t \\ X_s \cdot X_t & X_t \cdot X_t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}
\]
Notice

\[ N \cdot N = 1 \]

Now consider a tangent vector \( \mathbf{V} \)

\[ \frac{\partial}{\partial s} + b \frac{\partial}{\partial t} \]

if \( \mathbf{V} = a \mathbf{X}_s + b \mathbf{X}_t \)

\[ \mathbf{D}_\mathbf{V} \]

\[ \mathbf{D}_\mathbf{V} (N \cdot N) = 0 \]
So \((D_v N) \cdot N = 0\)

So \(D_v N\) is a tangent vector

for \(u\) a tangent vector,

\[(D_v \cdot N) \cdot u = II(u, v)\]

\(II(u, v)\) is a quadratic form in \(u, v\)

\[K = \frac{\det II}{\det I}\]
Our form is second fundamental form

$u, v$ are conjugate dirs if $II(u, v) = 0$