Splines

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Core ideas: Assembly, Continuity

- We "join up" pieces of curve to meet various constraints
  - result is a spline
- Continuity
  - Parametric
    - C^k : Curve and derivatives up to k are continuous
      - as a function of parameter value
    - Useful for (for example) animation
  - Geometric
    - G^k: a reparametrisation exists that would achieve C^k
    - Useful, because we often don't require parametric continuity
    - e.g. take two Hermite curves, both parametrised by [0, 1], identify endpoints and derivatives

Simple cases

- Join up two point Hermite curves
  - endpoints the same, vectors not - G^0
  - endpoints, vectors the same - G^1 (easy)
  - endpoints the same, vectors same direction - G^1 (harder)
- Subdivide a Bezier curve
  - result is G^infinity if we reparametrize each segment as we should
    - but not necessarily if we move the control points!
- Join up Bezier curves
  - endpoints join - G^0
  - endpoints join, end segments collinear - G^1

Cubic interpolating splines

- n+1 points P_i
- X_i(t) is curve between P_i, P_i+1

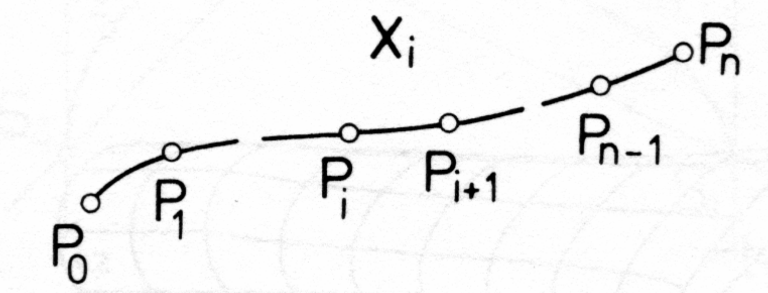


Fig. 3.11. The spline segment $\boldsymbol{X}_i$.

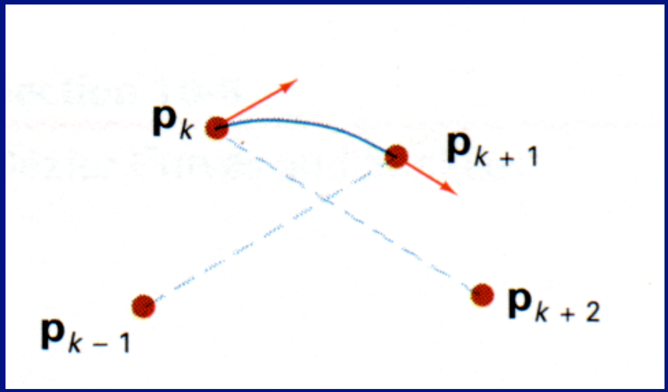
Interpolating Cubic splines, G^1

- join a series of Hermite curves with equal derivatives.
- But where are the derivative values to come from?
  - Measurements
  - Cardinal splines

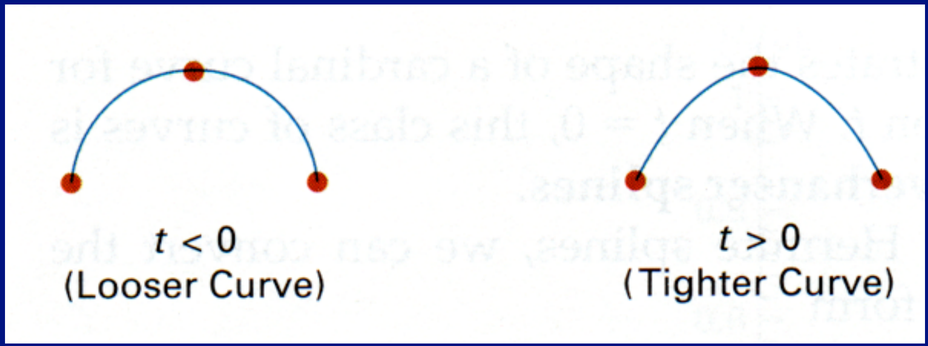
$$\frac{d\mathbf{X}_i}{dt}(0) = \frac{1}{2}(1-t)(\mathbf{P}_{i+1} - \mathbf{P}_{i-1})$$

    - average points
    - t is “tension”
    - specify endpoint tangents
      - or use difference between first two, last two points

Tension



$\mathbf{p}_k$
$\mathbf{p}_{k+1}$
$\mathbf{p}_{k-1}$
$\mathbf{p}_{k+2}$



$t < 0$
(Looser Curve)
$t > 0$
(Tighter Curve)

Interpolating Cubic splines: C^2

- One parametrization for the whole curve
  - divided up into intervals, called knots

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b.$$

$$\Delta t_i := t_{i+1} - t_i.$$

  - In each segment, there is a cubic curve FOR THAT SEGMENT

$$\mathbf{A}_i(t - t_i)^3 + \mathbf{B}_i(t - t_i)^2 + \mathbf{C}_i(t - t_i) + \mathbf{D}_i$$

  - And we must make this lot C^2

$$t_i \leq t < t_{i+1}$$

Continuity

- at interval endpoints, curves must be
  - Continuous

$$\mathbf{X}_i(t_i) = \mathbf{X}_{i-1}(t_i) \qquad \mathbf{X}_i(t_{i+1}) = \mathbf{X}_{i+1}(t_{i+1})$$

  - have continuous derivative

$$\frac{d\mathbf{X}_i}{dt}(t_i) = \frac{d\mathbf{X}_{i-1}}{dt}(t_i)$$

  - have continuous second derivative

$$\frac{d^2\mathbf{X}_i}{dt^2}(t_i) = \frac{d^2\mathbf{X}_{i-1}}{dt^2}(t_i)$$

Curves

- Assume we KNOW the derivative at each point
  - write derivatives with ‘

$$\mathbf{X}_i(t_i) = \mathbf{P}_i = \mathbf{D}_i$$

$$\frac{d\mathbf{X}_i}{dt}(t_i) = \mathbf{X}'_i(t_i) = \mathbf{P}'_i = \mathbf{C}_i$$

$$\mathbf{X}_i(t_{i+1}) = \mathbf{P}_{i+1} = \mathbf{A}_i \Delta t_i^3 + \mathbf{B}_i \Delta t_i^2 + \mathbf{C}_i \Delta t_i + \mathbf{D}_i$$

$$\mathbf{X}'_i(t_{i+1}) = \mathbf{P}'_{i+1} = 3\mathbf{A}_i \Delta t_i^2 + 2\mathbf{B}_i \Delta t_i + \mathbf{C}_i$$

Curves

$$\begin{aligned}
\mathbf{X}_i(t) &= \mathbf{P}_i\left(2\frac{(t-t_i)^3}{(\Delta t_i)^3} - 3\frac{(t-t_i)^2}{(\Delta t_i)^2} + 1\right) + \\
&\quad \mathbf{P}_{i+1}\left(-2\frac{(t-t_i)^3}{(\Delta t_i)^3} + 3\frac{(t-t_i)^2}{(\Delta t_i)^2}\right) + \\
&\quad \mathbf{P}'_i\left(\frac{(t-t_i)^3}{(\Delta t_i)^2} - 2\frac{(t-t_i)^2}{(\Delta t_i)} + (t-t_i)\right) + \\
&\quad \mathbf{P}'_{i+1}\left(\frac{(t-t_i)^3}{(\Delta t_i)^2} - \frac{(t-t_i)^2}{(\Delta t_i)}\right)
\end{aligned}$$

C^2 Continuity supplies derivatives

- Second derivative is continuous

$$\mathbf{X''}_{i-1}(t_i) = \mathbf{X}_i(t_i)$$

- Differentiate curves, rearrange to get

$$\Delta t_i \mathbf{P'}_{i-1} + 2(\Delta t_{i-1} + \Delta t_i)\mathbf{P'}_i + \Delta t_{i-1}\mathbf{P'}_{i+1} = 3\frac{\Delta t_{i-1}}{\Delta t_i}(\mathbf{P}_{i+1} - \mathbf{P}_i) + 3\frac{\Delta t_i}{\Delta t_{i-1}}(\mathbf{P}_i - \mathbf{P}_{i-1})$$

- This is a linear system in tridiagonal form
  - can see as recurrence relation - we need two tangents to solve

C^2 cubic splines

- Recurrence relations
  - d(n-1) equations in d(n+1) unknowns (d is dimension)
- Options:
  - give P'_0, P'_1 (easiest, unnatural)
  - second derivatives vanish at each end (natural spline)
  - give slopes at the boundary
    - vector from first to second, second last to last
  - parabola through first three, last three points
  - third derivative is the same at first, last knot

More general splines

- We would like to retain continuity, local control
  - but drop interpolation
- Mechanism
  - get clever with blending functions
  - continuity of curve=continuity of blending functions
  - we will need to “switch” on or off pieces of function
    - e.g. linear example

B-splines

- Knot vector

$$t_0 < t_1 < \ldots < t_{n+k}$$

- Curve

$$\mathbf{X}(t) = \sum_{k=0}^{n} \mathbf{P}_i \mathbf{N}_{i,d}(t)$$

- d is order

$$2 \leq d \leq n+1$$

Recursive definition

- Switches=base case

$$N_{i,1} = \begin{cases} 1 & t_i \le t \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

- Spline

$$\begin{aligned} N_{i,d} \quad = \quad & \left(\frac{t - t_i}{t_{i+d-1} - t_i} \right) N_{i,d-1}(t) + \\ & \left(\frac{t_{i+d} - t}{t_{i+d} - t_{i+1}} \right) N_{i+1,d-1}(t) \end{aligned}$$

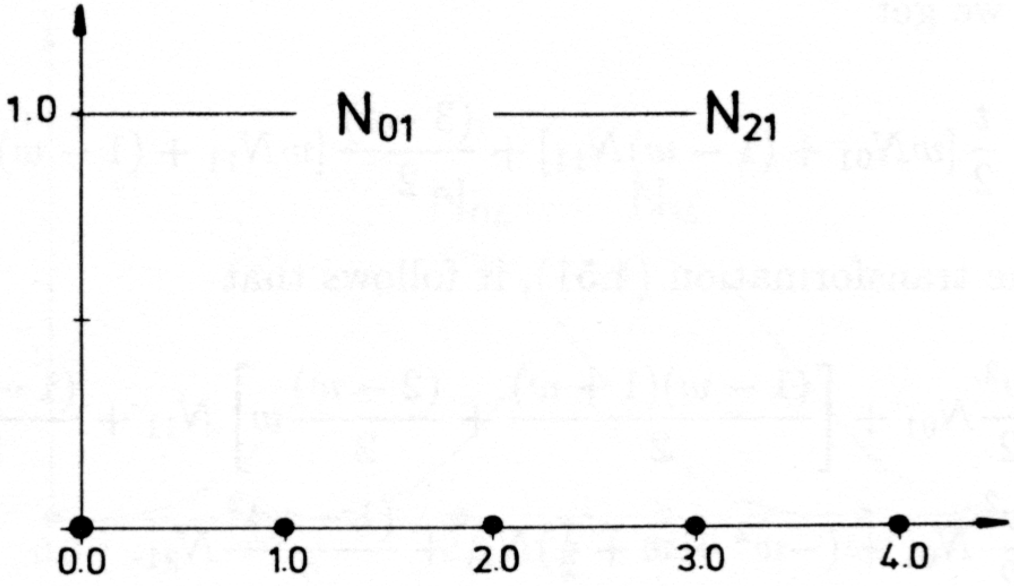


Fig. 4.22c. The B-splines N_{01}, N_{21}.

These figures show blending functions with a uniform knot vector, knots at 0, 1, 2, etc.

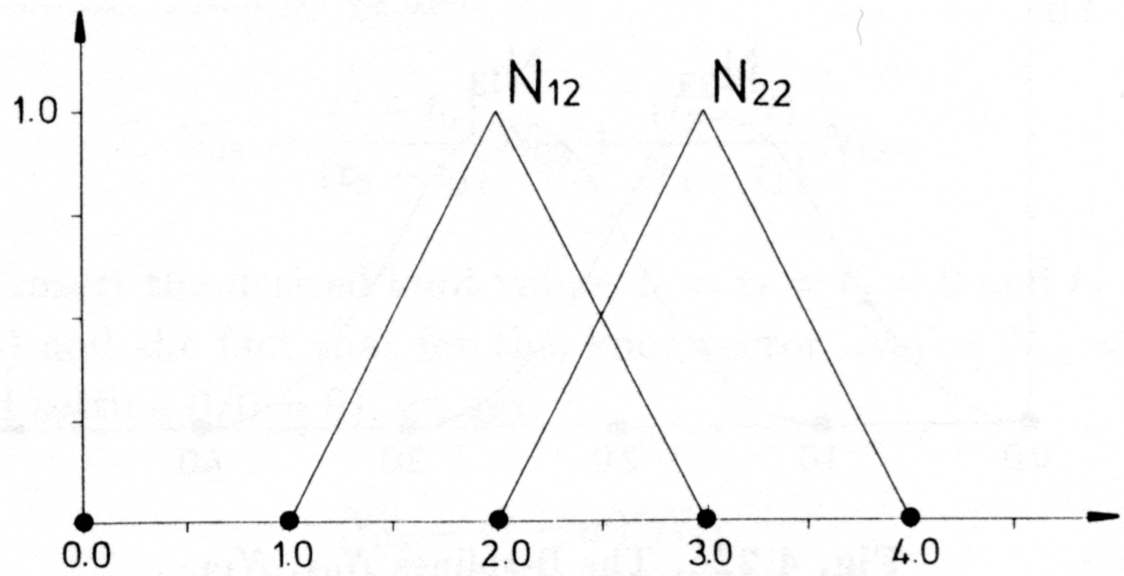


Fig. 4.22d. The B-splines N_{12}, N_{22}.

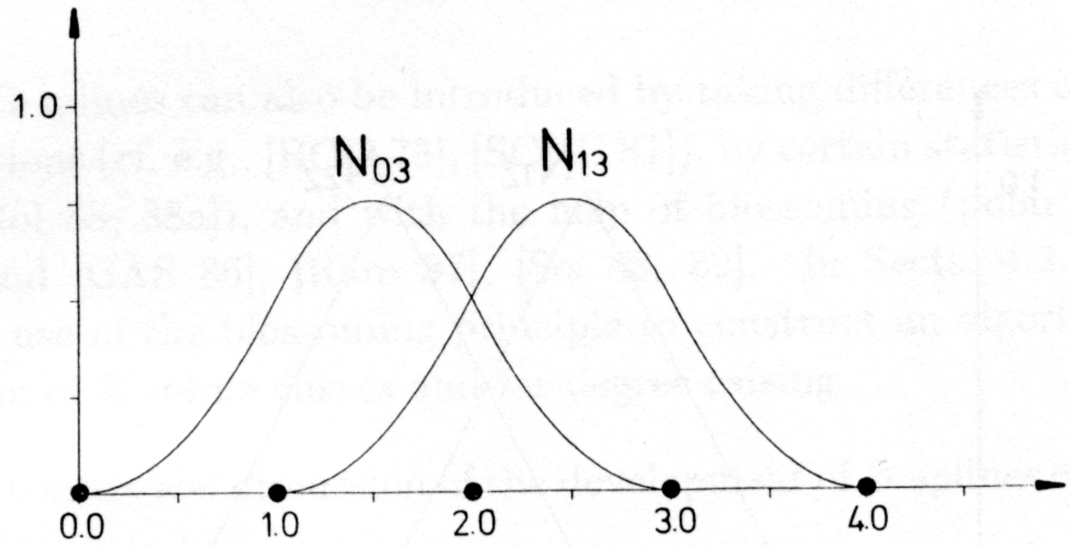
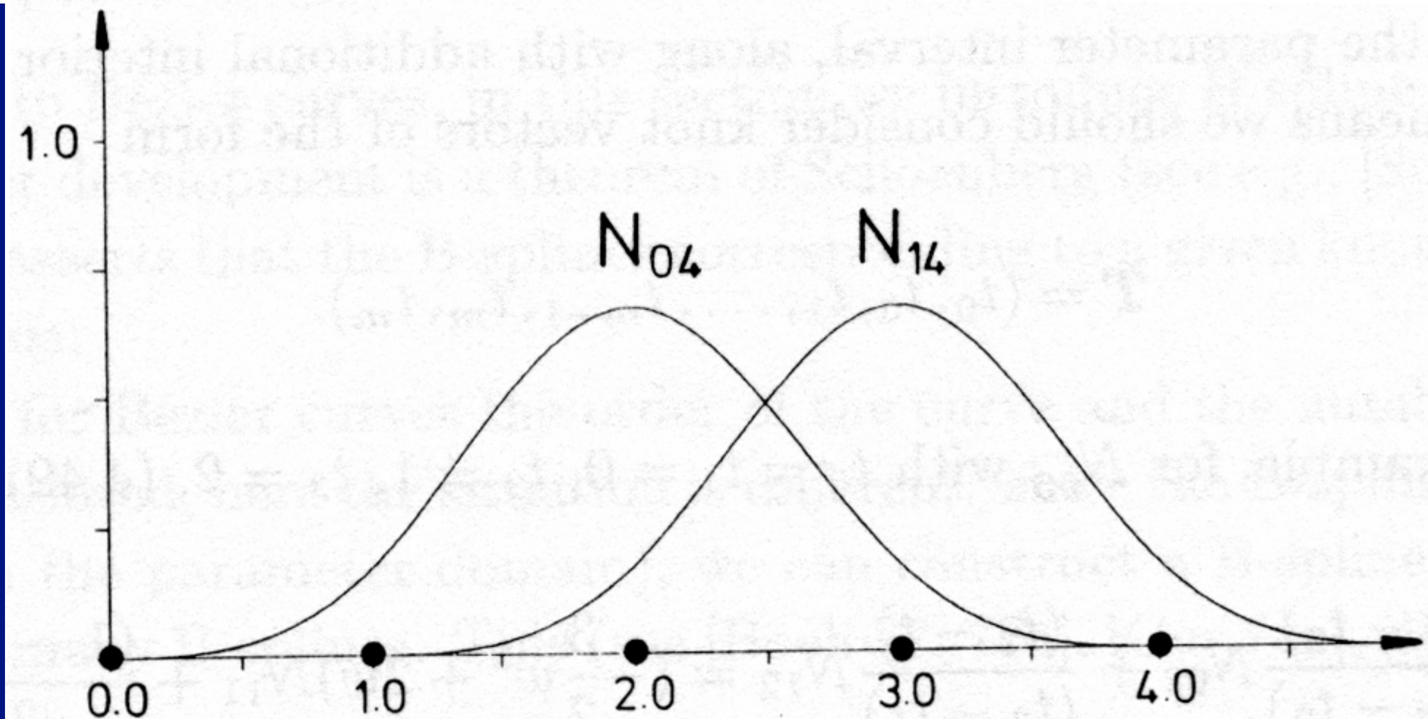


Fig. 4.22e. The B-splines N_{03}, N_{13}.



The B

Closed B-Splines

- Periodically extend the control points and the knots

$$\mathbf{P}_{n+1} = \mathbf{P}_0$$
$$t_{n+1} = t_0$$

- etc

Fig. 4.26a.

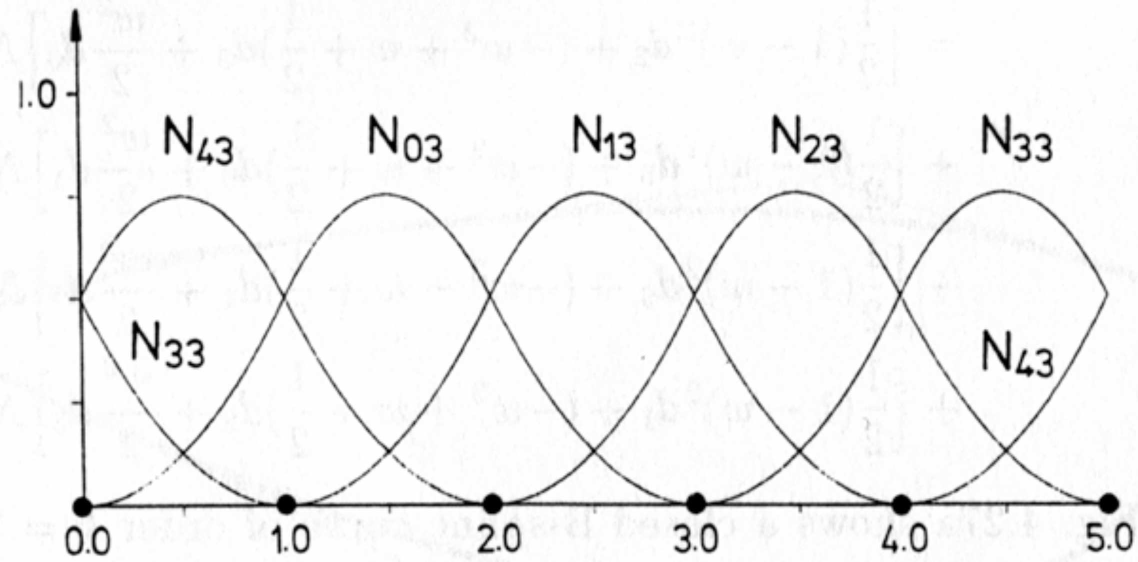


Fig. 4.26b.

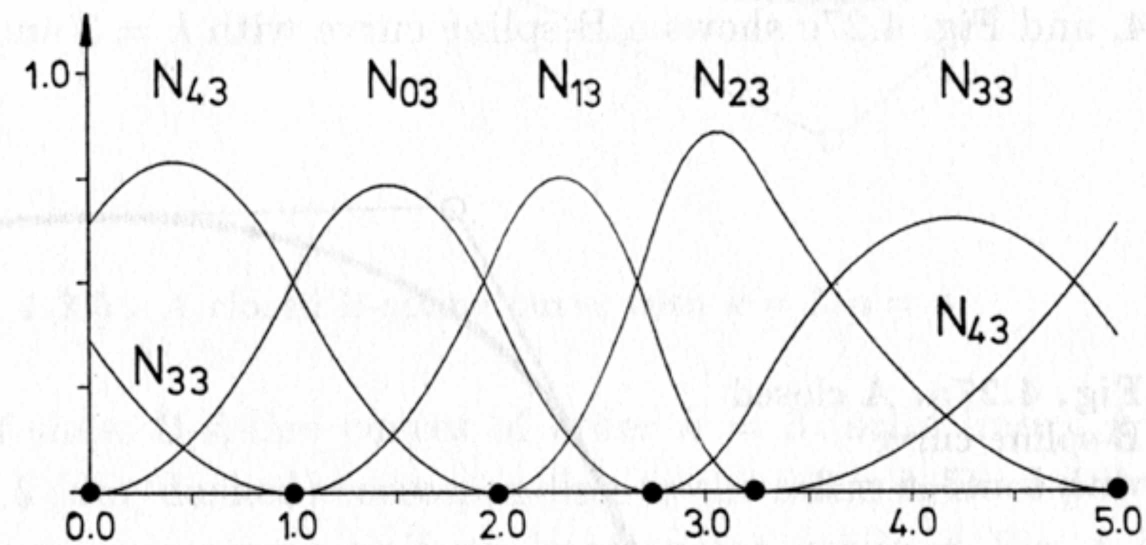


Fig. 4.26. B-splines with uniform and non-uniform knot vectors for a closed B-spline curve.

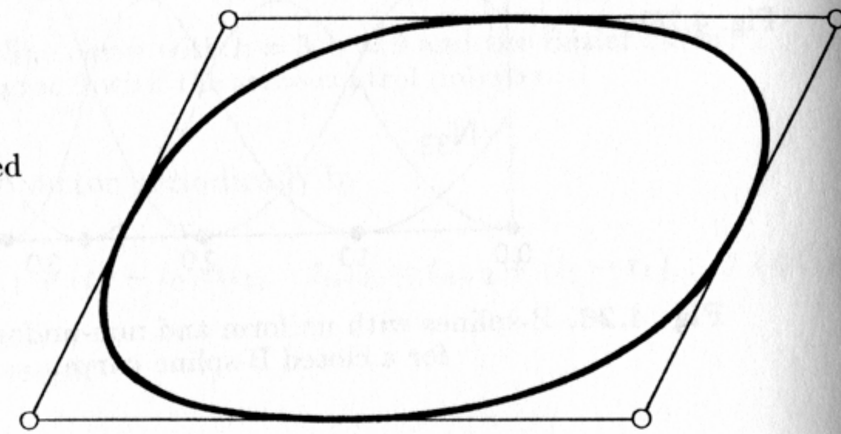


Fig. 4.27a. A closed B-spline curve with $k = 3, n = 3$.

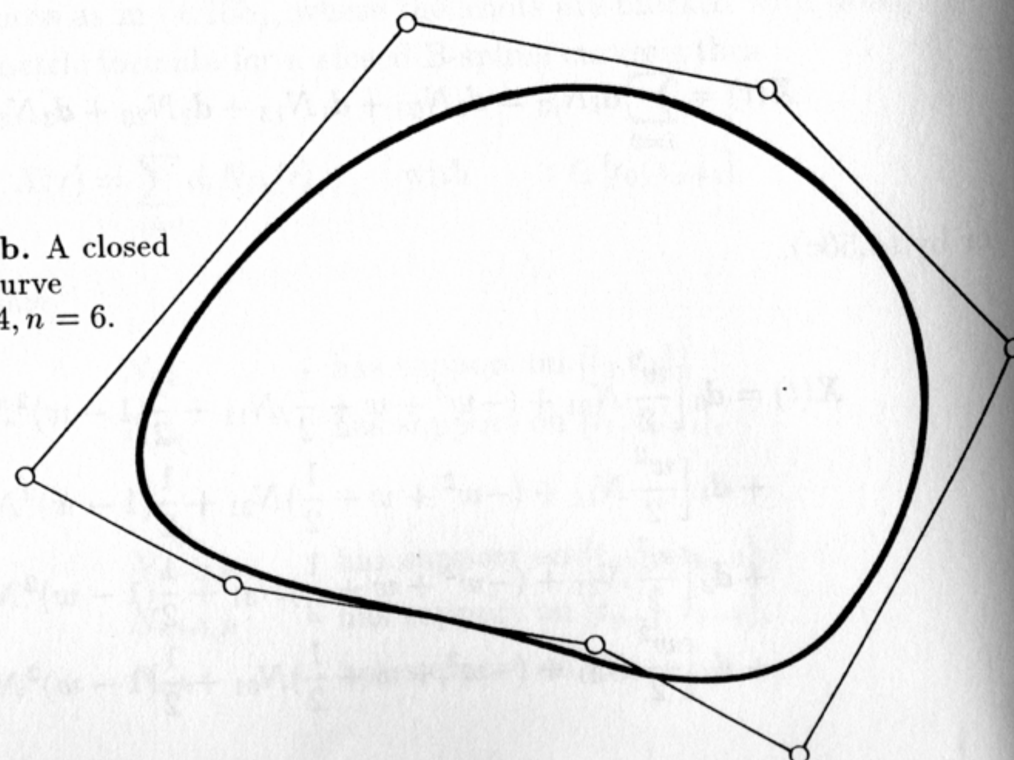


Fig. 4.27b. A closed B-spline curve with $k = 4, n = 6$.

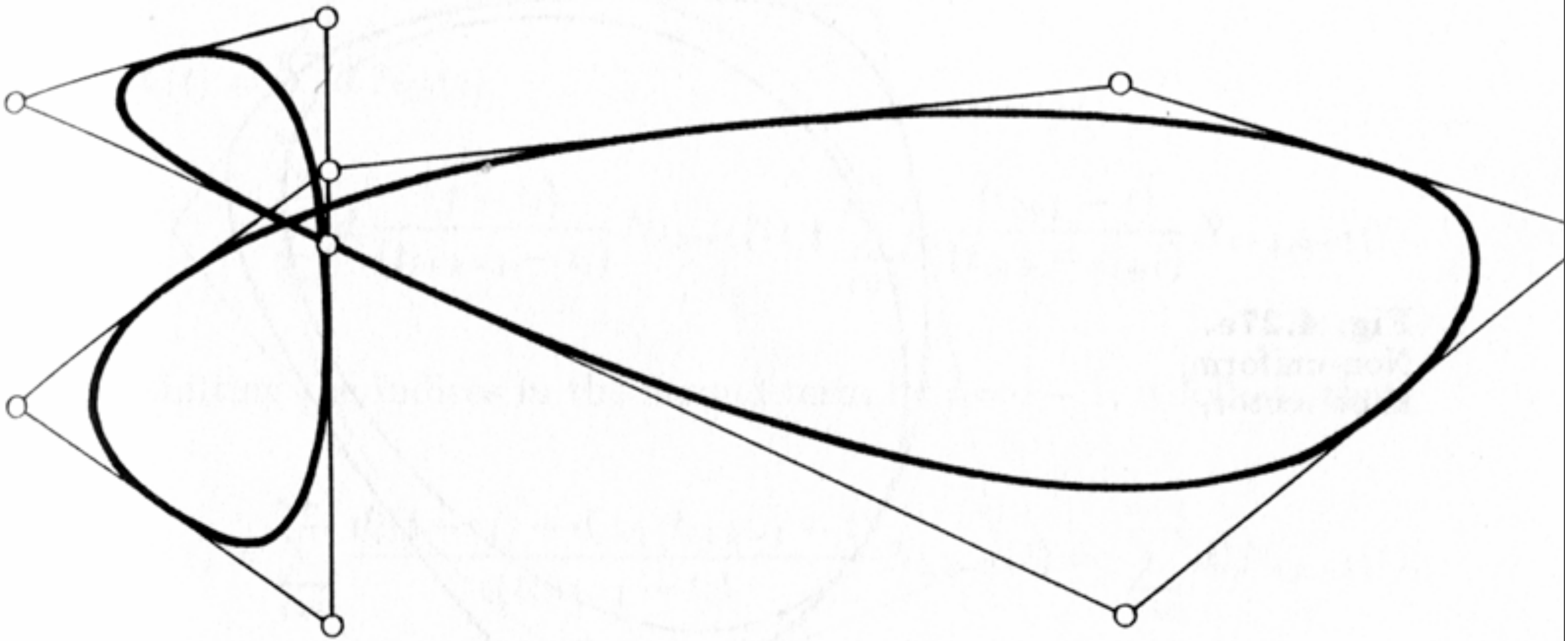
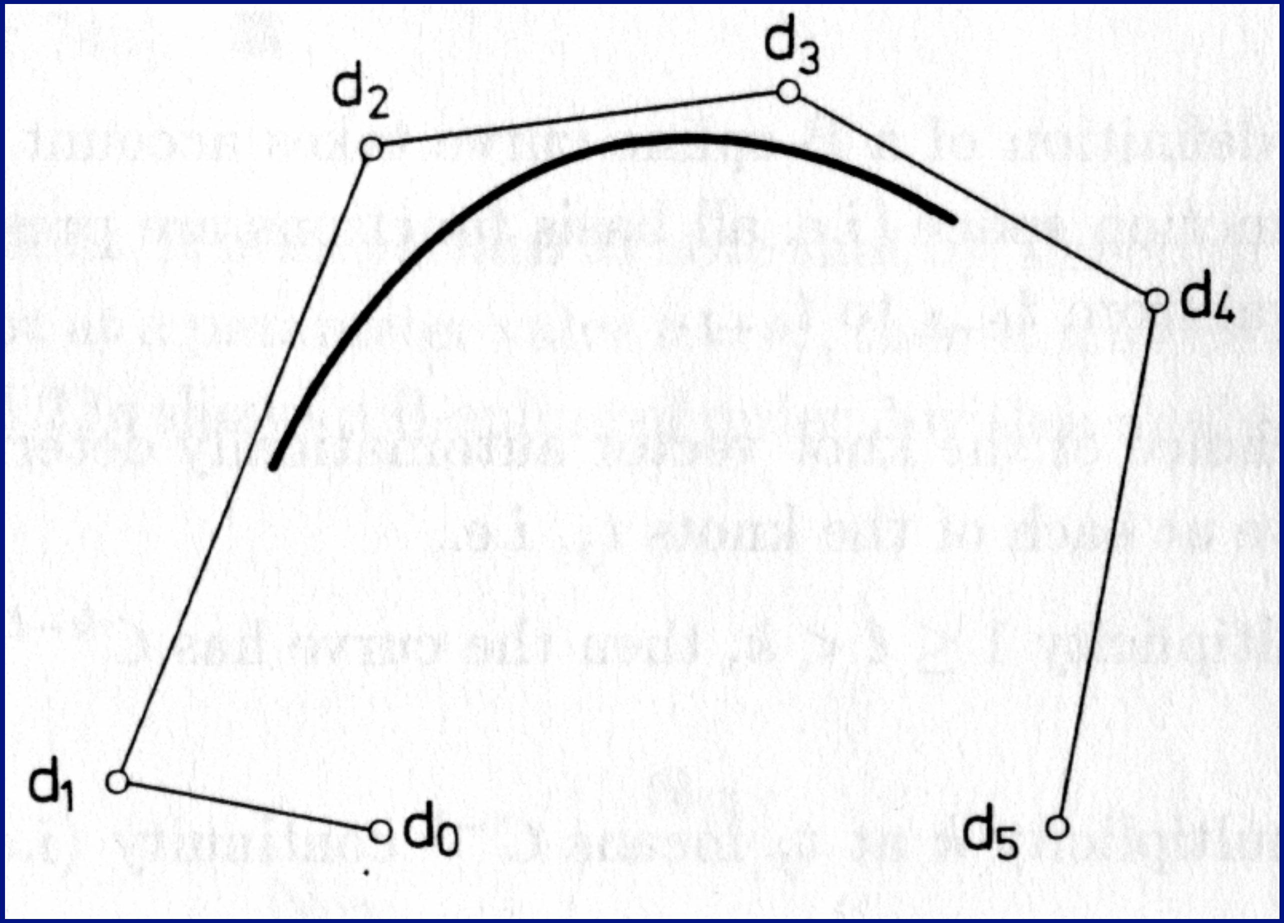


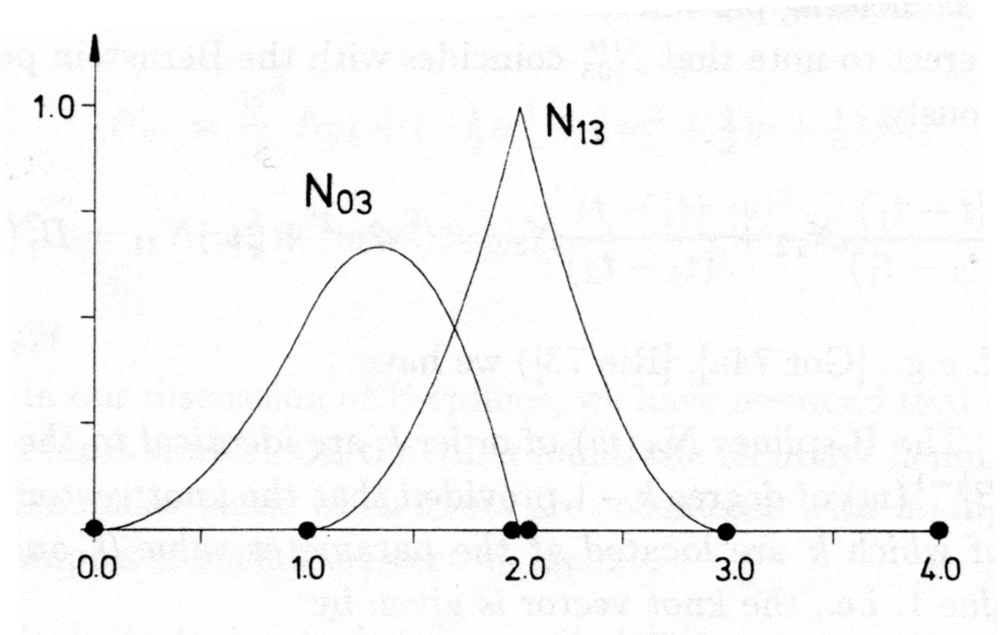
Fig. 4.27c. A closed B-spline curve with $k = 3, n = 8$.



A B-spline curve, with knots at 0,1,... and order 5

Repeated knots

- Definition works for repeated knots
  - (if we are understanding about 0/0)
- Repeated knot reduces continuity.
  - A B-spline blending function has continuity Cd-2; if the knot is repeated m times, continuity is now Cd-m-1
- e.g. -> quadratic B-spline (i.e. order 3) with a double knot



Most useful case

- select the first d and the last d knots to be the same
  - we then get the first and last points lying on the curve
  - also, the curve is tangent to the first and last segment e.g. cubic case below
- Notice that a control point influences at most d parameter intervals - local control

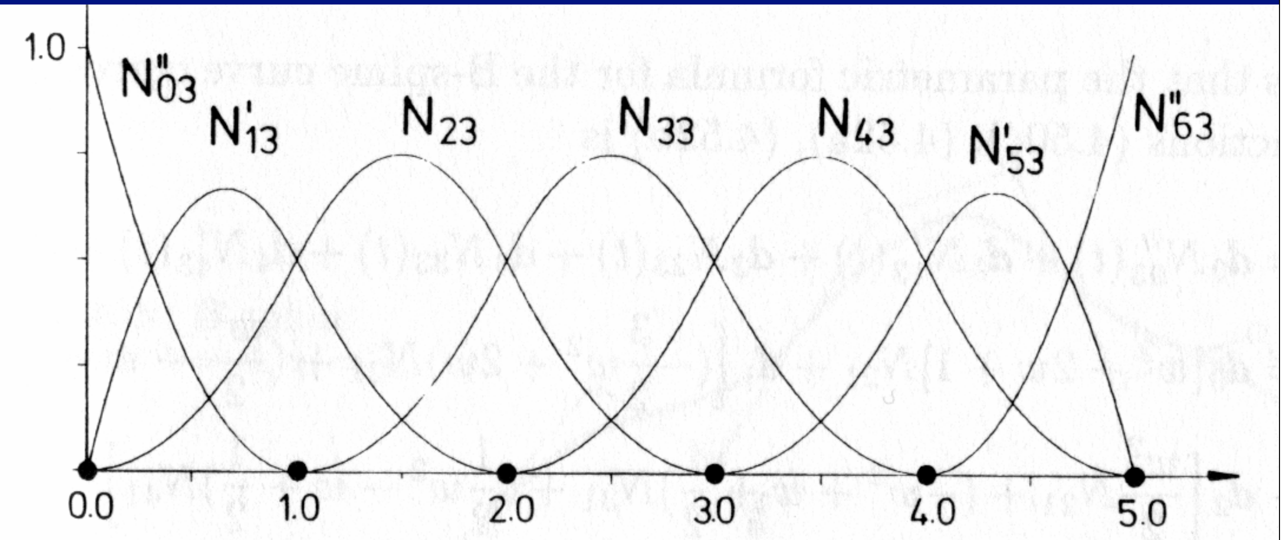


Fig. 4.24a. B-splines for an open B-spline curve with uniform knot vector.

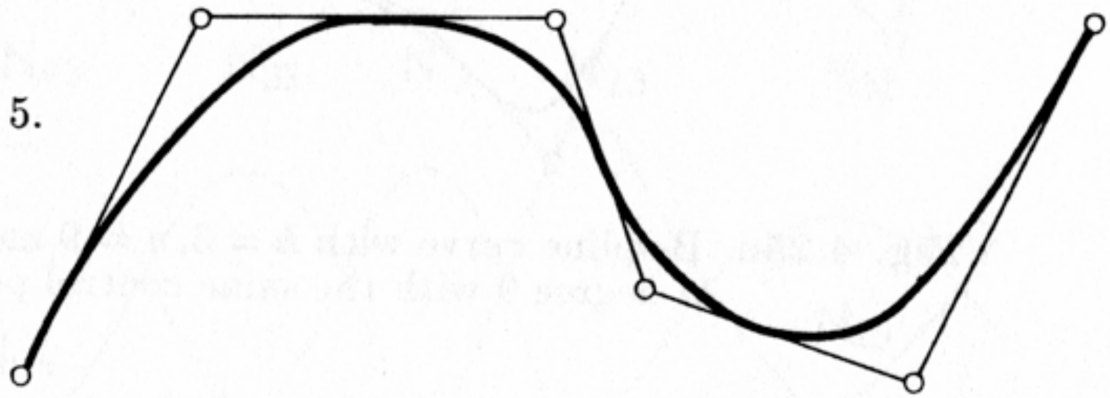


Fig. 4.25a. B-spline curve with $k = 3$, $n = 5$.

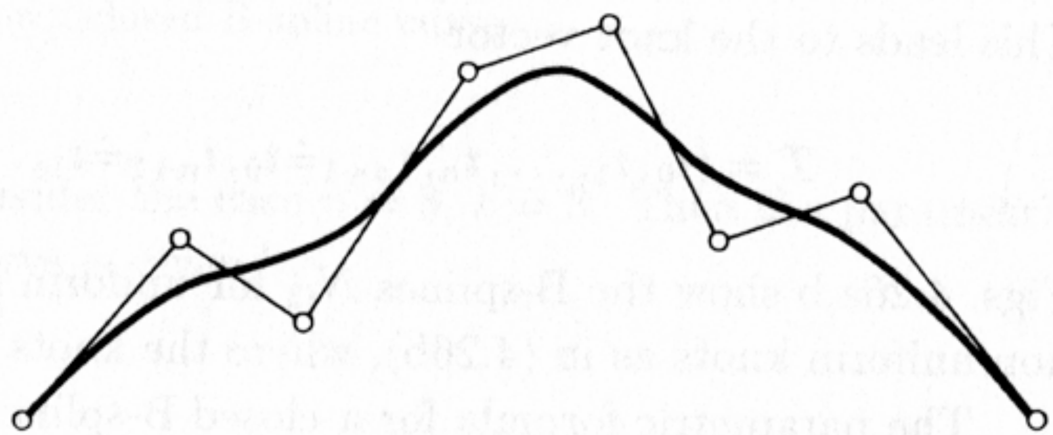


Fig. 4.25b. B-spline curve with $k = 4$, $n = 7$.

top curve has order 3, bottom order 4

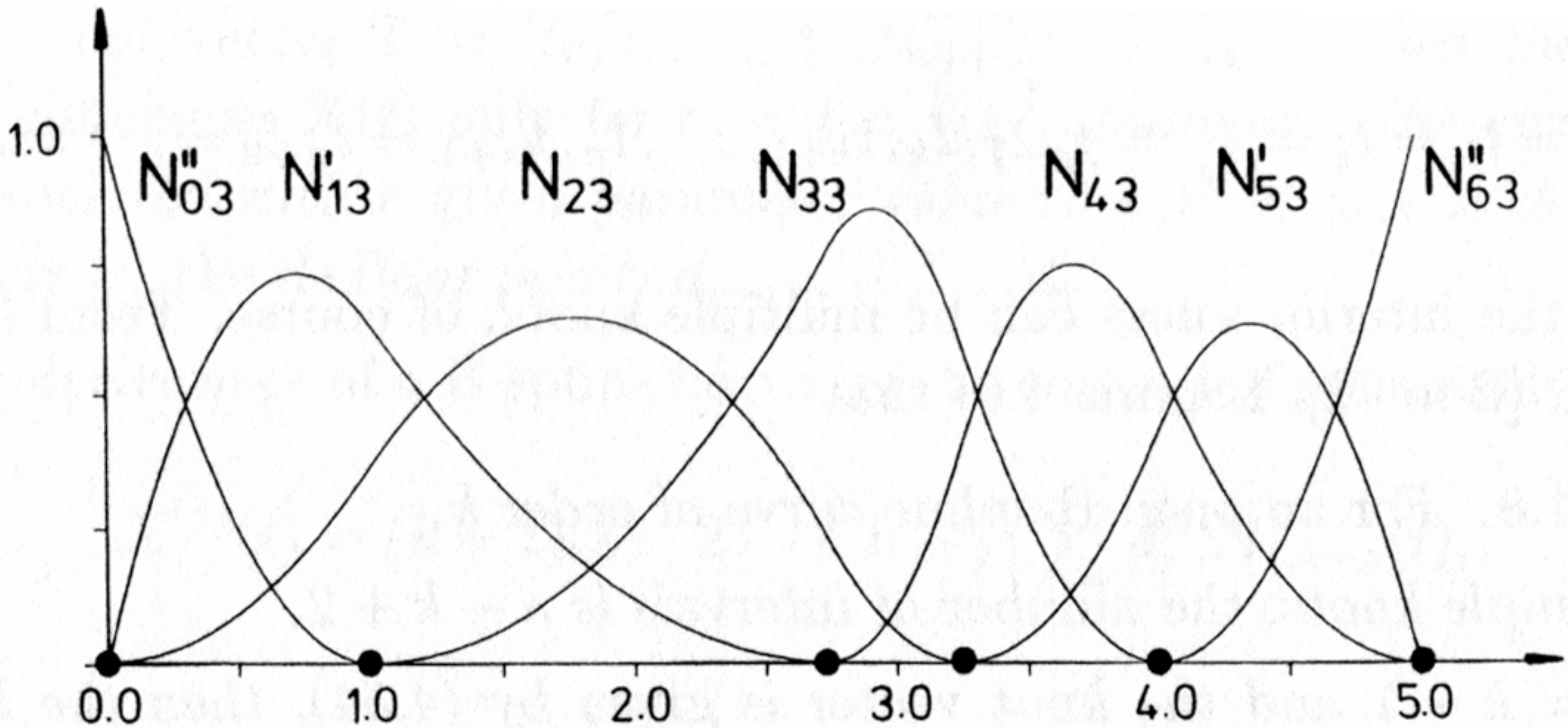


Fig. 4.24b. B-splines for an open B-spline curve with non-uniform knot vector.

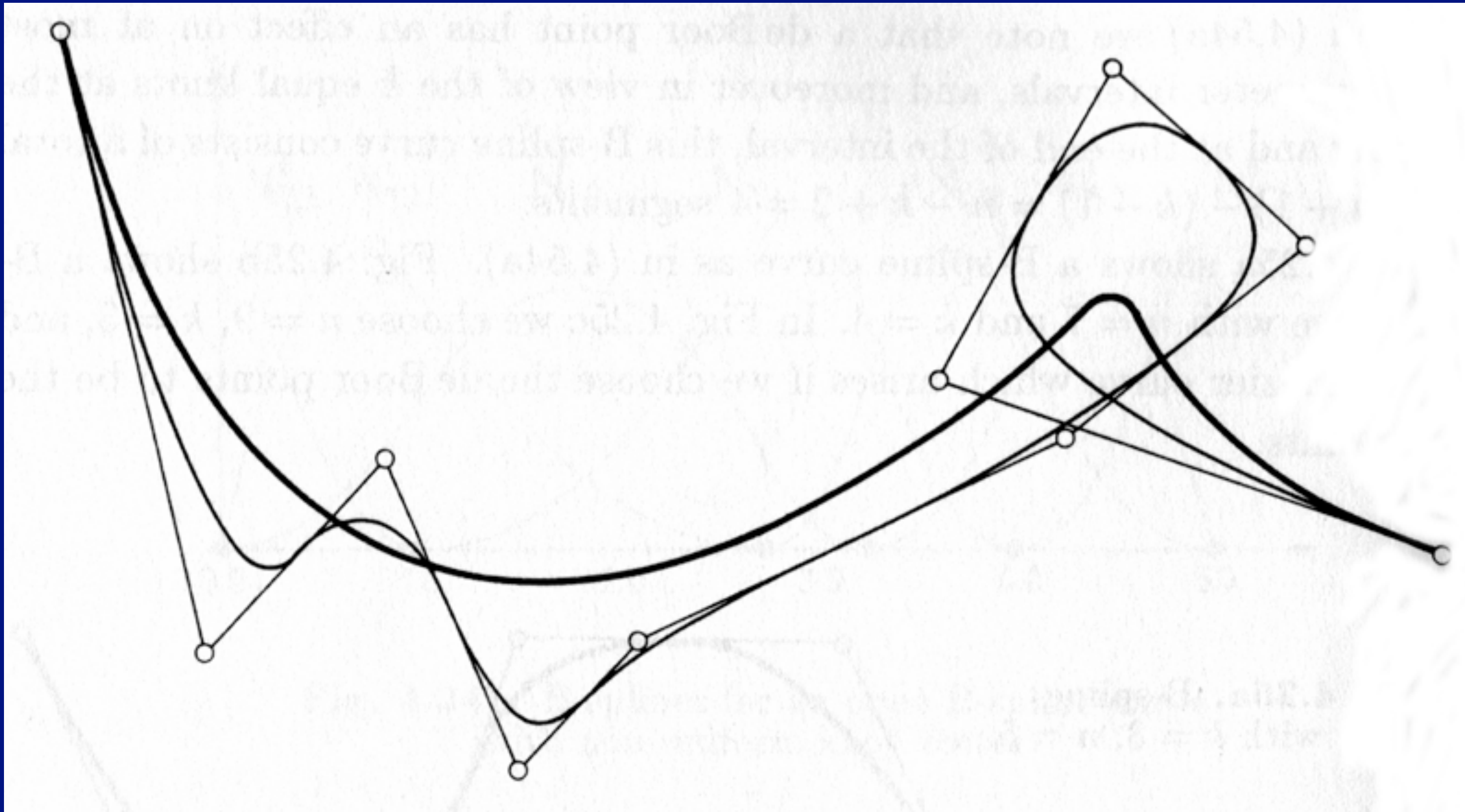


Fig. 4.25c. B-spline curve with $k = 3, n = 9$ and the Bézier curve of degree 9 with the same control polygon.

Bezier curve is the heavy curve

B-Spline properties

- For a B-spline curve of order d
  - if m knots coincide, the curve is C^{d-m-1} at the corresponding point
  - if d-1 points of the control polygon are collinear, then the curve is tangent to the polygon
  - if d points of the control polygon are collinear, then the curve and the polygon have a common segment
  - if d-1 points coincide, then the curve interpolates the common point and the two adjacent sides of the polygon are tangent to the curve
  - each segment of the curve lies in the convex hull of the associated d points

Recursive definition

- Switches=base case

$$N_{i,1} = \begin{cases} 1 & t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

- Spline

$$\begin{aligned} N_{i,d} &= \left(\frac{t - t_i}{t_{i+d-1} - t_i} \right) N_{i,d-1}(t) + \\ &\quad \left(\frac{t_{i+d} - t}{t_{i+d} - t_{i+1}} \right) N_{i+1,d-1}(t) \end{aligned}$$