Paths, diffuse interreflections, caching and radiometry

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How we got here

• We want to render diffuse interreflections
  • strategy: compute approximation $B$-hat, then gather

$$B = E + (\rho K)E + (\rho K)(\hat{B} - E)$$

Exitance

Source term

mostly zero

One or more bounces

Can change fast - shadows, etc.

Changes much more slowly, because $K$ smoothes, so we should approximate this
Gathering

- We gather radiosity from B-hat
  - Here $S$ is all the surfaces in the world

\[
(\rho K)(\hat{B} - E) = \rho(x) \int_{S} \frac{\cos \theta_i \cos \theta_s}{\pi r^2} \text{Vis}(x, u)(\hat{B}(u) - E(u))dA_s
\]

- Another integral
  - but not a good idea to integrate over $dA_s$
    - too much area, too many samples
  - instead, integrate over hemisphere
Remember Solid Angle

- By analogy with angle (in radians)
- The solid angle subtended by a patch area $dA$ is given by

$$d\omega = \frac{dA \cos \theta}{r^2}$$
Changing variables

- Rather than integrate over all area, integrate over hemisphere
  - equivalently, integrate over solid angle

\[
\cos \theta_s \frac{dA_s}{r^2} = d\omega_s
\]
Changing variables

• Start with:
  \[ \rho(x) \int_S \frac{\cos \theta_i \cos \theta_s}{\pi r^2} \text{Vis}(x, u)(\hat{B}(u) - E(u))dA_s \]

• Substitute:
  \[ \frac{\cos \theta_s}{r^2}dA_s = d\omega_s \]

• Get:
  \[ \rho(x) \frac{1}{\pi} \int_{\Omega} \cos \theta_i (\hat{B}(\omega) - E(\omega))d\omega \]

  Value at far end of ray through angle

  Incoming hemisphere
Evaluating integral

- **Procedure**
  - Generate N uniform random samples on hemisphere
    - procedure described on whiteboard
  - Find B-hat-E at far end of each ray
  - Average

- **How big should N be?**
  - **Variance**
    - estimate is a random variable, so must have variance
    - small N implies high variance, fast
    - large N implies low variance, slow
  - Variance will look like noise
    - but should be small, because the term is small
    - suggests small N is OK
Gathering from $B$-hat - $E$

\[ B = E + (\rho \mathcal{K}) E + (\rho \mathcal{K})(\hat{B} - E) \]
Alternative: B-hat via random paths

- Notice that B-hat is also an integral approximation to B
- Now from $B = E + (\rho K)B$
  - we expect $\hat{B} = E + (\rho K)\hat{B}$
  - so $\hat{B} - E = (\rho K)\hat{B}$
- expand by substituting to get $\hat{B} = E + (\rho K)(E + (\rho K)\hat{B})$
- ie $\hat{B} - E = (\rho K)(E + (\rho K)\hat{B})$
- substitute from above to get $\hat{B} - E = (\rho K)E + (\rho K)(\hat{B} - E)$
Alternative

- We could evaluate B-hat - E recursively

\[ \hat{B} - E = (\rho K)E + (\rho K)(\hat{B} - E) \]

\[ \begin{array}{c|c}
\text{Direct term} & \text{Indirect term} \\
\end{array} \]
Recursive evaluation

\[ \text{shade}(x) = E(x) + \rho(x)\text{direct}(x) + \text{RKBME}(x) \]
Recursive evaluation: direct term

\[ \text{direct}(x) = \sum_{l \in \text{luminaires}} \text{directfromL}(x, l) \]

\[ \text{directfromL}(x, L) \]

generate \( N \) uniform random samples \( u_i \) on luminaire \( L \) with area \( A_l \)

return \( \frac{A_l}{N} \sum_i \frac{\cos \theta_x \cos \theta_u}{\pi r^2} E(u_i) \)

We did this when we discussed area luminaires - no big mystery here
Recursive evaluation: Indirect term

This form isn’t yet practical, because the recursion is infinite!

RKBME(x)

Generate M points \( p_i \) uniformly at random on unit hemisphere at \( x \)
For each point \( p_i \), write \( u_i \) for the first hit on the ray from \( x \) to \( p_i \)
write \( \cos \theta_{si} \) for the cosine at \( x \) of the \( i \)'th direction

return \( \rho(x)2\pi \frac{1}{\pi} \frac{1}{M} \sum_i (\rho(u_i)\text{direct}(u_i) + \text{RKBME}(u_i)) \cos \theta_{si} \)
B-hat via random paths becomes a tree
B-hat via random paths becomes a tree

And so on...
Recursive evaluation: Indirect term

Recursion no longer infinite, but estimate must be (very slightly) too small

\[ \text{RKBME}(x, \text{depth}) \]

Generate \( M \) points \( p_i \) uniformly at random on unit hemisphere at \( x \)
For each point \( p_i \), write \( u_i \) for the first hit on the ray from \( x \) to \( p_i \)
write \( \cos \theta_{si} \) for the cosine at \( x \) of the \( i \)'th direction
if depth==0
    return 0
else
    return \( \rho(x)^2 \frac{1}{\pi} \frac{1}{M} \sum_i (\rho(u_i) \text{direct}(u_i) + \text{RKBME}(u_i, \text{depth} - 1)) \cos \theta_{si} \)
Recursive evaluation: Indirect term

Recursion no longer infinite, not as deep as previous, but estimate must still be (very slightly) too small

\[ \text{RKBME}(x, \rho_{acc}) \]

Generate \( M \) points \( p_i \) uniformly at random on unit hemisphere at \( x \)
For each point \( p_i \), write \( u_i \) for the first hit on the ray from \( x \) to \( p_i \)
write \( \cos \theta_{si} \) for the cosine at \( x \) of the \( i \)'th direction
if \( \rho_{acc} < \text{smallthresh} \)
return 0
else
return \( \rho(x)2\pi \frac{1}{\pi} \frac{1}{M} \sum_i (\rho(u_i)\text{direct}(u_i) + \text{RKBME}(u_i, \rho(x) \ast \rho_{acc})) \cos \theta_{si} \)
Russian roulette

- Consider a random process:
  - with probability p, return S
  - with probability 1-p, return 0

- Expected value:
  - p*S

- We can use this to prune paths at random, mainly pruning when albedo is low
Russian roulette

Notice what’s happened to the albedo term. When a path gets to low albedo surface, it has little chance of continuing. This is unbiased!

\[
\text{RKBME}(x)
\]

Generate \( v \) uniform random variable, \( v \in [0, 1] \)

if \( v > \rho(x) \)
return 0
else

Generate \( M \) points \( p_i \) uniformly at random on unit hemisphere at \( x \)
For each point \( p_i \), write \( u_i \) for the first hit on the ray from \( x \) to \( p_i \)
write \( \cos \theta_{si} \) for the cosine at \( x \) of the \( i \)'th direction

return \( 2\pi \frac{1}{\pi} \frac{1}{M} \sum_i (\rho(u_i)\text{direct}(u_i) + \text{RKBME}(u_i)) \cos \theta_{si} \)
Light path analysis

- We’ve now done LD*E
  - Russian roulette cleverly explores paths; if there’s lots of albedo, paths tend to be long; else short.
  - Russian roulette is a random process
    - Random choice of directions; random choice to prune
    - Unbiased
      - Expected value is the right answer
  - Variance
    - Because it’s random
    - Looks like image noise
    - Seen this before in lenses, motion blur
  - Control by
    - More rays (!)
    - Caching
    - Importance sampling (later)
Caching

Imagine we’ve hit this point before; why expand?

And so on...
Caching

RKBME(x)

Generate $v$ uniform random variable, $v \in [0, 1]$

if $v > \rho(x)$
return 0
else

Interrogate cache - do we have an RKBME value close to $x$?
if yes
return cache value
else

Generate $M$ points $p_i$ uniformly at random on unit hemisphere at $x$
For each point $p_i$, write $u_i$ for the first hit on the ray from $x$ to $p_i$
write $\cos \theta_{si}$ for the cosine at $x$ of the $i$’th direction

return $2\pi \frac{1}{\pi} \frac{1}{M} \sum_i (\rho(u_i) \text{direct}(u_i) + \text{RKBME}(u_i)) \cos \theta_{si}$
Irradiance cache vs path tracing, from Pharr + Humphreys, for the same amount of cpu
Light path analysis

• Main strategy
  • build and evaluate light paths
• We can do other kinds of path like this, too
  • requires extra radiometry
Ray tracing
add soft shadows
global illumination
Radiometry

• Questions:
  • how “bright” will surfaces be?
  • what is “brightness”?  
    • measuring light
    • interactions between light and surfaces

• Core idea - think about light arriving at a surface
• around any point is a hemisphere of directions
• Simplest problems can be dealt with by reasoning about this hemisphere
Lambert’s wall

- Overcast sky
- Infinitely high wall
- Infinite plane
More complex wall

- Overcast sky
- Infinitely high wall
- Infinite plane
- p
More complex wall
Solid Angle

- By analogy with angle (in radians)
- The solid angle subtended by a patch area \(dA\) is given by

\[
d\omega = \frac{dA \cos \theta}{r^2}
\]

- Another useful expression:

\[
d\omega = \sin \theta (d\theta)(d\phi)
\]
Radiance

- Measure the “amount of light” at a point, in a direction
- Property is: **Radiant power per unit foreshortened area per unit solid angle**
- Units: watts per square meter per steradian (\(\text{wm}^{-2}\text{sr}^{-1}\))
- Usually written as:

\[ L(x, \theta, \varphi) \]

- Crucial property: In a vacuum, radiance leaving \(p\) in the direction of \(q\) is the same as radiance arriving at \(q\) from \(p\) – hence the units
Radiance is constant along straight lines

- Power 1->2, leaving 1:
  \[ L(x_1, \theta, \varphi)(dA_1 \cos \theta_1) \left( \frac{dA_2 \cos \theta_2}{r^2} \right) \]

- Power 1->2, arriving at 2:
  \[ L(x_2, \theta, \varphi)(dA_2 \cos \theta_2) \left( \frac{dA_1 \cos \theta_1}{r^2} \right) \]
Irradiance

- How much light is arriving at a surface?
- Sensible unit is Irradiance $L(x, \theta, \phi) \cos \theta d\omega$
- Incident power per unit area not foreshortened
- This is a function of incoming angle.
- A surface experiencing radiance $L(x, \theta, \phi)$ coming in from $d\omega$ experiences irradiance

$$\int \limits_{\Omega} L(x, \theta, \phi) \cos \theta \sin \theta d\theta d\phi$$

- Crucial property:
  Total power arriving at the surface is given by adding irradiance over all incoming angles --- this is why it’s a natural unit
Surfaces and the BRDF

- Many effects when light strikes a surface -- could be:
  - absorbed; transmitted. reflected; scattered
- Assume that
  - surfaces don’t fluoresce
  - surfaces don’t emit light (i.e. are cool)
  - all the light leaving a point is due to that arriving at that point
- Can model this situation with the Bidirectional Reflectance Distribution Function (BRDF)
- the ratio of the radiance in the outgoing direction to the incident irradiance

\[ \rho_{bd}(x, \theta_o, \varphi_o, \theta_i, \varphi_i) = \frac{L_o(x, \theta_o, \varphi_o)}{L_i(x, \theta_i, \varphi_i) \cos \theta_i d\omega} \]
BRDF

- Units: inverse steradians (sr⁻¹)
- Symmetric in incoming and outgoing directions
- Radiance leaving in a particular direction:
  - add contributions from every incoming direction

\[
\int_{\Omega} \rho_{bd}(x, \vartheta_o, \varphi_o, \vartheta_i, \varphi_i) L_i(x, \vartheta_i, \varphi_i) \cos \vartheta_i \, d\omega_i
\]
Suppressing Angles - Radiosity

- In many situations, we do not really need angle coordinates
  - e.g. cotton cloth, where the reflected light is not dependent on angle
- Appropriate radiometric unit is radiosity
  - total power leaving a point on the surface, per unit area on the surface (Wm\(^{-2}\))
- Radiosity from radiance?
  - sum radiance leaving surface over all exit directions

\[ B(\mathbf{x}) = \int_{\Omega} L_o(\mathbf{x}, \vartheta, \varphi) \cos \vartheta d\omega \]
Radiosity

- Important relationship:
  - radiosity of a surface whose radiance is independent of angle (e.g. that cotton cloth)

\[
B(x) = \int_{\Omega} L_o(x, \vartheta, \varphi) \cos \vartheta d\omega \\
= L_o(x) \int_{\Omega} \cos \vartheta d\omega \\
= L_o(x) \int_{\vartheta / 2\pi}^{\pi / 2\pi} \int_{0}^{0} \cos \vartheta \sin \vartheta d\varphi d\theta \\
= \pi L_o(x)
\]
Directional hemispheric reflectance

- BRDF is a very general notion
  - some surfaces need it (underside of a CD; tiger eye; etc)
  - very hard to measure and very unstable
  - for many surfaces, light leaving the surface is largely independent of exit angle (surface roughness is one source of this property)

- Directional hemispheric reflectance:
  - the fraction of the incident irradiance in a given direction that is reflected by the surface (whatever the direction of reflection)
  - unitless, range 0-1

\[
\rho_{dh}(\theta_i, \phi_i) = \int_{\Omega} \frac{L_o(x, \theta_o, \phi_o) \cos \theta_o d\omega_o}{L_i(x, \theta_i, \phi_i) \cos \theta_i d\omega_i} = \int_{\Omega} \rho_{bd}(x, \theta_o, \phi_o, \theta_i, \phi_i) \cos \theta_o d\omega_o
\]
Lambertian surfaces and albedo

• For some surfaces, the DHR is independent of direction
  • cotton cloth, carpets, matte paper, matte paints, etc.
  • radiance leaving the surface is independent of angle
  • Lambertian surfaces (same Lambert) or ideal diffuse surfaces
  • Use radiosity as a unit to describe light leaving the surface
  • DHR is often called diffuse reflectance, or albedo

• for a Lambertian surface, BRDF is independent of angle, too.

• Useful fact:

\[ \rho_{brdf} = \frac{\rho_d}{\pi} \]
Specular surfaces

- Another important class of surfaces is specular, or mirror-like.
  - radiation arriving along a direction leaves along the specular direction
  - reflect about normal
  - some fraction is absorbed, some reflected
  - on real surfaces, energy usually goes into a lobe of directions
  - can write a BRDF, but requires the use of funny functions
Phong’s model

- There are very few cases where the exact shape of the specular lobe matters.
- Typically:
  - very, very small --- mirror
  - small -- blurry mirror
  - bigger -- see only light sources as “specularities”
  - very big -- faint specularities
- Phong’s model
  - reflected energy falls off with

$$\cos^n(\delta \Theta)$$
Lambertian + specular

- **Widespread model**
  - all surfaces are Lambertian plus specular component

- **Advantages**
  - easy to manipulate
  - very often quite close true

- **Disadvantages**
  - some surfaces are not
    - e.g. underside of CD’s, feathers of many birds, blue spots on many marine crustaceans and fish, most rough surfaces, oil films (skin!), wet surfaces
  - Generally, very little advantage in modelling behaviour of light at a surface in more detail -- it is quite difficult to understand behaviour of L+S surfaces
Area sources

- Examples: diffuser boxes, white walls.
- The radiosity at a point due to an area source is obtained by adding up the contribution over the section of view hemisphere subtended by the source.
  - change variables and add up over the source
Radiosity due to an area source

- \( \rho \) is albedo
- \( E \) is exitance
- \( r(x, u) \) is distance between points
- \( u \) is a coordinate on the source

\[
B(x) = \rho_d(x) \int_{\Omega} L_t(x, u \rightarrow x) \cos \theta_i \, d\omega
\]

\[
= \rho_d(x) \int_{\Omega} L_e(x, u \rightarrow x) \cos \theta_i \, d\omega
\]

\[
= \rho_d(x) \int_{\Omega} \left( \frac{E(u)}{\pi} \right) \cos \theta_i \, d\omega
\]

\[
= \rho_d(x) \int_{\text{source}} \left( \frac{E(u)}{\pi} \right) \cos \theta_i \left( \frac{\cos \theta_s}{r(x, u)^2} \frac{dA_u}{r(x, u)^2} \right)
\]

\[
= \rho_d(x) \int_{\text{source}} E(u) \frac{\cos \theta_i \cos \theta_s}{\pi r(x, u)^2} \, dA_u
\]