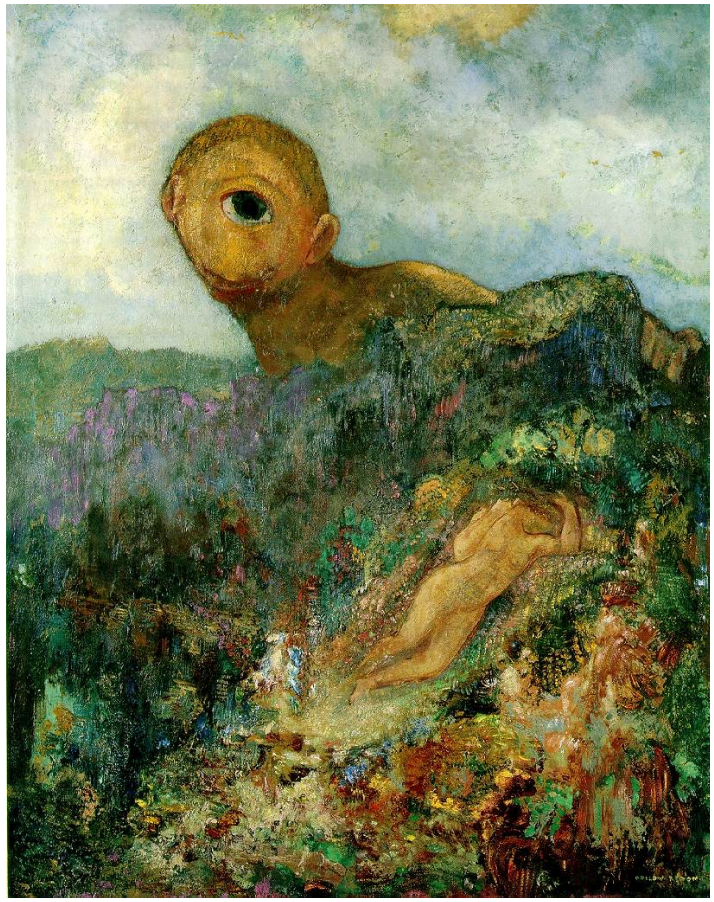
Camera calibration

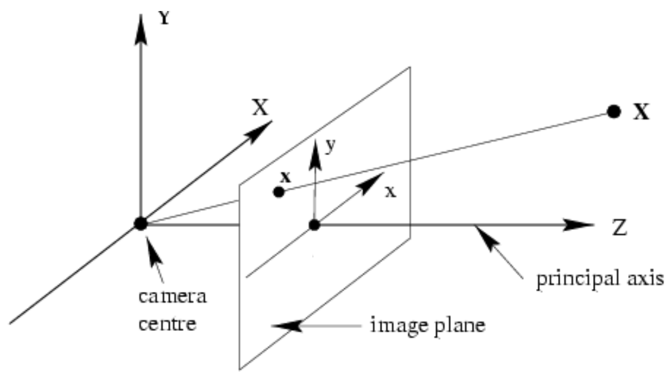


Odilon Redon, *Cyclops*, 1914

Overview

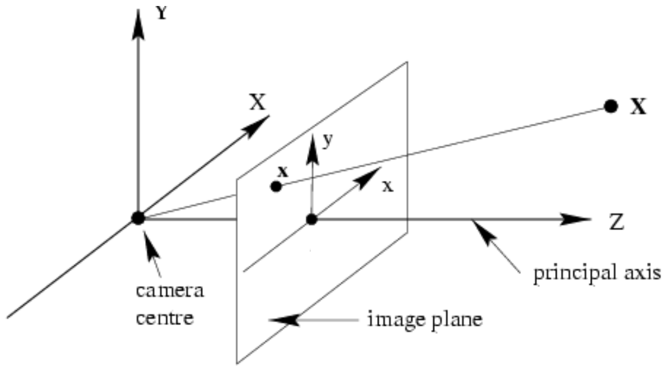
- Camera calibration
  - Intrinsic camera parameters
  - Extrinsic camera parameters
  - Estimation
- First taste of 3D reconstruction: triangulation

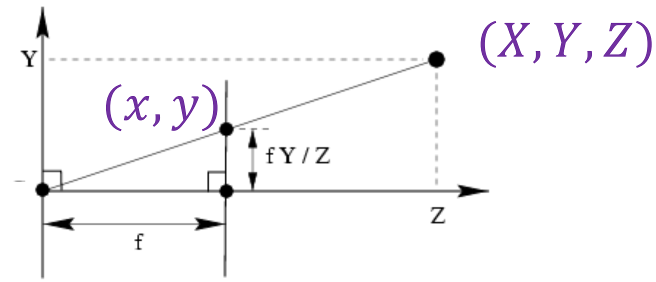
Perspective projection in normalized coordinates



- **Normalized (camera) coordinate system:** camera center is at the origin, the *principal axis* is the z-axis, x and y axes of the image plane are parallel to x and y axes of the world

Perspective projection in normalized coordinates





$$x = f\frac{X}{Z}, y = f\frac{Y}{Z}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$\boldsymbol{x}$ Homogeneous coord. vec. of image point

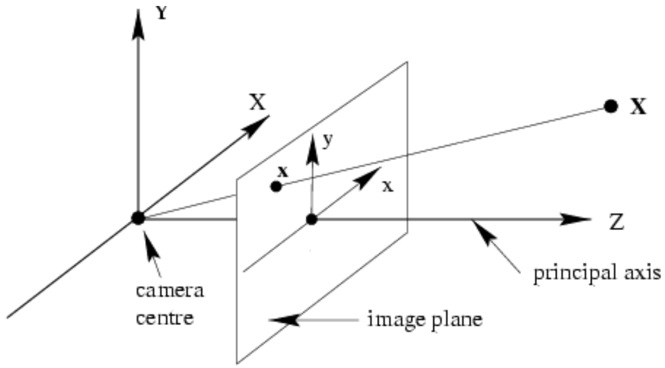
$\boldsymbol{P}$ Camera projection matrix

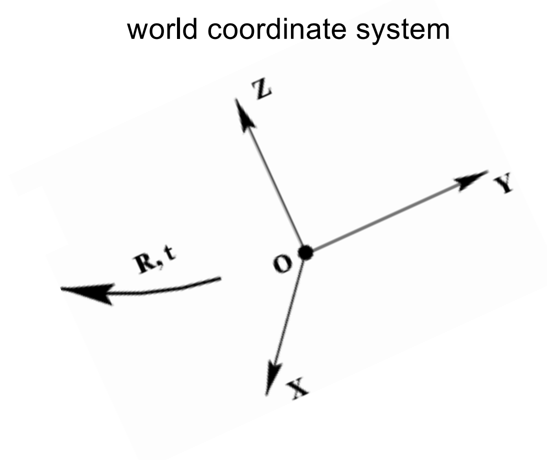
$\boldsymbol{X}$ Homogeneous coord. vec. of 3D point

$$\boldsymbol{x} \cong \boldsymbol{P}\boldsymbol{X}$$

Equality up to scale

Camera calibration: Overview





- **Camera calibration:** figuring out transformation from *world* coordinate system to *image* coordinate system

$$\begin{pmatrix}\text{2D}\\ \text{point}\\ \text{(3x1)}\end{pmatrix} \cong \begin{pmatrix}\text{Camera to}\\ \text{pixel coord.}\\ \text{trans. matrix}\\ \boldsymbol{K}\ \text{(3x3)}\end{pmatrix}\begin{pmatrix}\textit{Canonical}\\ \text{projection matrix}\\ [\boldsymbol{I}\mid\boldsymbol{0}]\ \text{(3x4)}\end{pmatrix}\begin{pmatrix}\text{World to}\\ \text{camera coord.}\\ \text{trans. matrix}\\ \begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0}^T & 1\end{bmatrix}\ \text{(4x4)}\end{pmatrix}\begin{pmatrix}\text{3D}\\ \text{point}\\ \text{(4x1)}\end{pmatrix}$$

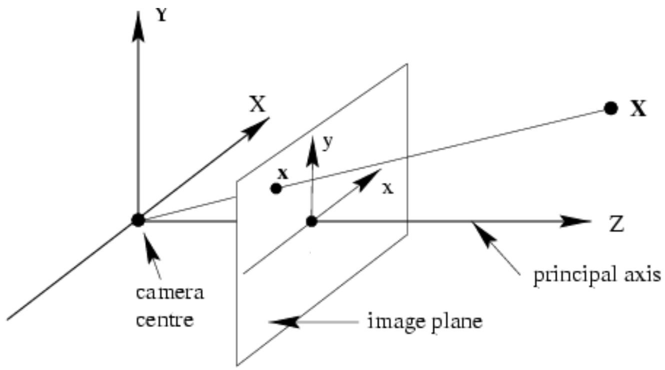
$\boldsymbol{x}$

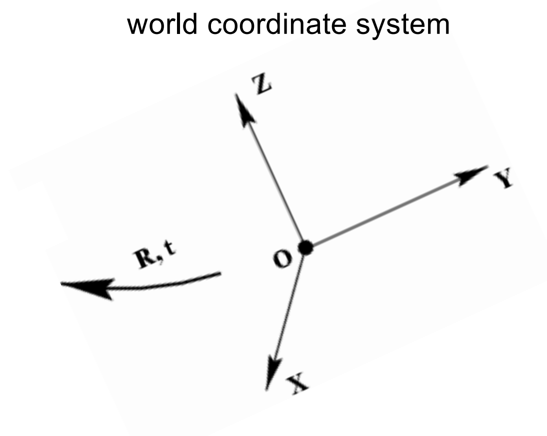
***Intrinsic* camera parameters**: principal point, scaling factors

***Extrinsic* camera parameters**: rotation, translation

$\boldsymbol{X}$

Camera calibration: Overview



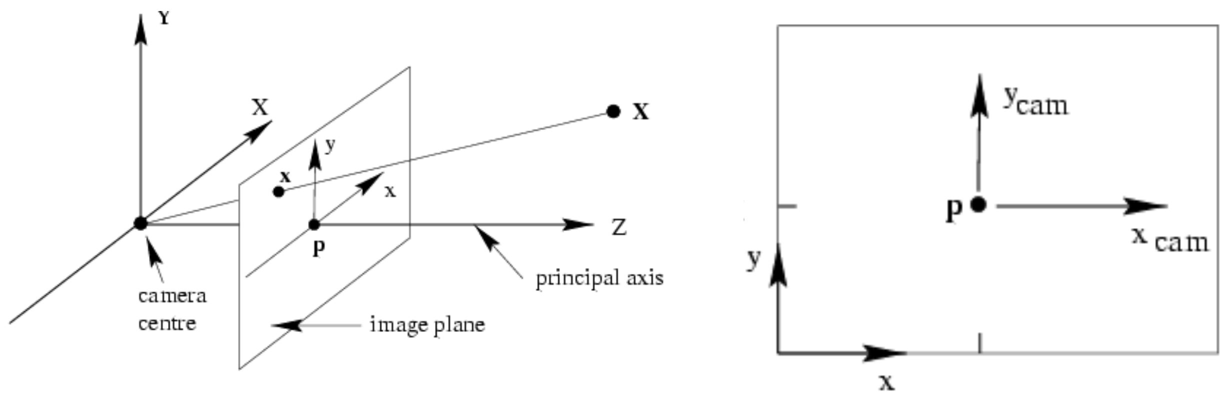


- **Camera calibration:** figuring out transformation from *world* coordinate system to *image* coordinate system

$$\underset{\boldsymbol{x}}{\begin{pmatrix}\text{2D}\\\text{point}\\\text{(3x1)}\end{pmatrix}} \cong \underbrace{\begin{pmatrix}\text{Camera to}\\\text{pixel coord.}\\\text{trans. matrix}\\ \boldsymbol{K}\ \text{(3x3)}\end{pmatrix}\begin{pmatrix}\textit{Canonical}\\\text{projection matrix}\\ [\boldsymbol{I}\mid\mathbf{0}]\ \text{(3x4)}\end{pmatrix}\begin{pmatrix}\text{World to}\\\text{camera coord.}\\\text{trans. matrix}\\ \begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \mathbf{0}^T & 1\end{bmatrix}\ \text{(4x4)}\end{pmatrix}}_{\boldsymbol{P}=\boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]} \underset{\boldsymbol{X}}{\begin{pmatrix}\text{3D}\\\text{point}\\\text{(4x1)}\end{pmatrix}}$$

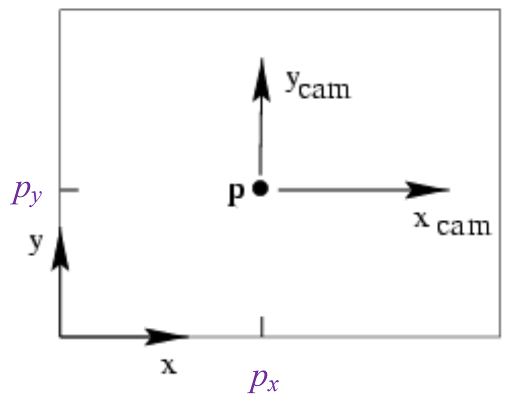
General camera projection matrix

Intrinsic parameters: Principal point



- **Principal point (p):** point where principal axis intersects the image plane
- In the *normalized* coordinate system, the origin of the image is at the principal point, or camera center
- In the *image* coordinate system: the origin is in the corner

Intrinsic parameters: Principal point

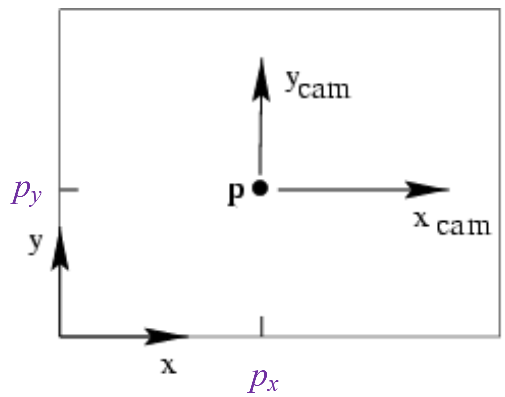


We want the principal point to map to (p_x, p_y) instead of $(0,0)$

$$x = f\frac{X}{Z} + p_x, \qquad y = f\frac{Y}{Z} + p_y$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Intrinsic parameters: Principal point



Principal point: (p_x, p_y)

$$\begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

calibration matrix $\boldsymbol{K}$

Canonical projection matrix $[\boldsymbol{I} \mid \boldsymbol{0}]$

$$\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{I}|\boldsymbol{0}]$$

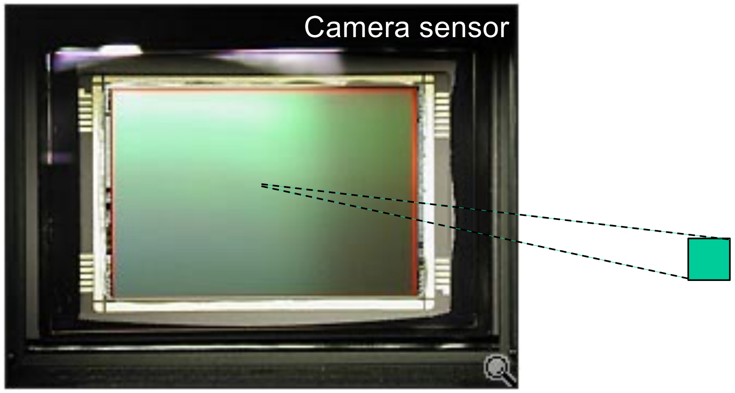
Intrinsic parameters: Principal point

- What are the units of the focal length f and principal point coordinates (p_x, p_y)?
  - Same as world units – presumably metric units
- What units do we want for measuring image coordinates?
  - Pixel units
- Thus, we need to introduce *scaling factors* for mapping from world to pixel units

$$\underbrace{\begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{calibration matrix } \boldsymbol{K}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Canonical projection matrix } [\boldsymbol{I} \mid \boldsymbol{0}]} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{I}|\boldsymbol{0}]$$

Intrinsic parameters: Scaling factors



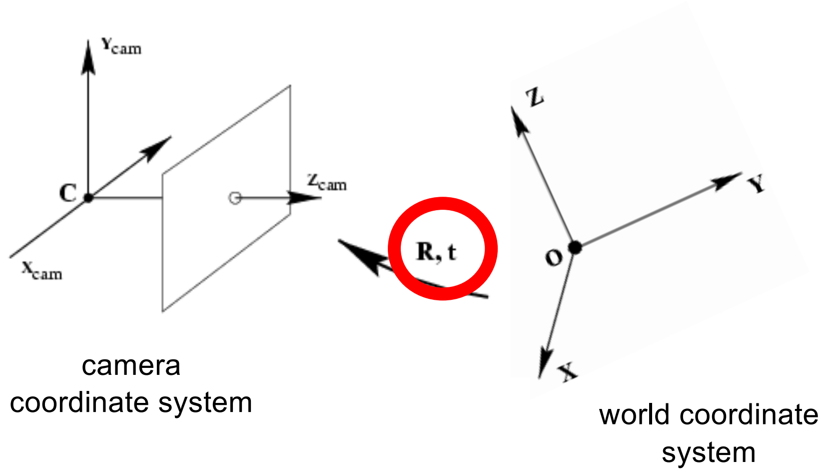
m_x pixels/m in horizontal direction,
m_y pixels/m in vertical direction

Pixel size (m): $\frac{1}{m_x} \times \frac{1}{m_y}$

Scaling factors — Calibration matrix $\boldsymbol{K}$ in metric units — Calibration matrix $\boldsymbol{K}$ in pixel units

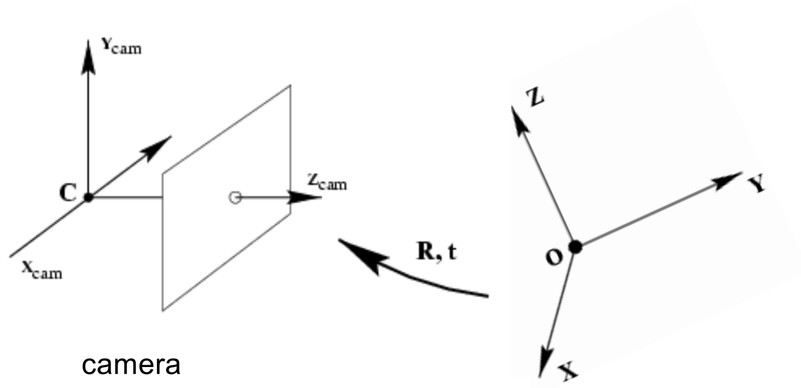
$$\underbrace{\begin{bmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{pixels/m}} \underbrace{\begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{m}} = \boxed{\underbrace{\begin{bmatrix} \alpha_x & 0 & \beta_x \\ 0 & \alpha_y & \beta_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{pixels}}}$$

Extrinsic parameters: Rotation and translation



- In general, the camera coordinate frame will be related to the world coordinate frame by a rotation and a translation

Extrinsic parameters: Rotation and translation



camera
coordinate system

world coordinate
system

- In general, the camera coordinate frame will be related to the world coordinate frame by a rotation and a translation

- In non-homogeneous coordinates, the transformation from world to normalized camera coordinate system is given by:

$$\widetilde{\boldsymbol{X}}_{\text{cam}} = \boldsymbol{R}\left(\widetilde{\boldsymbol{X}} - \widetilde{\boldsymbol{C}}\right) = \boldsymbol{R}\widetilde{\boldsymbol{X}} + \boldsymbol{t}$$

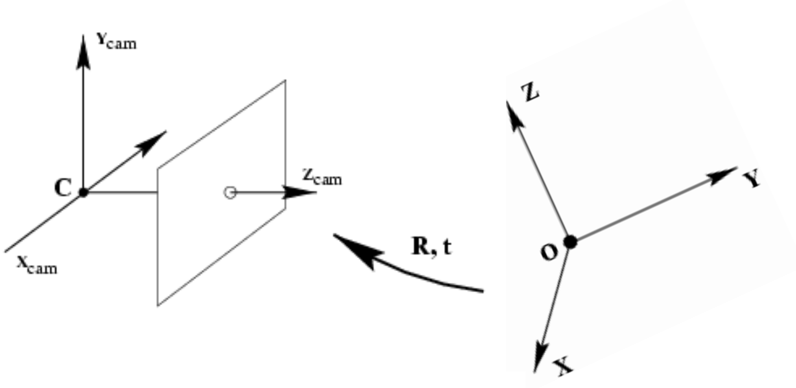
coords. of point in normalized camera frame

3x3 rotation matrix

coords. of a point in world frame

coords. of camera center in world frame

Extrinsic parameters: Rotation and translation



In *non-homogeneous* coordinates:

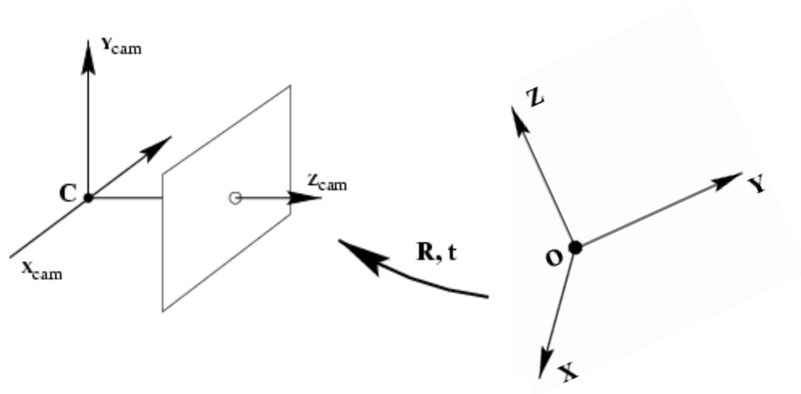
$$\widetilde{X}_{\text{cam}} = R\widetilde{X} + t$$

In *homogeneous* coordinates:

$$\begin{pmatrix} \widetilde{X}_{\text{cam}} \\ 1 \end{pmatrix} = \begin{bmatrix} R & t \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{pmatrix} \widetilde{X} \\ 1 \end{pmatrix}$$

3D transformation matrix (4 x 4)

Extrinsic parameters: Rotation and translation



In *non-homogeneous* coordinates:

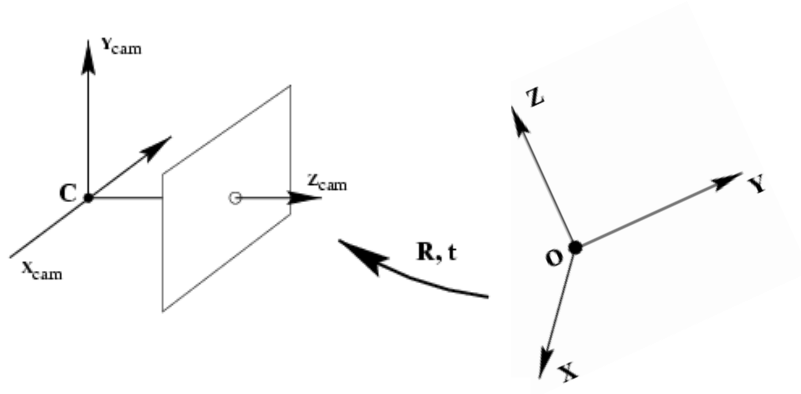
$$\tilde{\boldsymbol{X}}_{\text{cam}} = \boldsymbol{R}\tilde{\boldsymbol{X}} + \boldsymbol{t}$$

In *homogeneous* coordinates:

$$\boldsymbol{X}_{\text{cam}} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \boldsymbol{X}$$

3D transformation matrix (4 x 4)

Extrinsic parameters: Rotation and translation



In *non-homogeneous* coordinates:

$$\widetilde{\boldsymbol{X}}_{\text{cam}} = \boldsymbol{R}\widetilde{\boldsymbol{X}} + \boldsymbol{t}$$

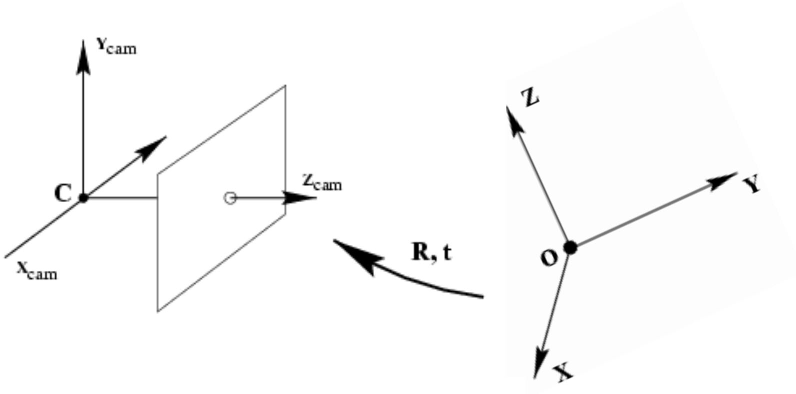
In *homogeneous* coordinates:

$$\boldsymbol{X}_{\text{cam}} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \boldsymbol{X}$$

Transformation from normalized 3D coordinates to pixel image coordinates:

$$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{I}|\boldsymbol{0}]\boldsymbol{X}_{\text{cam}}$$

Extrinsic parameters: Rotation and translation



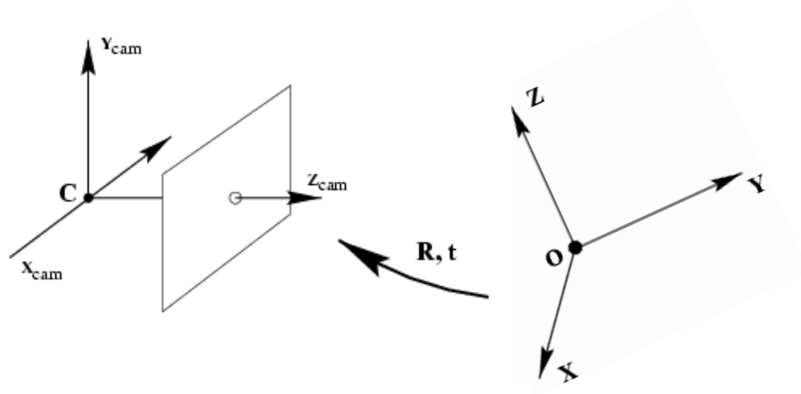
In *homogeneous* coordinates:

$$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{I}|\boldsymbol{0}]\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0}^{T} & 1\end{bmatrix}\boldsymbol{X}$$

Finally:

$$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\boldsymbol{X} \qquad \boldsymbol{t} = -\boldsymbol{R}\widetilde{\boldsymbol{C}}$$

Extrinsic parameters: Rotation and translation



$$x \cong K[R|t]X \qquad t = -R\tilde{C}$$

coords. of camera center in world frame

- What is the projection of the focal point in world coordinates?

$$PC = K[R \mid -R\tilde{C}]\begin{pmatrix}\tilde{C}\\ 1\end{pmatrix} = K(R\tilde{C} - R\tilde{C}) = \mathbf{0}$$

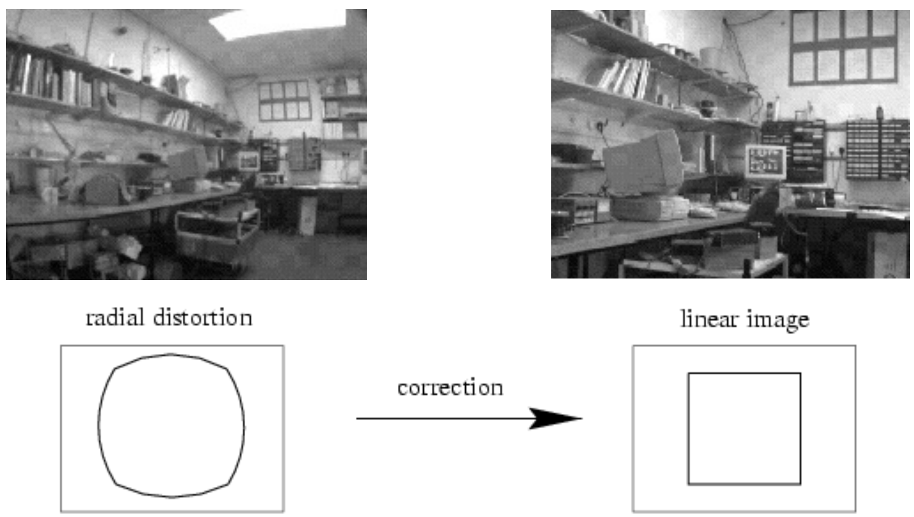
- The focal point is the null space of the projection matrix!

Camera parameters: Summary

$$\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]$$

- Intrinsic parameters
  - Principal point coordinates
  - Focal length
  - Pixel magnification factors
  - *Skew (non-rectangular pixels) – tends to get ignored in practice*
  - *Radial distortion – important in practice!*

$$\boldsymbol{K} = \begin{bmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_x & 0 & \beta_x \\ 0 & \alpha_y & \beta_y \\ 0 & 0 & 1 \end{bmatrix}$$



Camera extrinsics

$$\mathcal{T}_i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathcal{T}_e$$

Intrinsics Extrinsics

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \text{Transformation} \\ \text{mapping image} \\ \text{plane coords to} \\ \text{pixel coords} \end{bmatrix} \mathcal{C}_p \begin{bmatrix} \text{Transformation} \\ \text{mapping world} \\ \text{coords to camera} \\ \text{coords} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Rotation matrix - orthornormal, det=1

$$\begin{bmatrix} \mathcal{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Camera intrinsics

$$\mathcal{T}_i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathcal{T}_e$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \text{Transformation} \\ \text{mapping image} \\ \text{plane coords to} \\ \text{pixel coords} \end{bmatrix}}_{\text{Intrinsics}} \mathcal{C}_p \underbrace{\begin{bmatrix} \text{Transformation} \\ \text{mapping world} \\ \text{coords to camera} \\ \text{coords} \end{bmatrix}}_{\text{Extrinsics}} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

$$\begin{bmatrix} as & k & c_x \\ 0 & s & c_y \\ 0 & 0 & 1/f \end{bmatrix}$$

Overview

- Camera calibration
  - Intrinsic camera parameters
  - Extrinsic camera parameters
  - Estimation

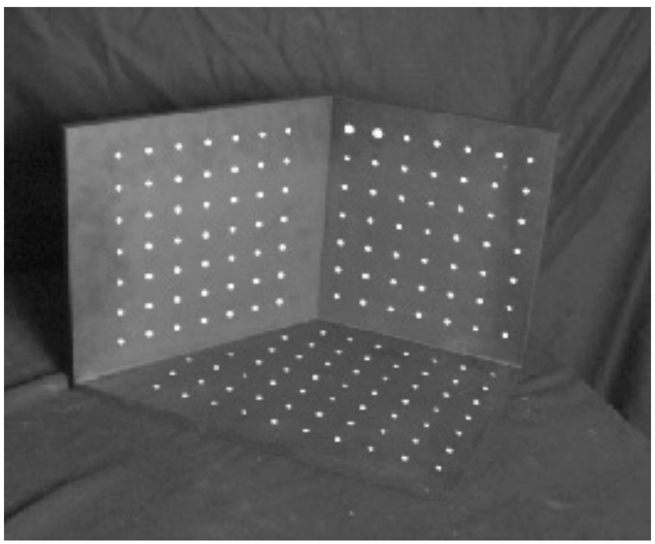
Camera calibration

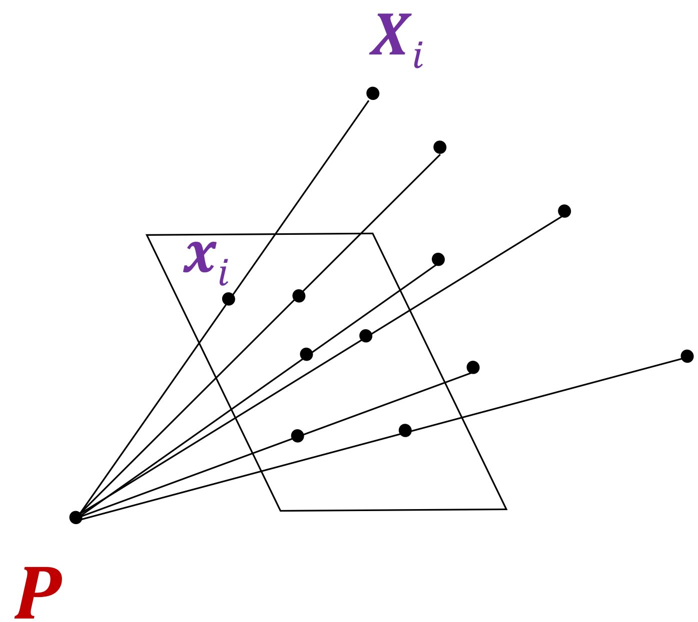
$$\boldsymbol{x} \cong \boldsymbol{K}[\boldsymbol{R} \quad \boldsymbol{t}]\boldsymbol{X}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Camera calibration

- Given n points with known 3D coordinates $\boldsymbol{X}_i$ and known image projections $\boldsymbol{x}_i$, estimate the camera parameters





Camera calibration

- Given n points with known 3D coordinates $\boldsymbol{X}_i$ and known image projections $\boldsymbol{x}_i$, estimate the camera parameters

Known 2D image coords	Known 3D locations
880 214	312.747 309.140 30.086
43 203	305.796 311.649 30.356
270 197	307.694 312.358 30.418
886 347	310.149 307.186 29.298
745 302	311.937 310.105 29.216
943 128	311.202 307.572 30.682
476 590	307.106 306.876 28.660
419 214	309.317 312.490 30.230
317 335	307.435 310.151 29.318
783 521	308.253 306.300 28.881
235 427	306.650 309.301 28.905
665 429	308.069 306.831 29.189
655 362	309.671 308.834 29.029
427 333	308.255 309.955 29.267
412 415	307.546 308.613 28.963
746 351	311.036 309.206 28.913
434 415	307.518 308.175 29.069
525 234	309.950 311.262 29.990
716 308	312.160 310.772 29.080
602 187	311.988 312.709 30.514

Image credit: J. Hays

Camera calibration: non-linear method

N reference points with known position in 3D $\mathbf{s}_i = [s_{x,i}, s_{y,i}, s_{z,i}]$

Predict the locations of the known points in camera using $\mathbf{t}_i$
estimated camera parameters

Compare to observed locations $\hat{\mathbf{t}}_i = [\hat{t}_{x,i}, \hat{t}_{y,i}]$

$$\hat{\mathbf{t}}_i = \mathbf{t}_i + \xi_i$$

Minimize least-squares error

$$\sum_i \xi_i^T \xi_i.$$

Camera calibration: non-linear method

Two issues:

write out equations for optimization problem

good start point for optimization

Camera extrinsics

$$\mathcal{T}_i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathcal{T}_e$$

Intrinsics

Extrinsics

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \text{Transformation} \\ \text{mapping image} \\ \text{plane coords to} \\ \text{pixel coords} \end{bmatrix} \mathcal{C}_p \begin{bmatrix} \text{Transformation} \\ \text{mapping world} \\ \text{coords to camera} \\ \text{coords} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Rotation matrix - orthornormal, det=1

$$\begin{bmatrix} \mathcal{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Camera intrinsics

$$\mathcal{T}_i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathcal{T}_e$$

Intrinsics

Extrinsics

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \text{Transformation} \\ \text{mapping image} \\ \text{plane coords to} \\ \text{pixel coords} \end{bmatrix} \mathcal{C}_p \begin{bmatrix} \text{Transformation} \\ \text{mapping world} \\ \text{coords to camera} \\ \text{coords} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

$$\begin{bmatrix} as & k & c_x \\ 0 & s & c_y \\ 0 & 0 & 1/f \end{bmatrix}$$

Camera calibration: non-linear, equations

$$\sum_i \xi_i^T \xi_i = \sum_i (t_{x,i} - p_{x,i})^2 + (t_{y,i} - p_{y,i})^2$$

The i's are intrinsic parameters (remember, this is upper triangular)

$$\begin{aligned} p_{x,i} &= \frac{i_{11} g_{x,i} + i_{12} g_{y,i} + i_{13} g_{z,i}}{g_{z,3}} \\ p_{y,i} &= \frac{i_{22} g_{x,i} + i_{23} g_{z,i}}{g_{z,i}} \end{aligned}$$

The e's are extrinsic parameters (they are constrained)

$$\begin{aligned} g_{x,i} &= e_{11} s_{x,i} + e_{12} s_{y,i} + e_{13} s_{z,i} + e_{14} \\ g_{y,i} &= e_{21} s_{x,i} + e_{22} s_{y,i} + e_{23} s_{z,i} + e_{24} \\ g_{z,i} &= e_{31} s_{x,i} + e_{32} s_{y,i} + e_{33} s_{z,i} + e_{34} \end{aligned}$$

$$1 - \sum_v e_{j,1v}^2 = 0 \text{ and } 1 - \sum_v e_{j,2v}^2 = 0 \text{ and } 1 - \sum_v e_{j,3v}^2 = 0$$

$$\sum_v e_{j,1v} e_{j,2v} = 0 \text{ and } \sum_v e_{j,1v} e_{j,3v} = 0 \text{ and } 1 - \sum_v e_{j,2v} e_{j,3v} = 0 \quad .$$

Camera calibration: non-linear

Strategy:

chuck it into a constrained optimizer and run

this fails – you need a good starting point

Start point:

neat linear construction

Camera calibration: Linear method

Write $\mathbf{C}_j^T$ for the j'th row of the camera matrix, and $\mathbf{S}_i = [s_{x,i}, s_{y,i}, s_{z,i}, 1]^T$ for homogeneous coordinates representing the i'th point in 3D. Then, assuming no errors in measurement, we have

$$\hat{t}_{x,i} = \frac{\mathbf{C}_1^T \mathbf{S}_i}{\mathbf{C}_3^T \mathbf{S}_i} \text{ and } \hat{t}_{y,i} = \frac{\mathbf{C}_2^T \mathbf{S}_i}{\mathbf{C}_3^T \mathbf{S}_i}, \tag{24.3}$$

which we can rewrite as

$$\mathbf{C}_3^T \mathbf{S}_i \hat{t}_{x,i} - \mathbf{C}_1^T \mathbf{S}_i = 0 \text{ and } \mathbf{C}_3^T \mathbf{S}_i \hat{t}_{y,i} - \mathbf{C}_2^T \mathbf{S}_i = 0. \tag{24.4}$$

x component of location of i'th point in image

One match gives two linearly independent constraints on the camera matrix

Calibration: Linear method

N points gives 2N homogeneous equations

$$\begin{pmatrix} -\mathbf{S}_1^T & 0 & \mathbf{S}_1^T t_{x,1} \\ 0 & -\mathbf{S}_1^T & \mathbf{S}_1^T t_{y,1} \\ \dots & \dots & \dots \\ -\mathbf{S}_N^T & 0 & \mathbf{S}_N^T t_{x,N} \\ 0 & -\mathbf{S}_N^T & \mathbf{S}_N^T t_{y,N} \end{pmatrix} \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}$$

The camera matrix is 3 x 4 but scale doesn't matter so there are 11 degrees of freedom – we can estimate it with 6 points

Camera calibration: Linear method

- Final linear system:

$$\begin{pmatrix} -\mathbf{S}_1^T & 0 & \mathbf{S}_1^T t_{x,1} \\ 0 & -\mathbf{S}_1^T & \mathbf{S}_1^T t_{y,1} \\ \cdots & \cdots & \cdots \\ -\mathbf{S}_N^T & 0 & \mathbf{S}_N^T t_{x,N} \\ 0 & -\mathbf{S}_N^T & \mathbf{S}_N^T t_{y,N} \end{pmatrix} \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix}$$

- What if all the n 3D points are *coplanar*, i.e., there exists a set of line parameters $\boldsymbol{\Pi}^T = (a, b, c, d)^T$ such that $\boldsymbol{\Pi}^T \boldsymbol{X}_i = 0$ for all i?
  - Then we will get *degenerate solutions* $(\boldsymbol{\Pi}, \mathbf{0}, \mathbf{0})$, $(\mathbf{0}, \boldsymbol{\Pi}, \mathbf{0})$, or $(\mathbf{0}, \mathbf{0}, \boldsymbol{\Pi})$

Camera parameters from camera matrix

Remember this: *Given a 3×4 camera matrix $\mathcal{P}$, the homogeneous coordinates of the focal point of that camera are given by* $\mathbf{X}$, *where* $\mathcal{P}\mathbf{X} = [0, 0, 0]^T$

Camera matrix is:

$$\mathcal{T}_i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathcal{T}_e$$

Camera parameters from camera matrix

Remember this: *Given a 3×4 camera matrix $\mathcal{P}$, the homogeneous coordinates of the focal point of that camera are given by $\mathbf{X}$, where $\mathcal{P}\mathbf{X} = [0,0,0]^T$*

Camera matrix is:

$$\mathcal{T}_i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathcal{T}_e$$

Remember this: *Assume camera matrix $\mathcal{P}$ has null space $\lambda\mathbf{u} = \lambda\left[\mathbf{f}^T, 1\right]^T$. Then we must have $\mathcal{T}_e\mathbf{u} = [0,0,0,1]^T$, so we must have*

$$\mathcal{T}_e = \begin{bmatrix} \mathcal{R} & -\mathcal{R}\mathbf{f} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{24.5}$$

Parameters from camera matrix

This means that, if we know $\mathcal{R}$, we can recover the translation from the focal point. We must now recover the intrinsic transformation and $\mathcal{R}$ from what we know.

$$\lambda \mathcal{P} = \mathcal{T}_i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{R} & -\mathcal{R}\mathbf{f} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{T}_i \mathcal{R} & -\mathcal{T}_i \mathcal{R}\mathbf{f} \end{bmatrix} \tag{24.6}$$

Parameters from camera matrix

We do not know λ, but we do know $\mathcal{P}$. Now write $\mathcal{P}_l$ for the left 3×3 block of $\mathcal{P}$, and recall that $\mathcal{T}_i$ is upper triangular and $\mathcal{R}$ orthonormal. The first question is the sign of λ. We expect $\mathsf{Det}(\mathcal{R}) = 1$, and $\mathsf{Det}(\mathcal{T}_i) > 0$, so $\mathsf{Det}(\mathcal{P}_l)$ should be positive. This yields the sign of λ – choose a sign $s \in \{-1, 1\}$ so that $\mathsf{Det}(s\mathcal{P}_l)$ is positive.

We can now factor $s\mathcal{P}_l$ into an upper triangular matrix $\mathcal{T}$ and an orthonormal matrix $\mathcal{Q}$. This is an RQ factorization (Section 35.2). Recall we could not distinguish between scaling caused by the focal length and scaling caused by pixel scale, so that

$$\mathcal{T}_i = \begin{bmatrix} as & k & c_x \\ 0 & s & c_y \\ 0 & 0 & 1 \end{bmatrix} \tag{24.7}$$

In turn, we have $\lambda = s(1/t_{33})$, $c_y = (t_{23}/t_{33})$, $s = (t_{22}/t_{33})$, $c_x = (t_{13}/t_{33})$, $k = (t_{12}/t_{33})$, and $a = (t_{11}/t_{22})$.

Camera calibration: nonlinear

- *non-linear* methods are preferred
  - Can include radial distortion and constraints such as known focal length, orthogonality, visibility of points

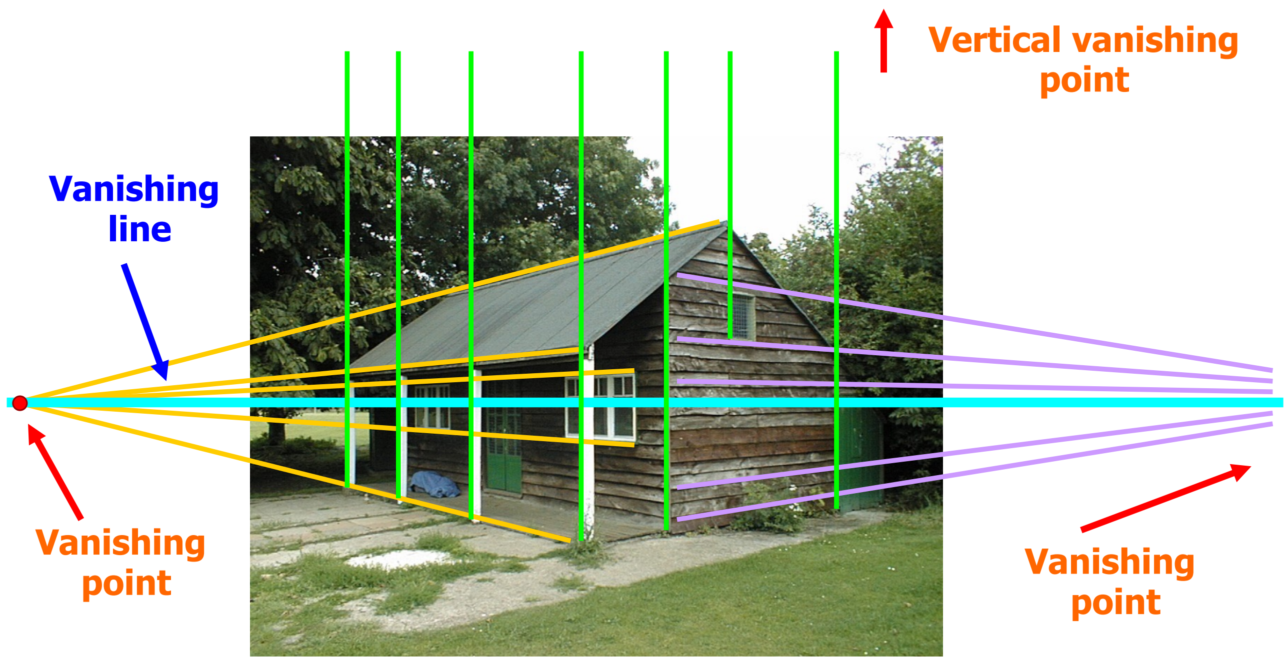
Camera calibration using vanishing points

- If world coordinates of reference 3D points are not known, in special cases, we may be able to use vanishing points

Source: A. Efros, A. Criminisi

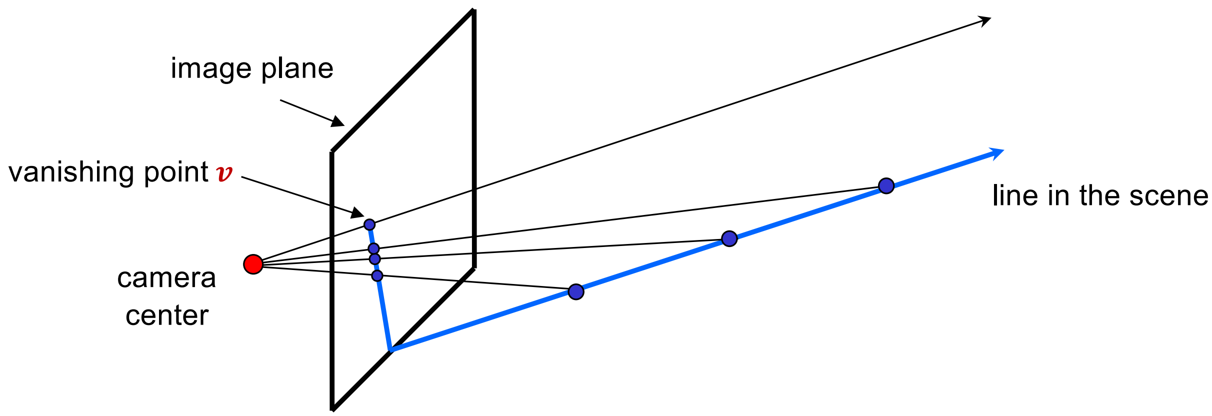
Camera calibration using vanishing points

- If world coordinates of reference 3D points are not known, in special cases, we may be able to use vanishing points



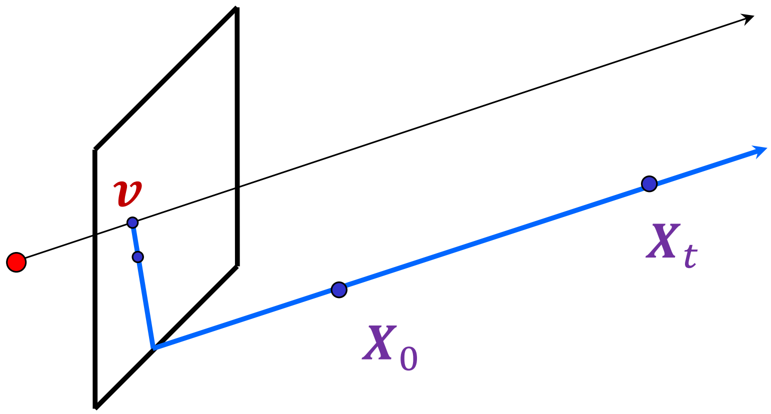
Source: A. Efros, A. Criminisi

Review: Vanishing points



- All lines having the same direction share the same vanishing point

Computing vanishing points



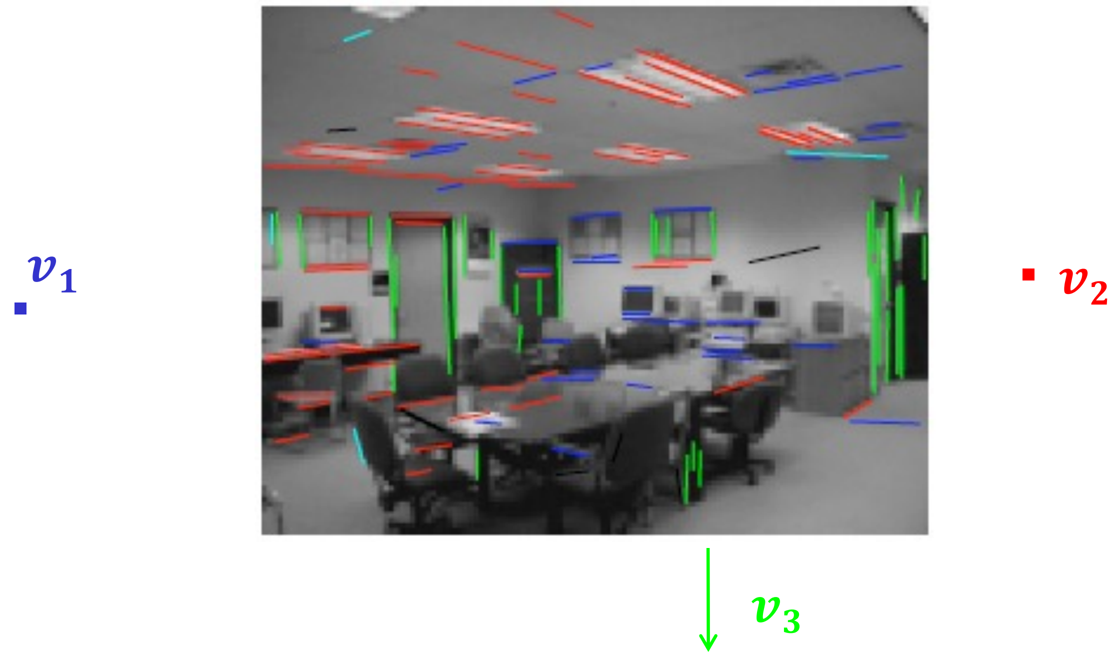
- Let's parameterize the line using point $\boldsymbol{X}_0 = (X_0, Y_0, Z_0, 1)^T$ and direction vector $\boldsymbol{D} = (D_1, D_2, D_3)^T$:

$$\boldsymbol{X}_t = \begin{pmatrix} X_0 + tD_1 \\ Y_0 + tD_2 \\ Z_0 + tD_3 \\ 1 \end{pmatrix} \cong \begin{pmatrix} X_0/t + D_1 \\ Y_0/t + D_2 \\ Z_0/t + D_3 \\ 1/t \end{pmatrix} \qquad \boldsymbol{X}_\infty = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ 0 \end{pmatrix}$$

- $\boldsymbol{X}_\infty$ is a *point at infinity,* $\boldsymbol{v}$ is its projection: $\boldsymbol{v} \cong \boldsymbol{P}\boldsymbol{X}_\infty$

Calibration from vanishing points

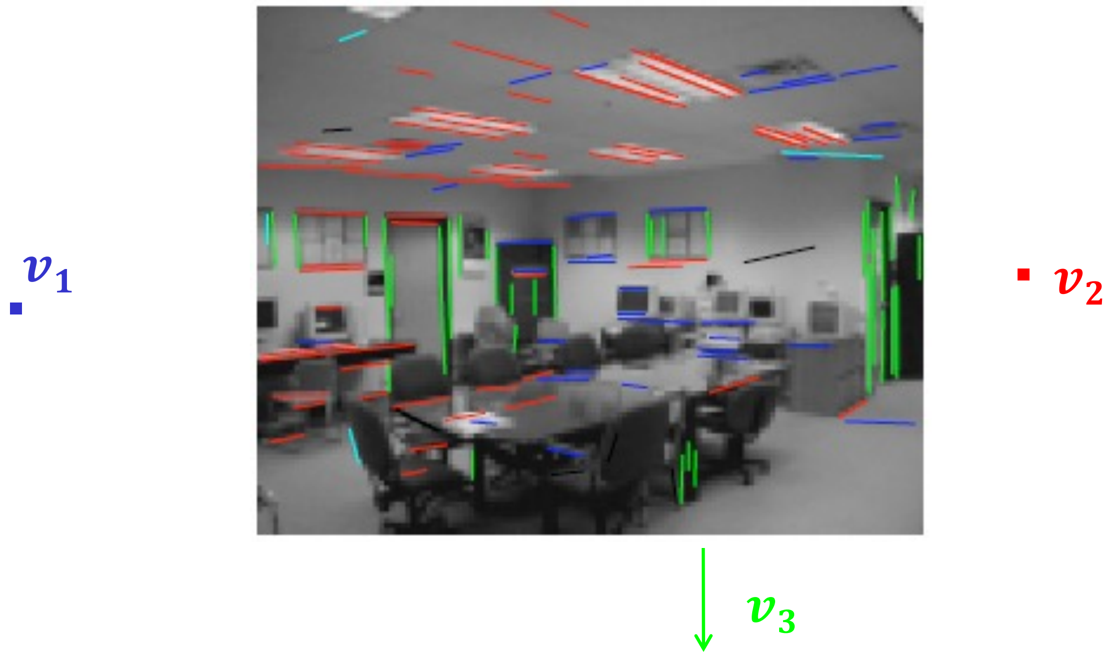
- Consider a scene with three orthogonal vanishing directions:



- Note: v_1, v_2 are *finite* vanishing points and v_3 is an *infinite* vanishing point

Calibration from vanishing points

- Consider a scene with three orthogonal vanishing directions:



- We can align the world coordinate system with these directions

Projection of the world coordinate system

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}$$

Projection of the world coordinate system

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{p}_1$$

$$\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \boldsymbol{p}_3 \quad \boldsymbol{p}_4$$

Projection of the world coordinate system

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{p}_2$$

$$\begin{array}{cccc} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 \end{array}$$

Projection of the world coordinate system

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \boldsymbol{p}_3$$

$$\underbrace{\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \boldsymbol{p}_3}_{\text{Vanishing points in } x,\, y,\, z \text{ directions (i.e., } \boldsymbol{v}_1,\, \boldsymbol{v}_2,\, \boldsymbol{v}_3)} \quad \boldsymbol{p}_4$$

Projection of the world coordinate system

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{p}_4$$

$$\underbrace{\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \boldsymbol{p}_3}_{\substack{\text{Vanishing points in} \\ x,\ y,\ z \text{ directions} \\ (\text{i.e., } \boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3)}} \quad \boldsymbol{p}_4$$

Projection of world origin

- Problem: this only gives us the four columns up to independent scale factors, additional constraints needed to solve for them

Calibration from vanishing points

- Let us align the world coordinate system with three orthogonal vanishing directions in the scene:

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{v}_i \cong \boldsymbol{K}[\boldsymbol{R}|\boldsymbol{t}]\begin{pmatrix} \boldsymbol{e}_i \\ 0 \end{pmatrix}$$

Calibration from vanishing points

- Let us align the world coordinate system with three orthogonal vanishing directions in the scene:

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{v}_i \cong \boldsymbol{K}\boldsymbol{R}\boldsymbol{e}_i$$

$$\boldsymbol{e}_i \cong \boldsymbol{R}^T\boldsymbol{K}^{-1}\boldsymbol{v}_i$$

- Orthogonality constraint: $\boldsymbol{e}_i^T\boldsymbol{e}_j = 0$

$$\underbrace{\boldsymbol{v}_i^T\boldsymbol{K}^{-T}\boldsymbol{R}}_{\boldsymbol{e}_i^T}\underbrace{\boldsymbol{R}^T\boldsymbol{K}^{-1}\boldsymbol{v}_j}_{\boldsymbol{e}_j} = 0$$

Calibration from vanishing points

- Let us align the world coordinate system with three orthogonal vanishing directions in the scene:

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{v}_i \cong \boldsymbol{K}\boldsymbol{R}\boldsymbol{e}_i$$

$$\boldsymbol{e}_i \cong \boldsymbol{R}^T\boldsymbol{K}^{-1}\boldsymbol{v}_i$$

- Orthogonality constraint: $\boldsymbol{e}_i^T\boldsymbol{e}_j = 0$

$$\boldsymbol{v}_i^T\boldsymbol{K}^{-T}\boldsymbol{K}^{-1}\boldsymbol{v}_j = 0$$

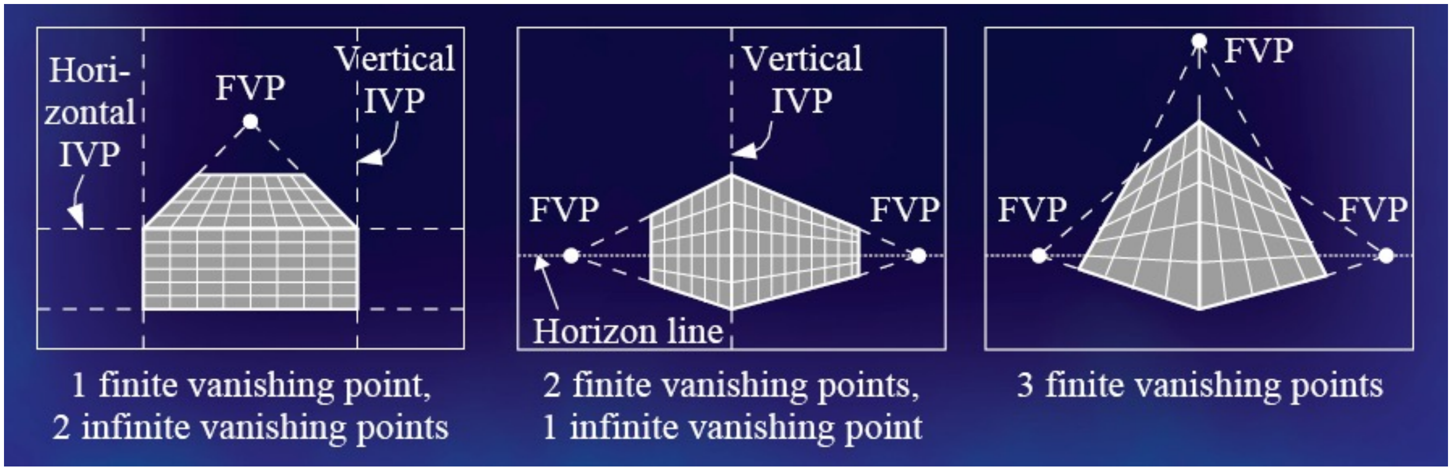
- Extrinsic parameter matrix ($\boldsymbol{R}$) disappears and we are left with constraints on the calibration matrix!

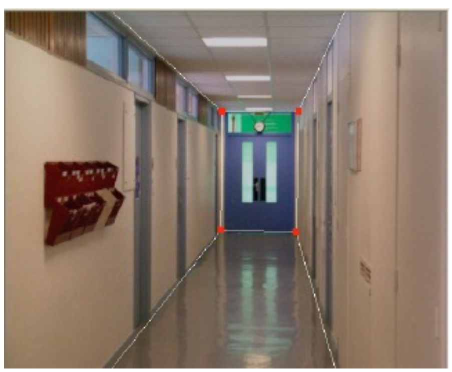
Calibration from vanishing points

$$\boldsymbol{v}_i^T \boldsymbol{K}^{-T} \boldsymbol{K}^{-1} \boldsymbol{v}_j = 0$$

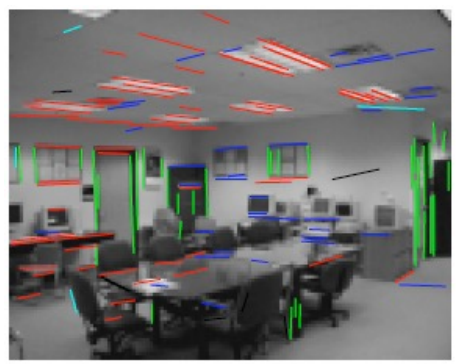
- How many constraints do we get?
  - Three: one for each pair of vanishing points
- How many unknown parameters does $\boldsymbol{K}$ have?
  - Three: f, p_x, p_y
- A couple of complications:
  - The constraints are nonlinear, but it's not hard to do the algebra (omitted)
  - At least two *finite* vanishing points are needed to solve for both focal length and principal point

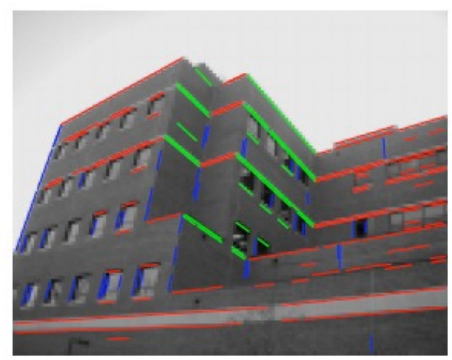
Calibration from vanishing points





Cannot recover focal length, principal point is the third vanishing point





Can solve for focal length, principal point

Rotation from vanishing points

- Constraints on vanishing points: $\boldsymbol{v}_i \cong \boldsymbol{KRe}_i$
- We just used orthogonality constraints to solve for $\boldsymbol{K}$
- Now we have:

$$\boldsymbol{K}^{-1}\boldsymbol{v}_i \cong \boldsymbol{Re}_i$$

Notice: $\boldsymbol{Re}_1 = [\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \boldsymbol{r}_3]\begin{pmatrix}1\\0\\0\end{pmatrix} = \boldsymbol{r}_1$

Thus, $\boldsymbol{r}_i \cong \boldsymbol{K}^{-1}\boldsymbol{v}_i$

- The scale ambiguity goes away since we require $\|\boldsymbol{r}_i\|^2 = 1$

Calibration from vanishing points: Summary

1. Solve for intrinsic parameters (focal length, principal point) using three orthogonal vanishing points
2. Get extrinsic parameters (rotation) directly from vanishing points once calibration matrix is known

- Advantages
  - No need for calibration chart, 2D-3D correspondences
  - Could be completely automatic
- Disadvantages
  - Only applies to certain kinds of scenes
  - It is tricky to accurately localize vanishing points
  - Need at least two finite vanishing points

A camera views a plane

$$\mathbf{x} = \mathcal{K}\,[\mathcal{I} \mid \mathbf{0}] \begin{bmatrix} \mathcal{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{X}$$

Put the plane at Z=0

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ 0 \\ 1 \end{bmatrix}$$

Simplify

A camera views a plane

Transformation simplifies to

$$\mathbf{x} = \mathcal{K}\left[\mathbf{r}_1 \mathbf{r}_2 \mathbf{t}\right] \left[\begin{array}{c} U \\ V \\ 1 \end{array} \right]$$

This is a homography

but may not be a general projective transformation

A camera rotates about the focal point

Camera 1 $$\mathbf{x} = \mathcal{K}\left[\mathcal{I} \mid \mathbf{0}\right]\mathbf{X}$$

Camera 2 $$\mathbf{x}' = \mathcal{K}\left[\mathcal{I} \mid \mathbf{0}\right]\left[\begin{array}{cc} \mathcal{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{array}\right]\mathbf{X}$$

But t must be zero, because the focal point doesn't change (no translation)
Check – 0, 0, 0, 1 will be fp

A camera rotates about the focal point

Camera 1 $$\mathbf{x} = \mathcal{K}\,[\mathcal{I} \mid \mathbf{0}]\,\mathbf{X}$$

Camera 2 $$\mathbf{x}' = \mathcal{K}\,[\mathcal{I} \mid \mathbf{0}] \begin{bmatrix} \mathcal{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{X} = \mathcal{K}\,[\mathcal{R} \mid \mathbf{0}]\,\mathbf{X}$$

A camera rotates about the focal point

Camera 1 $$\mathbf{x} = \mathcal{K}\,[\mathcal{I} \mid \mathbf{0}]\,\mathbf{X}$$

Camera 2

$$\begin{aligned}
\mathbf{x}' &= \mathcal{K}\,[\mathcal{I} \mid \mathbf{0}] \begin{bmatrix} \mathcal{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{X} \\
&= \mathcal{K}\,[\mathcal{R} \mid \mathbf{0}]\,\mathbf{X} = \mathcal{K}\mathcal{R}^T\mathcal{K}^{-1}\mathcal{K}\,[\mathcal{I} \mid \mathbf{0}]\,\mathbf{X} \\
&= \mathcal{K}\mathcal{R}^T\mathcal{K}^{-1}\mathbf{x}
\end{aligned}$$

Rotate about the focal point == apply a particular homography to image
Notice intrinsics matter here, entirely appropriate

Multi-view geometry “Bible”