Quick and dirty Fourier theory - I

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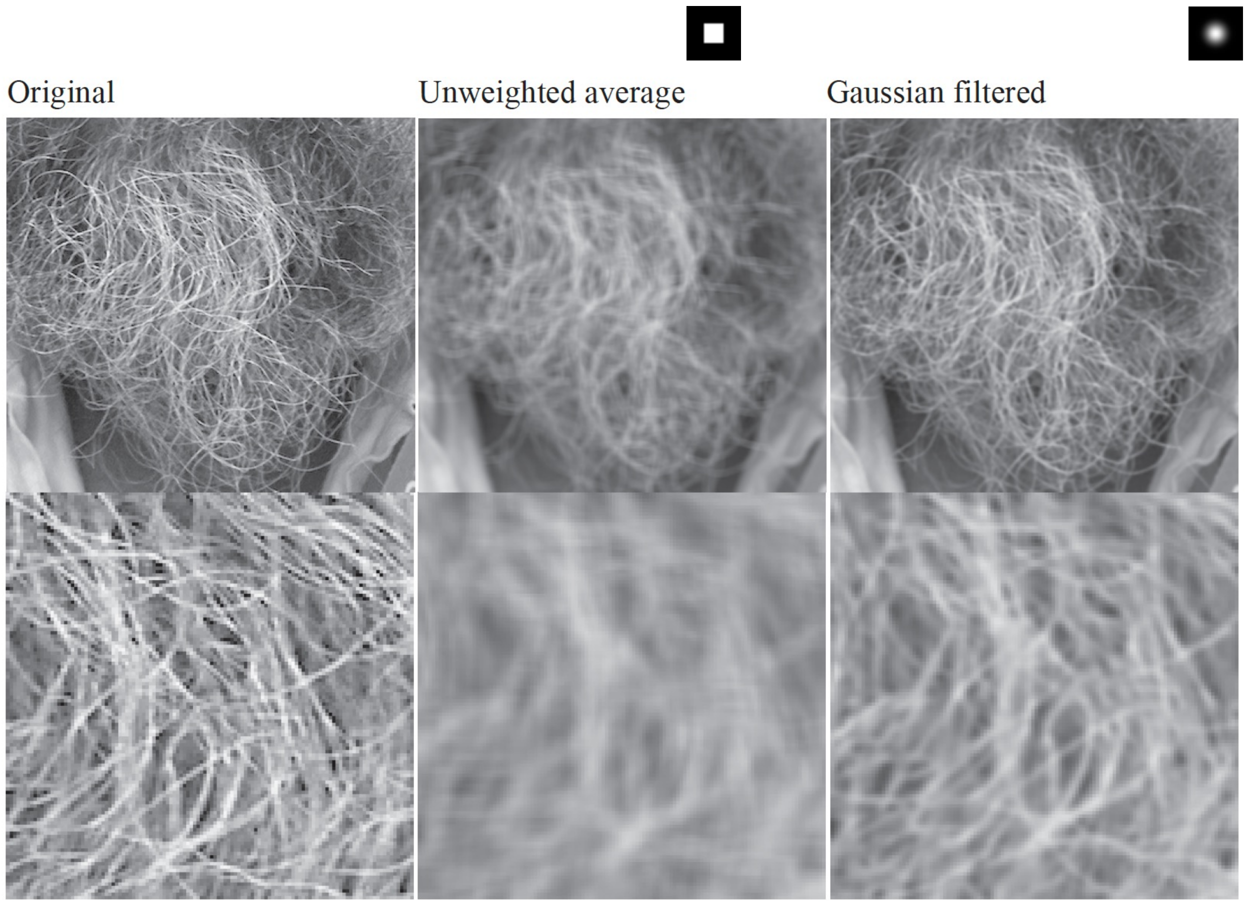
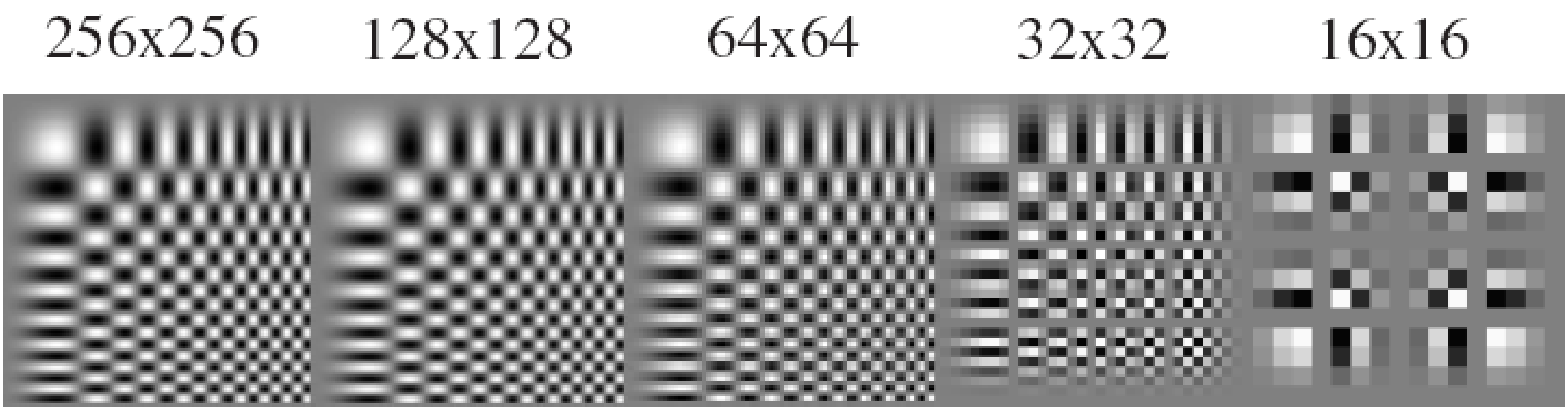
Mystery 1

Why does filtering with a Gaussian give a nice smooth image, but filtering with a box filter gives artifacts?



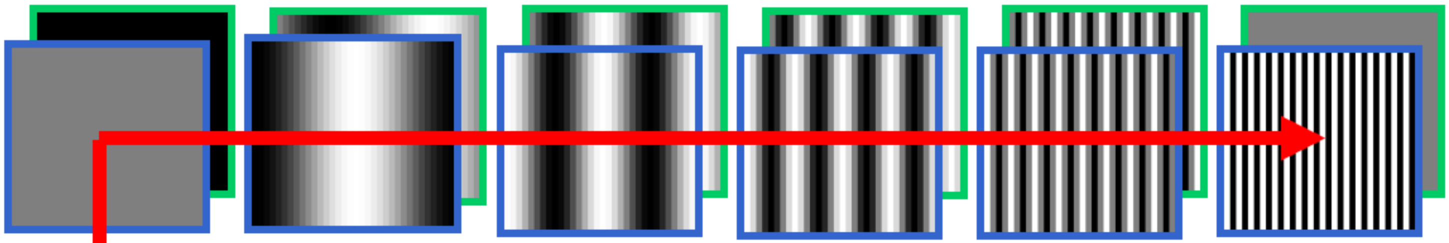
Mystery 2

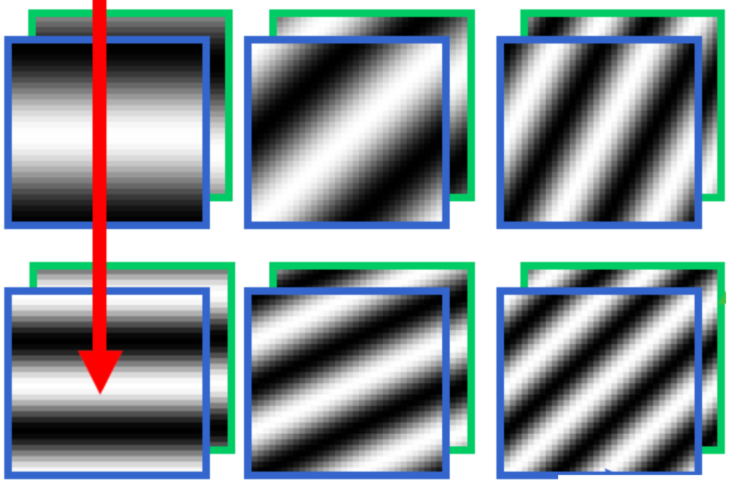
- Why can downsampling sometimes lead to aliasing?



Fourier analysis

- To understand such phenomena, we need a representation of images that allows us to tease ap





Fourier series

- Generally, we have for a (reasonable) periodic f(t)

$$f(t) \sim A_0 + \sum_{i=1}^{\infty} \left[A_i \cos(i2\pi t) + B_i \sin(i2\pi t) \right]$$

- And we need to figure out the weights for a given f(t).

Fourier series: useful facts

$$\int_0^1 \cos(i2\pi t)\,dt = \int_0^1 \sin(i2\pi t)\,dt = 0 \text{ for } i \text{ integer, } i > 0$$

Fact 1

$$\int_0^1 \cos(i2\pi t)\sin(j2\pi t)\,dt = 0 \text{ for } i,j \text{ integer, } i \neq j,\ i > 0,\ j > 0$$

$$\int_0^1 \cos(i2\pi t)\cos(j2\pi t)\,dt = 0 \text{ for } i,j \text{ integer, } i \neq j,\ i > 0,\ j > 0$$

Fact 2

$$\int_0^1 \sin(i2\pi t)\sin(j2\pi t)\,dt = 0 \text{ for } i,j \text{ integer, } i \neq j,\ i > 0,\ j > 0$$

$$\int_0^1 \sin^2(i2\pi t)\,dt = 1/2 \text{ for } i \text{ integer}$$

Fact 3

$$\int_0^1 \cos^2(i2\pi t)\,dt = 1/2 \text{ for } i \text{ integer}$$

Fourier series: using facts

- If:

$$f(t) \sim A_0 + \sum_{i=1}^{\infty} \left[A_i \cos\left(i2\pi t\right) + B_i \sin\left(i2\pi t\right)\right]$$

$$\int_0^1 f(t)dt = A_0$$

(fact 1 makes all the cosine/sine terms go away!)

$$\int_0^1 f(t)\sin(i2\pi t)dt = \frac{A_i}{2}$$

$$\int_0^1 f(t)\cos(i2\pi t)dt = \frac{B_i}{2}$$

(fact 2 makes all the other terms go away!
And fact 3 sets the scale)

Fourier series: issues

- A’s and B’s are inelegant -> complex exponentials
- Did NOT show that the series converges to the function
  - Read Korner’s wonderful book Fourier Analysis
  - We’re OK for anything we care about
- In principle, we can go forward
  - Function -> A’s, B’s
- Or backward
  - A’s, B’s -> Function
- Is this right? (mostly yes, but details…)
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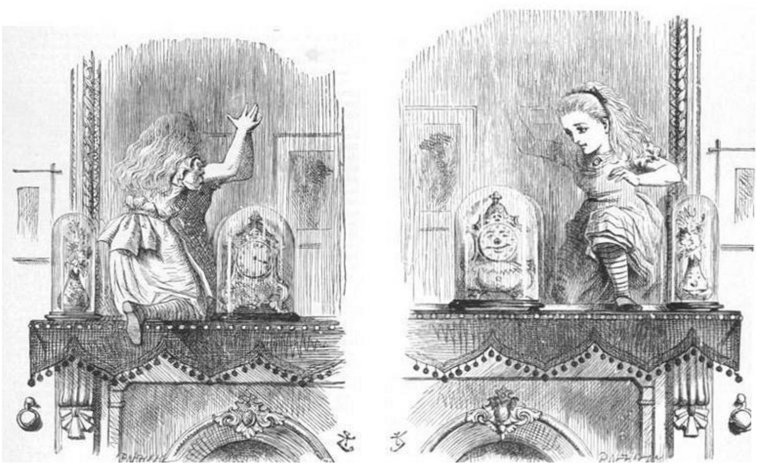
Complex exponentials

This i is the square root of -1 !!!

$$e^{i2k\pi t} = \cos(2k\pi t) + i\sin(2k\pi t)$$

$$f(t) \sim \sum_{k=0}^{\infty} c_k e^{i2k\pi t}$$



Advantage:

if the function is complex, can represent cleanly

don’t need to remember which is A, which B

Complex exponentials: compact facts

$$\int_0^1 e^{i2k\pi t} e^{-i2n\pi t} dt = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

k, n integers

This minus sign matters!

Fourier series with complex exponentials: using fact

- If: $$f(t) \sim \sum_{k=0}^{\infty} c_k e^{i2k\pi t}$$

$$c_k = \int_0^1 f(t) e^{-i2k\pi t} dt$$

Using the fact! (this is analogous to an orthonormal basis in linear algebra)

Fourier series with complex exponentials: issues

- But this is just for a periodic function on [0, 1]
  - Easy to extend to other intervals
  - Easy to extend to the circle
- But what about functions on [-infinity, infinity]?
  - These could wiggle often in numerous places
  - IDEA: use “more” basis elements
- The Fourier transform
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Things to think about...

11.1. What is the Fourier series for the function of one dimension with constant value 1?

11.2. What is the Fourier series for the function of one dimension $\sin 2\pi x$?

11.3. Look at the example of Section 11.1.2. Check that, in the limit $t \to \infty$, the solution is a constant. What is the constant?

11.4. What is the Fourier series for the function $\sin 2\pi(x + y)$?

11.5. What is the Fourier series for the function $\sin 2\pi(x + y)$?