Tensor product surfaces

- Natural way to think of a surface: curve is swept, and (possibly) deformed. Examples: ruled surface (line is swept), surface of revolution (circle is swept along line, grows and shrinks).
- Surface form:

$$\sum_{i,j} \underline{X_{ij}} f_i(u) g_j(v)$$

- Usually, domain is rectangular;
  - until further notice, all domains are rectangular.
- Classical tensor product interpolate is Gouraud shading on a rectangle; this gives a bilinear interpolate of the rectangles vertex values.
- Continuity constraints for surfaces are more interesting than for curves - see example

Tensor Product Bezier patches

- Tensor product of Bezier curves; write as:

$$\sum_{i=0}^{n}\sum_{j=0}^{m} \underline{b_{ij}} B_i^n(u) B_j^m(v)$$

- It follows from the tensor product form that:
  - interpolates four vertex points
  - tangent plane at each vertex is given by three points at that vertex
  - repeated de Casteljau (one direction, then the other) gives a point on the surface, tangent plane to surface

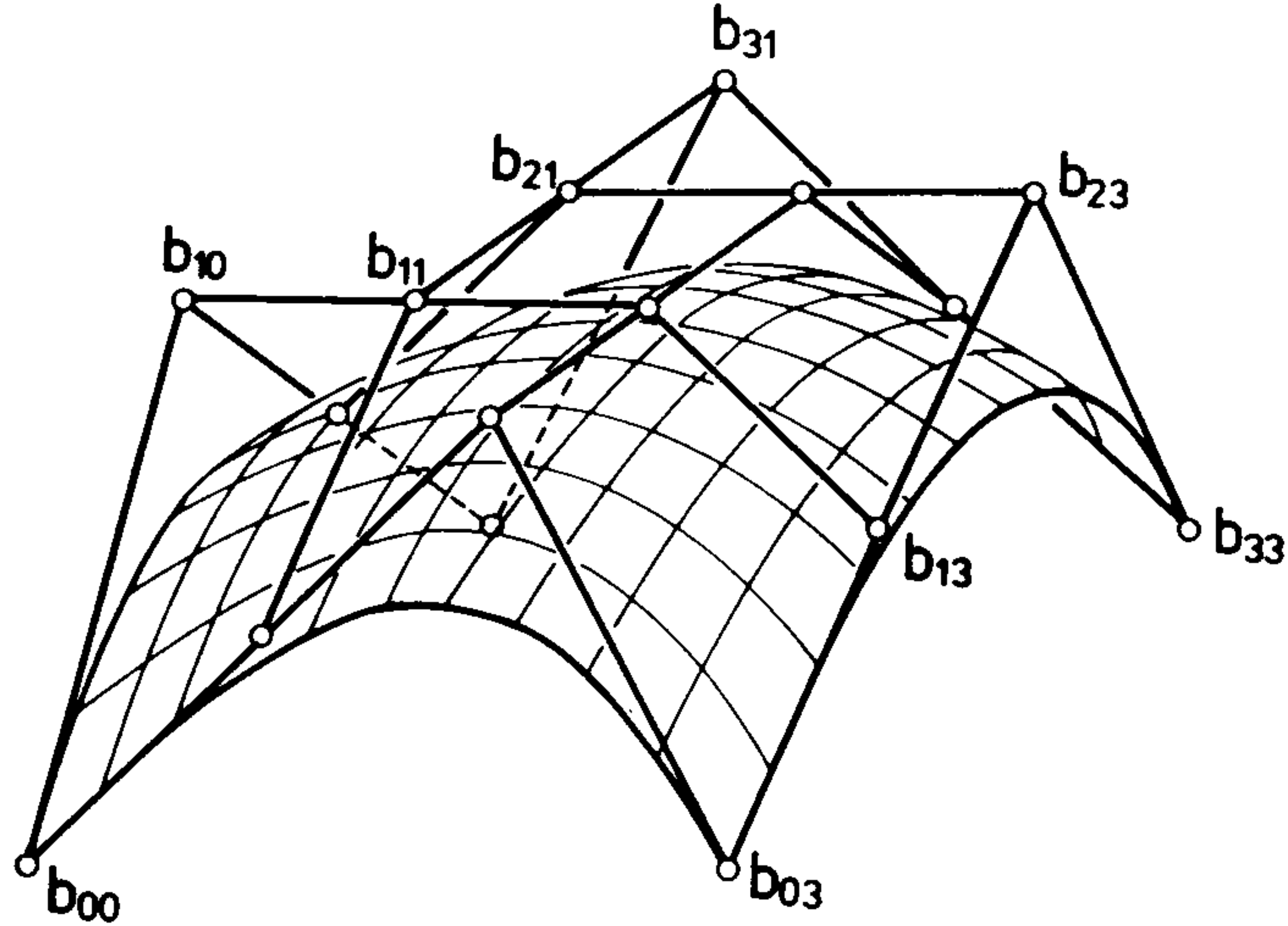


Fig. 6.3. Bézier surface of degree (3,3) and its Bézier net.

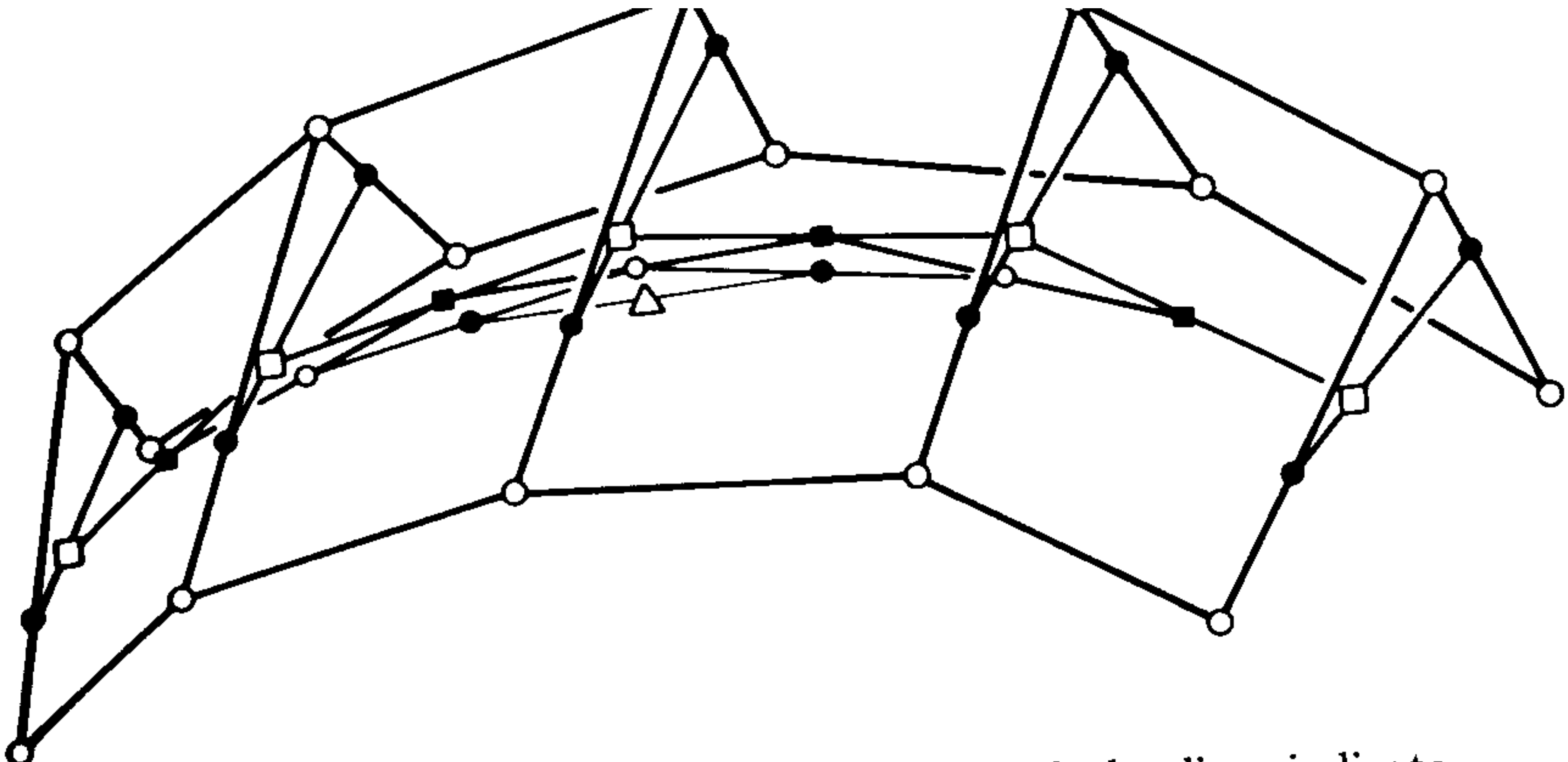


Fig. 6.7a. de Casteljau algorithm for surfaces: darker lines indicate interpolation in the v-direction.

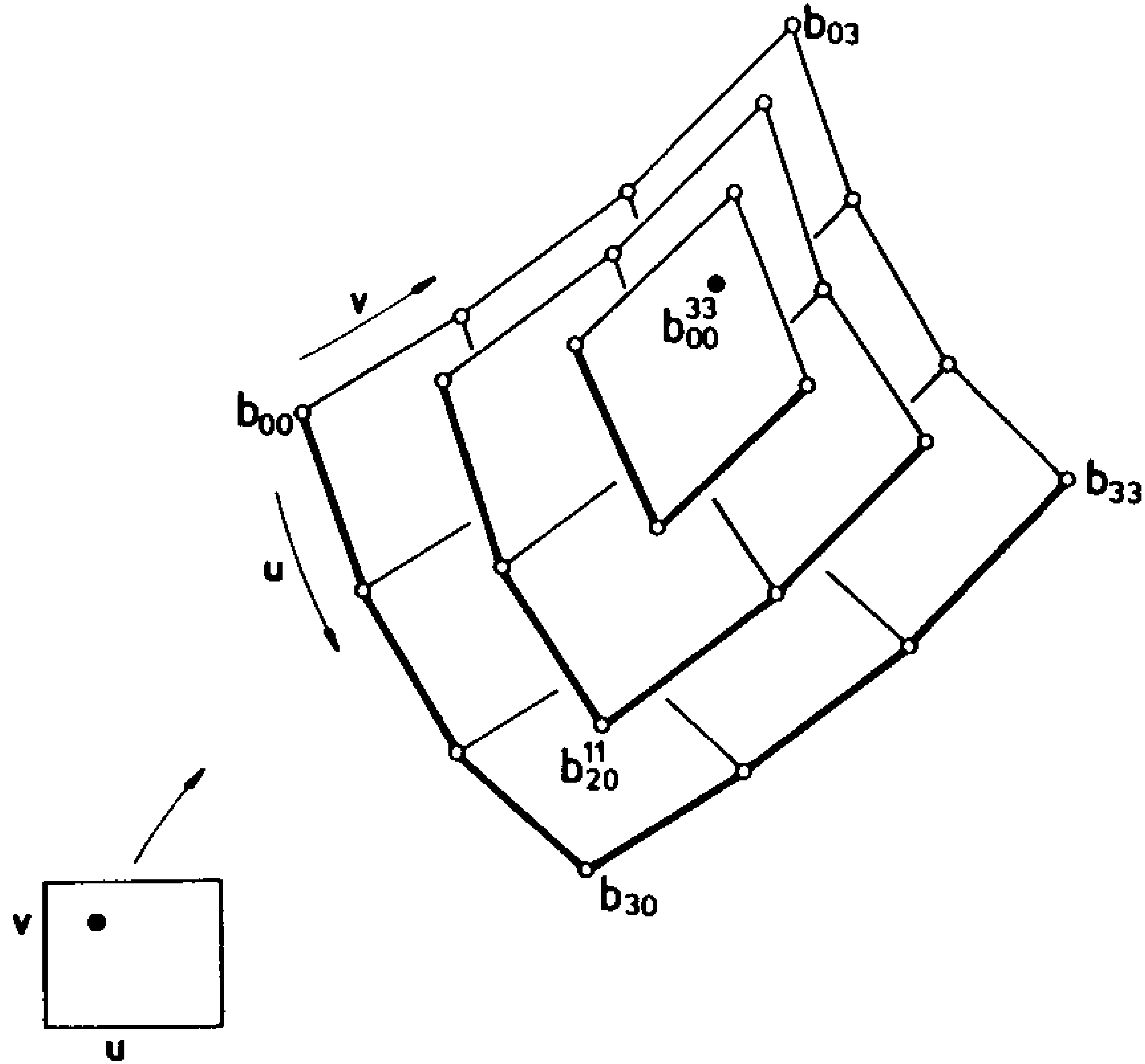


Fig. 6.7b. The de Casteljau algorithm viewed as bilinear interpolation.

Subdivision for Bezier curves

- Use De Casteljau (repeated linear interpolation) to identify points.
- Points as marked in figure give two control polygons, for two Bezier curves, which lie on top of the original.
- Repeated subdivision leads to a polygon that lies very close to the curve
- Limit of subdivision proces is a curve

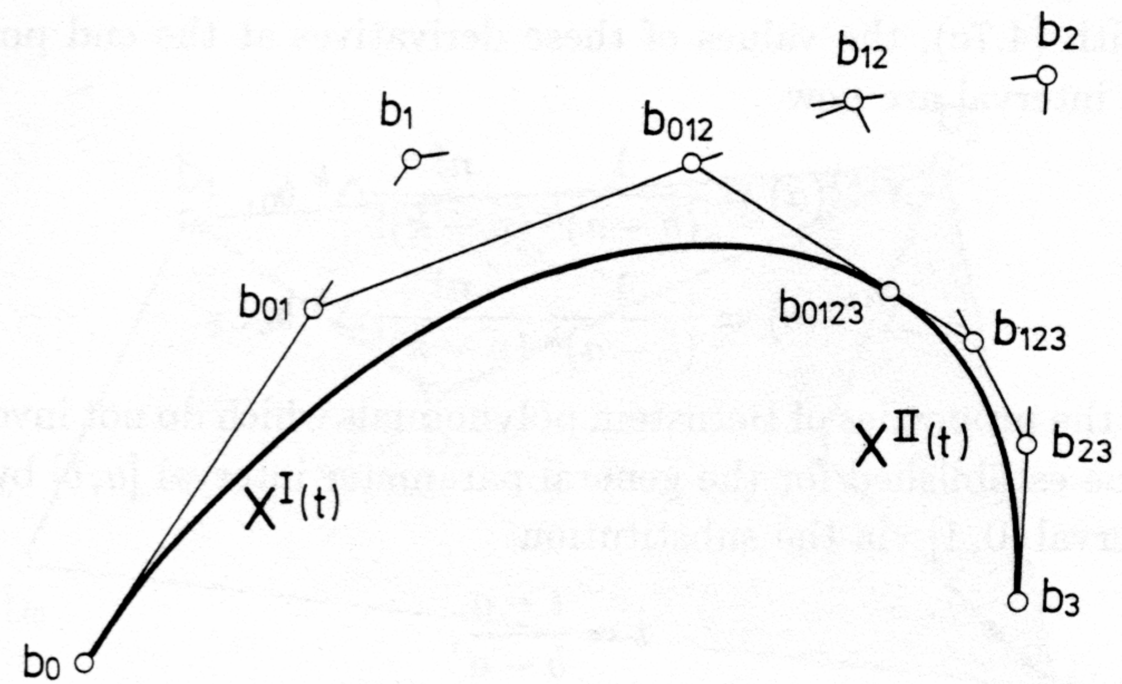


Fig. 4.5. Decomposition of a Bézier curve into two C^3 continuous curve segments (cf. Fig. 4.4).

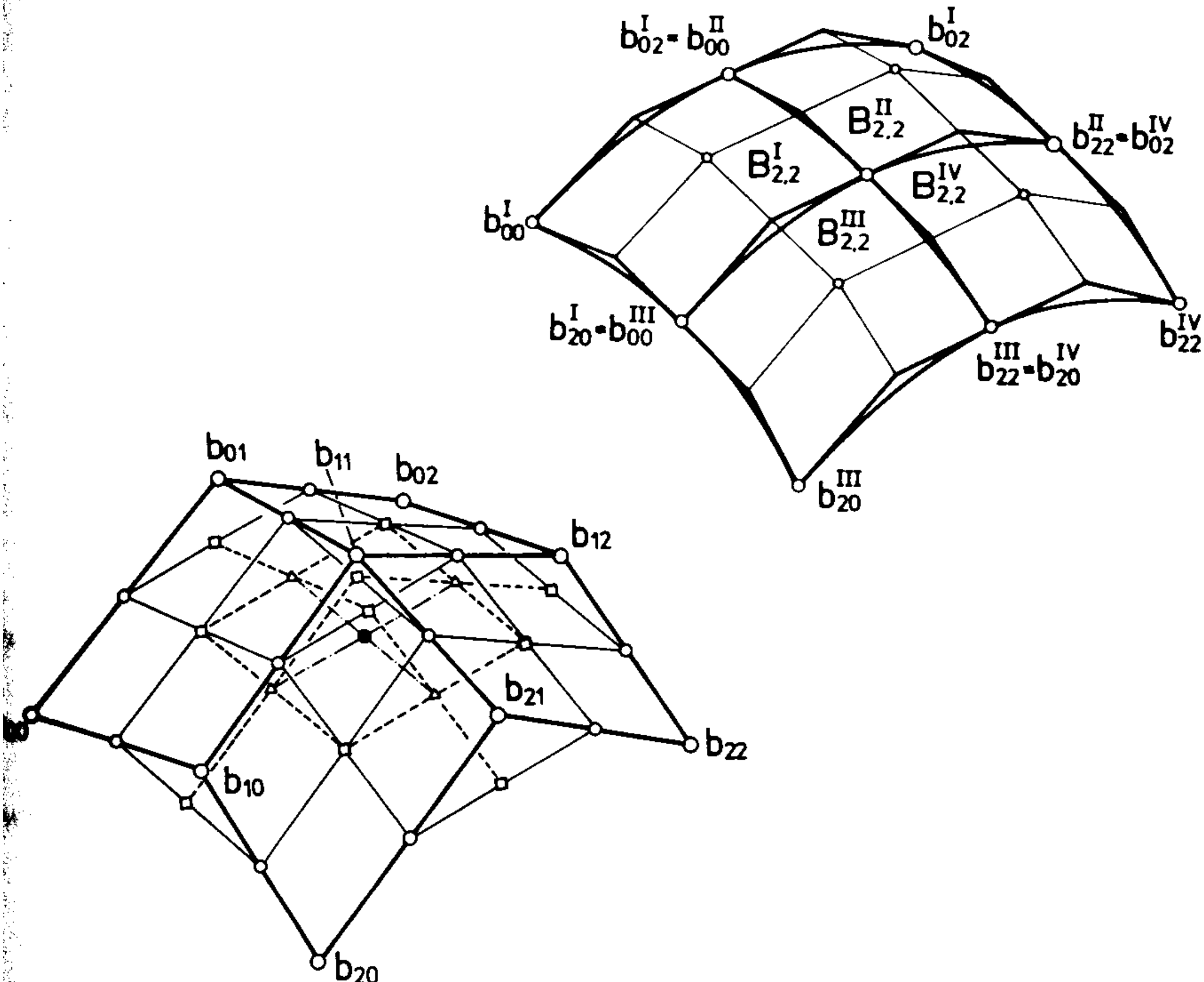


Fig. 6.8. Subdivision of a Bézier surface using the de Casteljau algorithm.

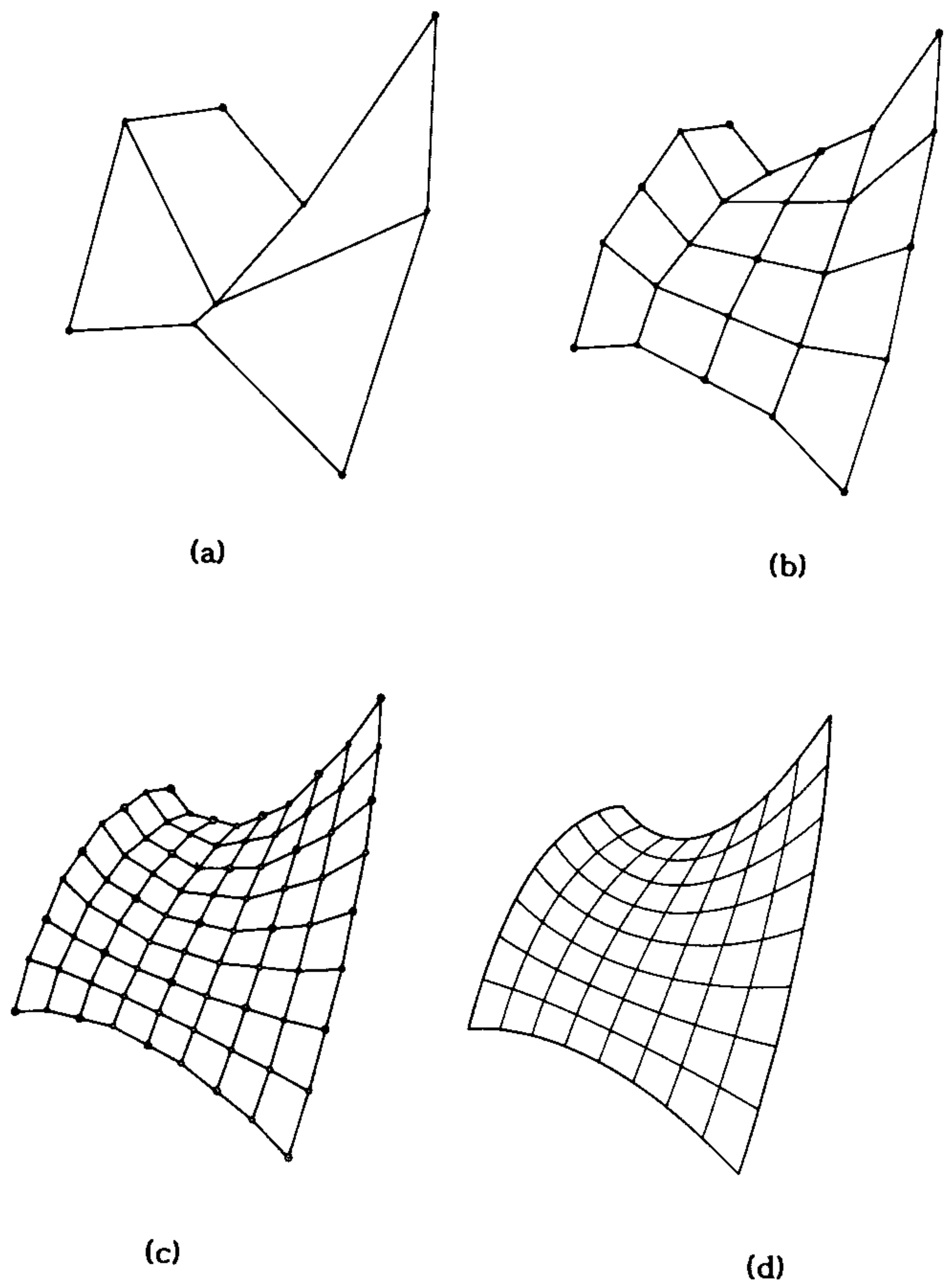
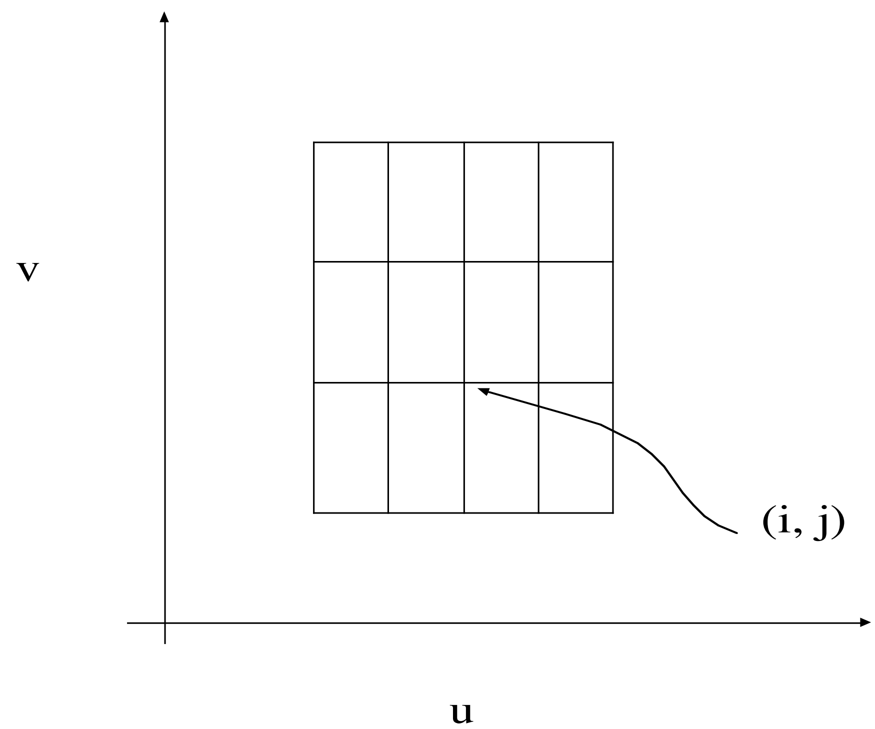


Fig. 6.9. Repeated refinement of a Bézier net (a) leads to approximations (b), (c) of the Bézier surface (d).

Example: bicubic interpolating surface

- Given a rectangular grid in the parameter domain, point values at each grid point, construct a surface that is locally cubic in u and in v separately, and interpolates. This means that, for fixed v, surface will be a piecewise cubic curve in u, and ditto.
- surface has form:



$$\underline{m}(x) := \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}$$

$$\underline{m}^T(u - u_i)\underline{\underline{A}}\underline{m}(v - v_i)$$

Bicubic Interpolate

- Construct surface so that surface, first partials, and mixed second partials are all continuous.
- write X_u=p, X_v=q, X_{uv}=r
- we can then exploit continuity conditions to obtain (here subscripts indicate the point at which the expression is evaluated)

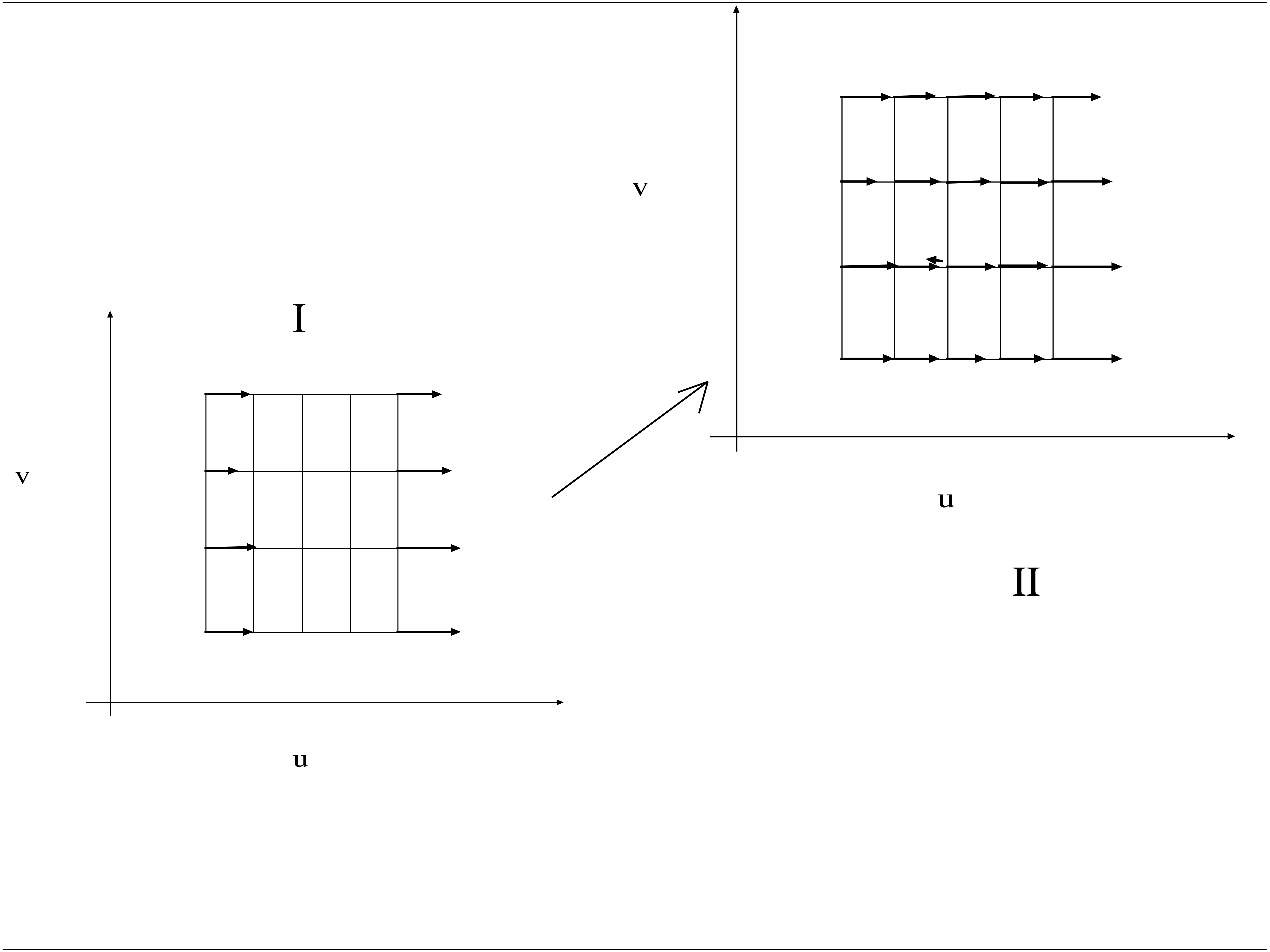
$$\underline{\underline{W}}_{ij} = \begin{pmatrix} \underline{P_{ij}} & \underline{q_{ij}} & \underline{P_{i,j+1}} & \underline{q_{i,j+1}} \\ \underline{p_{ij}} & \underline{r_{ij}} & \underline{p_{i,j+1}} & \underline{r_{i,j+1}} \\ \underline{P_{i+1,j}} & \underline{q_{i+1,j}} & \underline{P_{i+1,j+1}} & \underline{q_{i+1,j+1}} \\ \underline{p_{i+1,j}} & \underline{r_{i+1,j}} & \underline{p_{i+1,j+1}} & \underline{r_{i+1,j+1}} \end{pmatrix}$$

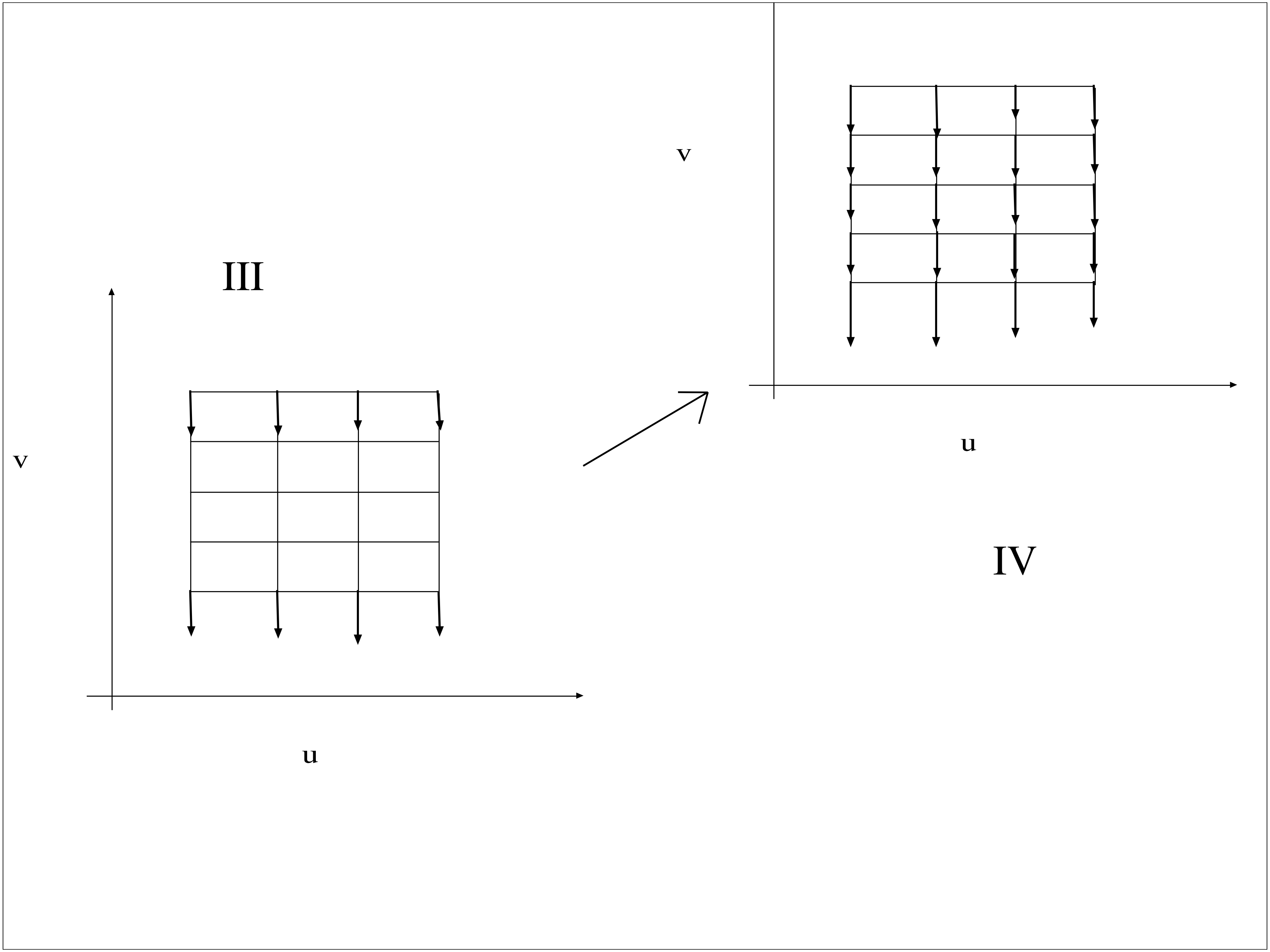
$$\underline{\underline{G}}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{pmatrix}$$

$$\underline{\underline{W}}_{ij} = \underline{\underline{G}}(\Delta u_i)\underline{\underline{A}}_{ij}\,\underline{\underline{G}}(\Delta v_i)$$

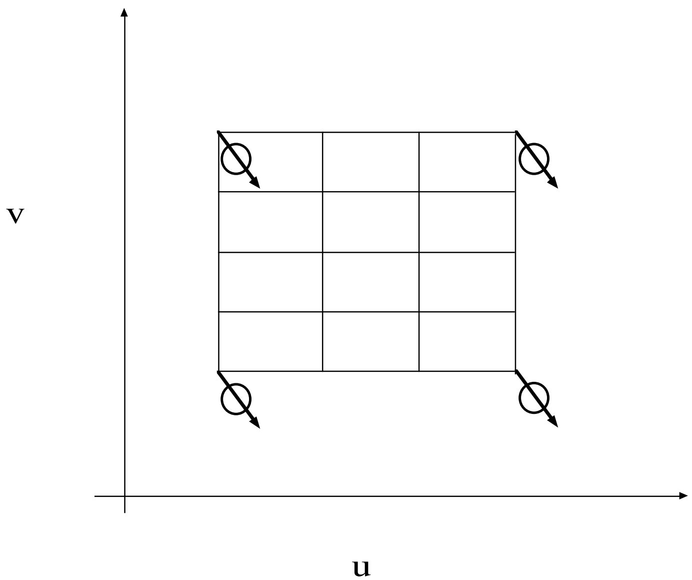
Bicubic Interpolate

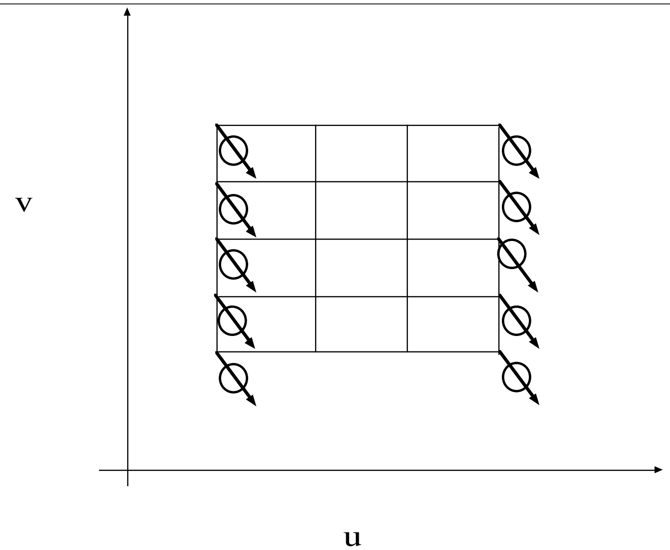
- Hence to construct surface, need only get first partials, mixed partials, at each vertex.
- These can not be freely chosen - they are constrained by the fact that, for fixed u, the surface is a cubic spline curve; ditto for fixed v.
- Hence, first partials in interior are constrained if those on boundary are known; ditto, mixed second partials.
- Estimate boundary first partial derivatives (e.g. using interpolate)
- Interior values for first partials follow from the fact that it's a cubic spline - recurrence relation on earlier slide.
- Notice that $q(u, v^*)$ is a cubic spline in u; ditto, $p(u^*, v)$ in v
- This means that, with estimates of mixed seconds in the corners, the mixed seconds at each grid point follow too.


I
v
u
v
u
II


III
v
u
v
u
IV

V





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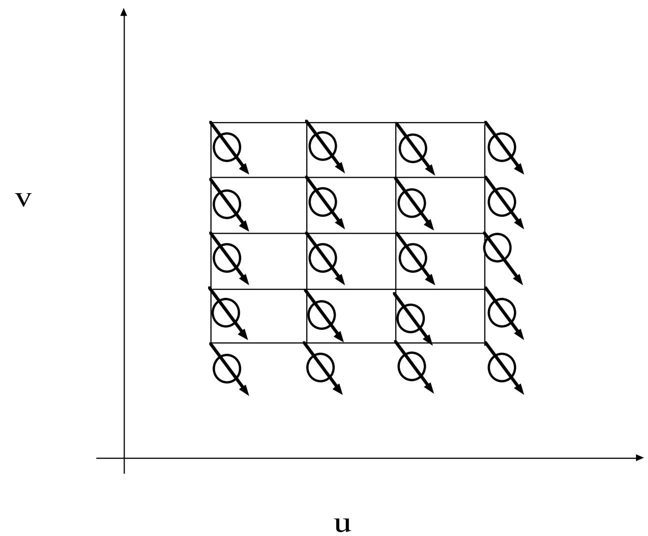

VII

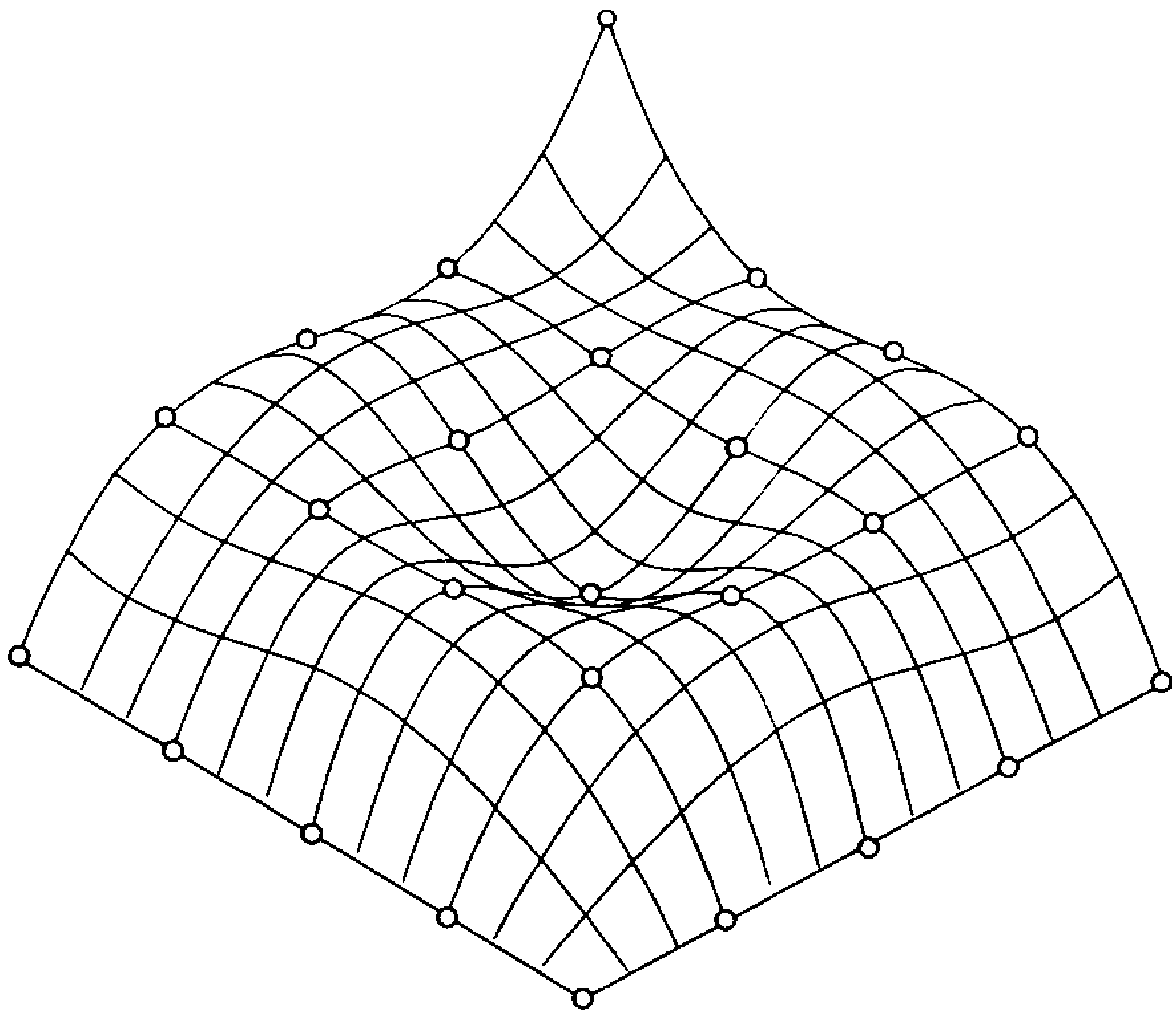

Fig. 6.2. Bicubic spline surface interpolating 5×5 points (circles).

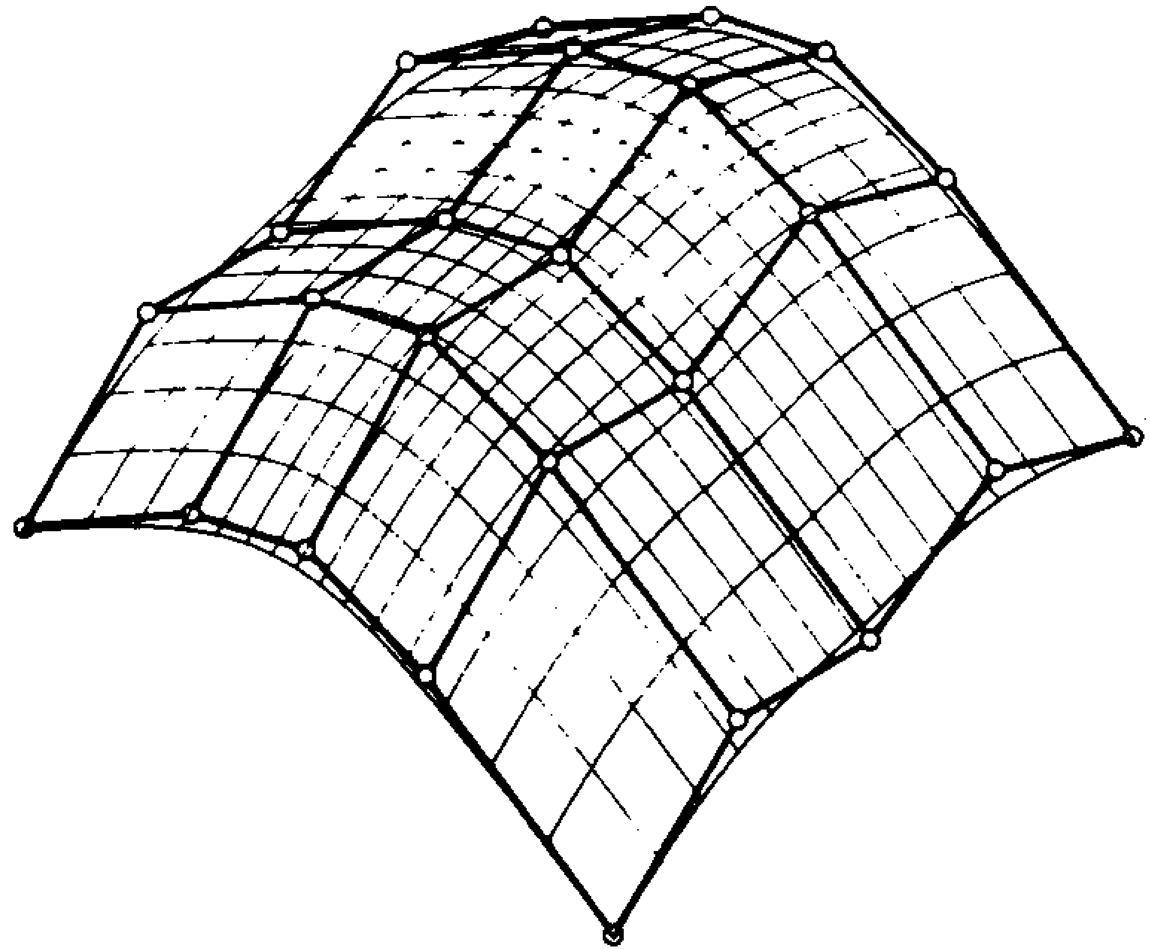

Fig. 6.13a. B-spline **surface** of order $k = 4$ **and** its de Boor net (**nonperiodic** basis **functions**).

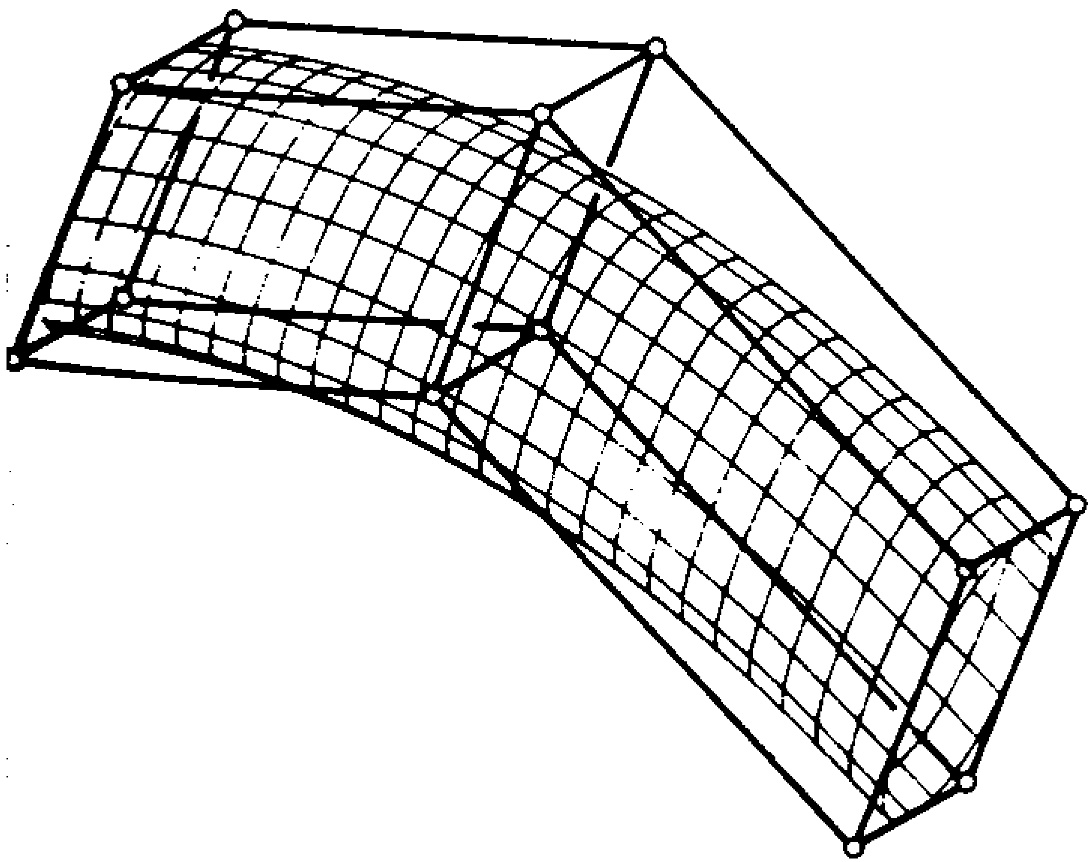

Fig. 6.13b. B-spline surface of order $k = 3$ with basis functions periodic in the u-direction.

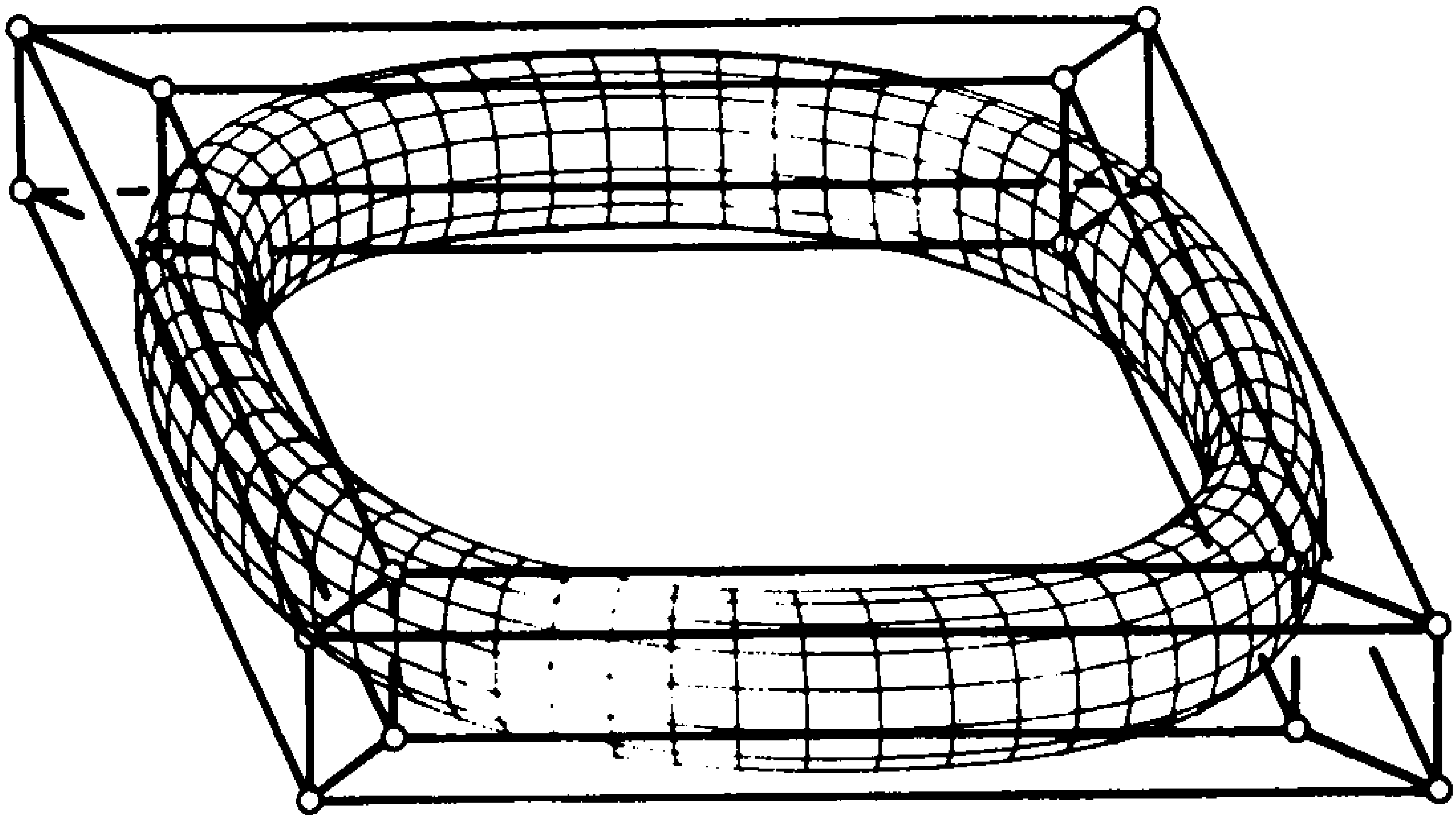

Fig. 6.13c. B-spline surface of order $k = 3$ with periodic basis functions in both the u- and v-directions.