Some Riemannian Geometry

Recall I claimed that one could compute Gaussian curvature from metric alone; we can work in terms of Christoffel symbols, etc.

→ recall also I gave an expression for C.S.'s in terms of metric \((I)\) and its derivative

→ we will generalize slightly

→ write \(g_{ij}\) for the \(i,j\)th component of the metric tensor

→ Symmetric, P.D. matrix, dimension \(d\), a function of coordinates \(x_1, \ldots, x_d\) (in the case of surfaces, \((x_1, x_2)\) are the parameters we discussed, \([g_{ij}] \cong I\) (first fff))

→ at each point of domain \(U \subset \mathbb{R}^d\), we have a Tangent vector space, spanned by \(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\)
We interpret $g_{ij}$ by considering $a, b \in T_p(U)$

$$\sum_{i,j} a_i \frac{\partial}{\partial x_i} \cdot \sum_{j} b_j \frac{\partial}{\partial x_j}$$

tangent space at $p$ of $U$.

Then

$$\langle a, b \rangle = \sum_{i,j} a_i \cdot b_j \cdot g_{ij}$$

is dot product of $a$ and $b$.

So $\int_{y}^{z} g \circ g$ is like $I$ -- a metric.

$$\int_{x}^{y} \left( \frac{2c}{dt} \cdot \frac{2c}{dt} \right)^{1/2} dt$$

is length of $C(t)$, from $t=e$ to $t=f$.

$$\int_{V \cup U} \det(g) \, dx \cdot dx_2 \cdot dx_4$$

is volume of $V$ (area).
Examples: (2D, for now)

1. \( g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)
2. \( g_{ij} = \begin{pmatrix} \frac{1}{x_1^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{pmatrix} \) for \( x_2 > 0 \)

3. \( g_{ij} = \begin{pmatrix} 4 \left( \frac{1 + x_1^2 + x_2^2}{1 + x_1^2 + x_2^2} \right) & 0 \\ 0 & 4 \left( \frac{1 + x_1^2 + x_2^2}{1 + x_1^2 + x_2^2} \right) \end{pmatrix} \) for \( x_1, x_2 \in \mathbb{R}^2 \)

4. \( g_{ij} = \begin{pmatrix} \cos^2 x_2 & 0 \\ 0 & 1 \end{pmatrix} \) for \( x_2 \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \)

- Notice these are symmetric 2D, but what are they?
- We can compute Gaussian curvature.

\[
K = -\frac{1}{g_{11}} \left[ \frac{\partial^2}{\partial x_1^2} \Gamma_{12}^2 - \frac{\partial^2}{\partial x_2^2} \Gamma_{12}^2 + \Gamma_{12}^1 \Gamma_{12}^2 - \Gamma_{12}^1 \Gamma_{12}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{12}^2 \Gamma_{12}^2 \right]
\]

(you could get this from derivations earlier in notes—
but notice (a) something of this sort must be true for surfaces because \( K \) is intrinsic AND 1-gene expression for C.S.'s in terms of I

(b) easily looked up)
recall I gave expressions for Christoffel symbols earlier.

case 0 \[ \Gamma^k_{ij} = 0 \quad ; \quad K = 0 \]

case 2\[ \begin{align*}
\Gamma^1_{11} &= 0 \\
\Gamma^1_{22} &= 1/x^2 \\
\Gamma^2_{12} &= -1/x^2 \\
\Gamma^1_{12} &= 0 \\
\Gamma^2_{11} &= 0 \\
\Gamma^2_{22} &= -1/x^2 \\
\end{align*} \]

case 3 should yield \( K = 1 \)

case 4 should also yield \( K = 1 \)

Now: cases 3, 4 suggest we have a problem.

- They appear to be spheres, but there are topological issues - case 1 is missing a point,
- case 2 at least 2 points.
This motivates the notion of a manifold.

Let \( M \) be a Hausdorff topological space necessary, but not really an issue in our lives.

A \( C^p \) atlas on \( M \) is given by:

- an open cover \( \{ U_i \} \) of \( M \)
- a family of homeomorphisms \( \phi_i : U_i \to \Omega_i \subset \mathbb{R}^n \), where \( \Omega_i \) open.

Such that for \( i, j \in I \)

\[ \phi_j \circ \phi_i^{-1} \text{ is } C^p \text{ diffeomorphism from } \phi_i(U_i \cap U_j) \text{ onto } \phi_j(U_i \cap U_j) \]

- we call \( U_i, \phi_i \) charts
Notice: it doesn't really matter which way \( \phi_g \circ \phi_i \) or \( \phi_i \circ \phi_g \) there are diffeo's the other way (i.e. \( \mathbb{R}^n \rightarrow U_i \)) which we can think of as coordinates on \( U_i \).

Notice: the big issue here is getting overlapping charts to be consistent.

- Use a overlap, we must be able to label points consistently.
Construction (stereographic proj.) projects every point on sphere (except \((0,0,1)\)) to a pt on plane.

**Exercise**

- what is this in coords?
- show case 3 is the metric in this proj.

This is one chart; the \( U \) is the \( \mathbb{R}^2 \) and the \( q \) is map as drama.

**Exercise:** how do we build a second chart?
- show they're consistent.
Case 4 is more interesting:

notice \( (0) \) \((-1) \) are bad, because map to \( \mathbb{C} \mathbb{R}^2 \) looks like

\[
\begin{align*}
\theta &= \text{atan2}(y, x) \\
\varphi &= \sin z
\end{align*}
\]

but when \( y = x = 0 \), any \( \theta \) is \( \theta \) OK.

Not a continuous map. Notice also \( \text{atan2} \) is only \( 1-1 \) if we restrict range, eg \( [0, 2\pi) \)

\[
\begin{align*}
\mathbf{U} &= (-\pi, 0) \times (-\pi/2, \pi/2) \\
\phi &\text{ is as above}
\end{align*}
\]
and we've omitted a "seam" of points

Ex: supply 2 other $U_i$, $\varphi_i$ to fix

Now all this defn works for $n > 2$ as well.

Some examples

- all matrices $d \times d$, st. $\det M \neq 0$
  - $\text{GL}(d)$, general linear group.
  
  Ex: what are good $U$, $\varphi$?

- all $d \times d$ matrices $M$ st
  
  $\det M = 1$
  $M^T M = 1d$

  Ex: for $d = 2$, what are good $U$, $\varphi$?
  
  $d = 3$, 

$\begin{cases}
\text{SO}(d) \\
\text{proper rotations,}
\text{special orthogonal group}
\end{cases}$
The circle on the plane

\[ \varphi_1 = \text{atan2}(y, x) \]

\[ \varphi_2 = \text{atan2}(x, y) \]

with constraints as above

\[ x^2 + y^2 = 1 \]
\[ x \in (-1, 1) \quad \text{open at this end} \]
\[ y \in [-1, 1] \quad \text{diff to} \]

Affine Group

\[ GL(n) \times \mathbb{R}^n \]

ex: \( U_e, \varphi_e \)

Real projective line: \( \mathbb{RP}^1 \)

This is important; it encodes lines through origin

Take \( (x, y) \in \mathbb{R}^2 - (0,0) \)

Identify \( (x, y) \) and \( \lambda (x, y) \) for \( \lambda \neq 0 \)
equivalent

\[ \phi_1 : (x, y) \rightarrow (x, y, 1) \]
for \( y \neq 0 \)

\[ \phi_2 : (x, y) \rightarrow (1, \frac{y}{x}) \]
for \( x \neq 0 \)

now there's only 1 coordinate in \( U_1 \), write \( \mathbb{Z}_1 \)

\[ U_2 \leftrightarrow \mathbb{Z}_2 \]
\[ \phi_2^{-1}: \mathbb{Z}_2 \to \ell(1, \mathbb{Z}_2) \]

Ray through pt in \( U_2 \)

\[ \phi \circ \phi^{-1} = \left( \frac{1}{z_2} \right) \quad z_2 \neq 0 \]

\[ \phi \circ \phi^{-1} = \frac{1}{z_1} \quad z_1 \neq 0 \]

so \( U_1 = \text{line pasted to } U_2 \text{ everywhere except at } 1\text{ pt } U_1 \),

\( U_2 = \text{line } \)

\( z_1 = 0 \) not in image of paste, but arbitrarily close

"paste"

Another way to think of this - we've attached a pt at \( \infty \) to line.
Real projective plane: $\mathbb{RP}^2$

- Important in camera, multiview geom
- $(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus (0,0,0)$
- Identify $(x_1, x_2, x_3)$ with $\lambda(x_1, x_2, x_3), \lambda \neq 0$
- Equivalently, all lines through origin

Each line pierces at least one of these planes
\( \varphi_{13} : (x_1, x_2, x_3) \rightarrow \left( \frac{x_1}{x_3}, \frac{x_2}{x_3}, 1 \right) \) for \( x_3 \neq 0 \)

\( \varphi_1, \varphi_2 \) follow leads us to:

\[
\begin{align*}
U_1 &= (a, b) \in \mathbb{R}^2 \\
U_2 &= (c, d) \\
U_3 &= (e, f) \\
\end{align*}
\]

\( \varphi_1 \circ \varphi_2^{-1} = \left( \frac{1}{c}, \frac{1}{c} \right) \) \text{ this pastes } U_1, U_2 \\
\text{together along } \\
\{U_2 - (a)\} \]

Notice I could build the whole thing with \( U \)'s and transitions.
real projective space, dimension $n$:

$\mathbb{RP}^n$ — all lines through origin in $\mathbb{R}^{n+1}$

Moebius band:

$\phi_1 \circ \phi_2^{-1} (\alpha, \beta) = (\frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta})$

$\phi_1 \circ \phi_2^{-1} (\frac{1}{r}, \frac{s}{r}) = (\frac{1}{\alpha}, \frac{s}{\alpha})$

$\beta \in (0, 1)$  Notice similarity to $\mathbb{RP}^1$

$s \in (0, 1)$  $\alpha \neq 0$

$cylinder, open at top, bottom, identity opposite points.
**EX:** $\mathbb{RP}^2$ can be obtained by pasting edge of disk to Moebius band (takes some thought).

Notice $J = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$ so $\det J < 0$ for $x_0$

Similar for $\varphi_2 \circ \varphi_1^{-1}$; we'll use this

The tangent space to a manifold:

- at each point $p$ in $M$, there is a tangent at $p$ space
- take a vector in
We can place functions on these objects straightforward, but must be consistent on \( \phi \)'s.

\[ \phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4 \]

\[ \phi_1 \circ \phi_2 = \phi_3 \circ \phi_4 \]

\[ f_2 : U_2 \rightarrow \text{Whatever} \]

\[ U_1 \quad f_1 : U_1 \rightarrow \text{Whatever} \]

must have

\[ f_1 \text{ on } \phi_1(M) = f_2 \circ (\phi_2 \circ \phi_1^{-1}) \text{ on } \phi_1(M) \]

Notice \( M \) is rather disappearing here — the real issue is the \( U \)'s, and \( \phi_2 \circ \phi_1^{-1} \).
We can place tangent vectors on $M$, too.

At each point $p$ of $M$, there is a tangent space (spanned by $\left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots \right)$).

A vector at $p$ is given by $u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + \ldots$

Interpret this as $\mathcal{J}_{\Phi^{-1}}(u)$ on $M$.

More interesting is a tangent vector field.

A smooth $f$ on $U_i$, consistent as above.

Ex: check that this gives a unique vector on $\cap V_i$.
More examples

The tangent vector space of a manifold $T(M)$ is all tangent vectors at every pt of $M$.

For $T(M)$ the $M$ has $U_i, \varphi_i$

$T(M)$ \ \ \ \ \ \ \ \ V_i, \psi_i$

$V_i = U_i \times \mathbb{R}^n$

$\psi_i = \varphi_i \times 1d$. 
The space of directed lines in 2D

- circle at origin gives dir
- point on tangent line gives translation $T(s')$

The space of undirected lines in 2D

$T(p')$

under directed " 3D

$T(S^2)$ (A D.)

under undirected " 3D

$T(P^2)$
Now we can (a) place a metric tensor on these objects (b) do geometry with it.

g_{ij} = \begin{pmatrix}
\frac{1}{x^2} & 0 \\
0 & \frac{1}{x^2}
\end{pmatrix}, \quad x_2 > 0

**Geodesics?** NOT all straight lines

Notice that to get from \((k, \varepsilon)\) to \((-k, \varepsilon)\) it's probably a good idea to go up a bit.

In this metric, distance from \((k, a)\) to \((-k, a)\) along a horizontal path is:

\[
\int_{-k}^{k} \sqrt{\left(\frac{x'}{y} \right)^2 + \left(\frac{y'}{y} \right)^2} \, dt = \frac{2k}{a}
\]

Big if \(a\) is small.
Alternative

path length: \(2 \times \text{length of segment } 01\)

\[= 2 \times \frac{K}{b}\]

because translating a vertical vector doesn't change its length

length of 01 = length of \([ (K, t) \text{ for } t \in (a, b) ] \)

\[= \int_a^b \sqrt{\left( \frac{d}{b} \right)^2 + \left( \frac{b}{y} \right)^2} \, dt = \int_a^b \frac{1}{y} \, dt = \log_b - \log_a\]

So path length is \(2 \log_b - 2 \log_a + 2 \frac{K}{b}\)

this is minimized when \(b = K\).

So: to go across, you should head in!
But by how much?

Geodesics are solutions to
\[ y_\dd = -2xy_1 = 0 \]
\[ y_\dd^2 + x^2 - y^2 = 0 \]

Should look easy to you:

**EX** check my calculation

**EX**: \( (r \sech t, r \tanh t) \) is a geodesic

**EX**: \( (0, e^t) \) is a geodesic

**EX**: if \( (x(t), y(t)) \) is a geodesic, then so is \( (x(t) + c, y(t)) \)

We now have all geodesics. (Notice each pt, dirn, identifies a geodesic.)