Some Riemannian Geometry

Recall I claimed that one could compute Gaussian curvature from metric alone; we can work in terms of Christoffel symbols, etc.

→ recall also I gave an expression for C.S.'s in terms of metric (I) and it's derivative

→ we will generalize slightly.

- write g_{ij} for the i,j th component of the metric tensor

- Symmetric, P.D. matrix, dimension d, a function of coordinates $x_1 \ldots x_d$ (in the case of surfaces, (x_1, x_2) are the parameters we discussed, $[g_{ij}]$ is I (first ff))

- at each point of domain $U \subset \mathbb{R}^d$, we have a Tangent vector space,

spanned by $\left\{ \frac{\partial}{\partial x_1} \quad \cdots \quad \frac{\partial}{\partial x_d} \right\}$

We interpret g_{ij} by.

consider $\underline{a}, \underline{b} \in T_p(U)$ ← tangent space at p of U.

$\underline{a} = \sum_i a_i \frac{\partial}{\partial x_i}$, $\underline{b} = \sum_j b_j \frac{\partial}{\partial x_j}$

then $\langle \underline{a}, \underline{b} \rangle = \sum_{ij} a_i b_j g_{ij}$

is dot product of $\underline{a}$ and $\underline{b}$

So $[g_{ij}] = g$ is like I — a metric.

$$\int_{e}^{f} \left\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial t} \right\rangle^{1/2} dt$$ is length of $c(t)$, from $t=e$ to $t=f$

$$\int_{V \subset U} \det(g)\, dx_1\, dx_2 \cdot dx_d$$ is volume of V (area)

~~Christo~~

Examples: (2D for now)

① $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ② $g_{ij} = \begin{pmatrix} \frac{1}{x_2^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{pmatrix}$ for $x_2 > 0$

③ $g_{ij} = \begin{pmatrix} \frac{4}{1+x_1^2+x_2^2} & 0 \\ 0 & \frac{4}{1+x_1^2+x_2^2} \end{pmatrix}$ $x_1, x_2 \in \mathbb{R}^2$

④ $g_{ij} = \begin{pmatrix} \cos^2 x_2 & 0 \\ 0 & 1 \end{pmatrix}$ $x_1 \in [0, 2\pi)$ $x_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

closed open.

→ notice these are symmetric, PD; but what are they?

→ We can compute Gaussian curvature.

$$K = -\frac{1}{g_{11}}\left[\frac{\partial}{\partial x_1}\Gamma^2_{12} - \frac{\partial}{\partial x_2}\Gamma^2_{11} + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\right]$$

(you could get this from derivations earlier in notes —
but notice (a) something of this sort must be true for surfaces because K is intrinsic AND I gave expression for C.S.'s in terms of I
(b) easily looked up
)

recall I gave expressions for Christoffel symbols earlier.

case 1 all $\Gamma_{ij}^{k} = 0$; $K = 0$

case 2
$$\Gamma_{11}^{1} = 0,\quad \Gamma_{11}^{2} = \frac{1}{x_2},\quad \Gamma_{12}^{1} = -\frac{1}{x_2},\quad \Gamma_{12}^{2} = 0,\quad \Gamma_{22}^{1} = 0,\quad \Gamma_{22}^{2} = -\frac{1}{x_2} \quad ; \quad K = -1$$

case 3 should yield $K = 1$

case 4 should also yield $K = 1$

<u>Now</u>: cases 3, 4 suggest we have a <u>problem</u>.

∴ they appear to be spheres, <u>but</u> there are topological issues — case 1 is missing a point, case 2 at least 2 points.

This motivates the notion of a <u>manifold</u>

let M be a <u>Hausdorff</u> topological space

necessary, but not really an issue in our lives.

a C^p atlas on M is given by:

- an open cover U_i $i \in I$ of M
- a family of homeomorphisms

$$\phi_i : U_i \to \Omega_i \subset \mathbb{R}^n \quad , \ \Omega_i \text{ open}.$$

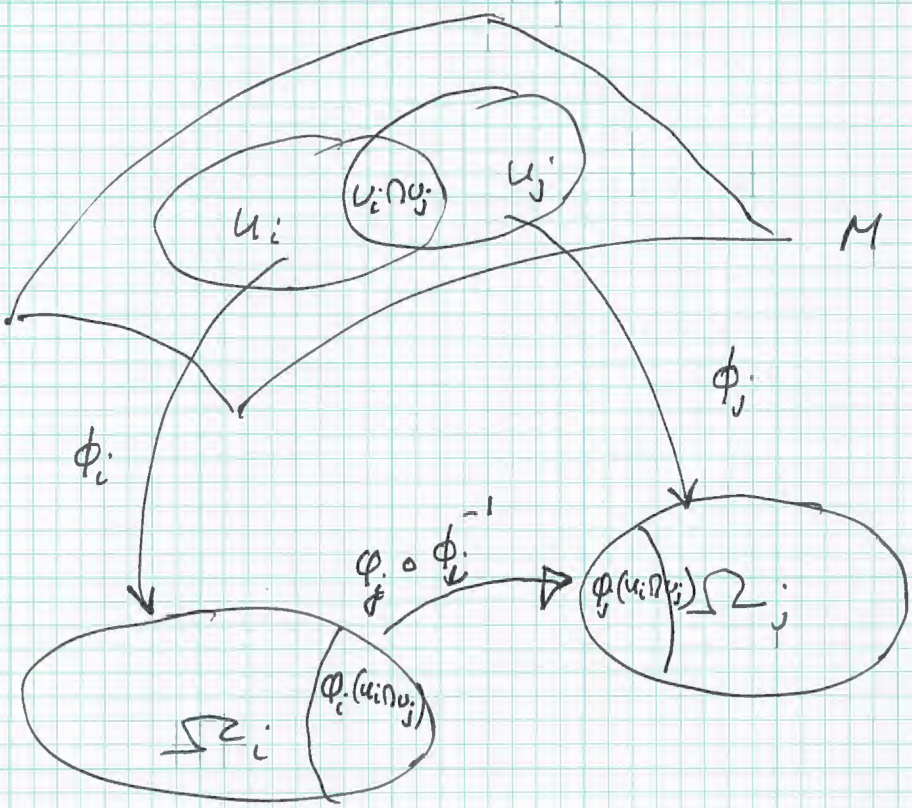
same n.

Such that for $i, j \in I$

$\phi_j \circ \phi_i^{-1}$ is C^p diffeomorphism from

$$\phi_i(U_i \cap U_j) \quad \text{onto} \quad \phi_j(U_i \cap U_j)$$

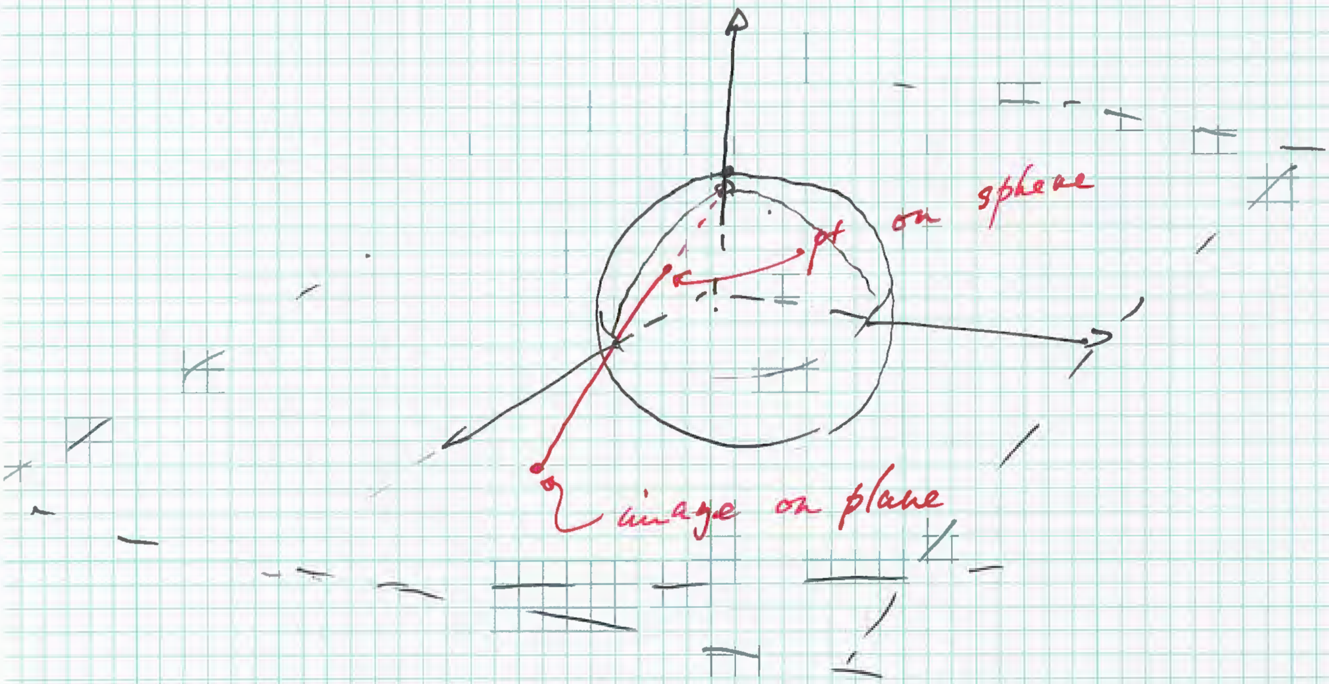
- we call U_i, ϕ_i <u>charts</u>



<u>Notice</u>: it doesn't really matter which way ϕ_i go — there are defn's the other way (i.e. $\Omega_i \to U_i$) which we can think of as coordinates on M

<u>Notice</u>: the big issue here is getting overlapping charts to be consistent

— in overlap, we must be able to label points consistently eg sphere



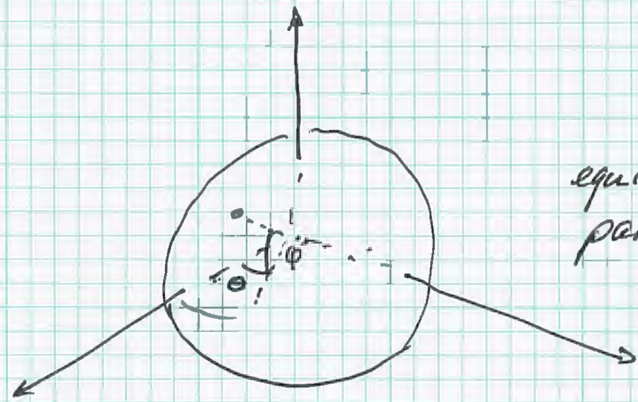
construction (stereographic proj'n) projects every point on sphere (except $(0,0,1)$) to a pt on plane.

Exercise
- what is this in coords?
- show case 3 is the metric in this proj?

this is one chart, the U is ~~the~~ $\mathbb{R}^2$ and the φ is map as drawn.

Exercise:
- how do we build a second chart?
- Show they're consistent.

Case 4 is more interesting:



equiv to param by.
$$\begin{pmatrix} \cos\varphi \cos\theta \\ \cos\varphi \sin\theta \\ \sin\varphi \end{pmatrix}$$

notice $\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\begin{pmatrix}0\\0\\-1\end{pmatrix}$ are bad, because

map to $U \subset \mathbb{R}^2$ looks like

$$\theta = \operatorname{atan2}(y, x)$$
$$\varphi = \operatorname{asin} z$$

but when $y = x = 0$, any θ is OK.

$\therefore$ not a continuous map.

notice also atan2 is only 1-1 if we restrict range, eg $[0, 2\pi)$

but closed at this end

So $U = (0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

ϕ is as above

and we've omitted a "seam" of points

Ex: • supply 2 other u_i, φ_i to fix

Now all this ~~a~~ defn works for $n > 2$ as well.

Some examples

- all matrices ~~d~~, $d \times d$, s.t. $\det M \neq 0$
  $Gl(d)$, general linear group.

  Ex what are good u, φ ?

- all $d \times d$ matrices ~~M~~ M st
  $\det M = 1$
  $M^T M = Id$

  $\leftarrow SO(d)$
  proper rotations; special orthogonal group

  Ex: for $d = 2$, what are good u, φ?
  $d = 3$, . . .

- The circle on the plane

$$\varphi_1 = \mathrm{atan2}(y, x)$$

$x^2 + y^2 = 1$,
$x \in (-1, 1]$
open at this end

1-1 with constraints as above

$$\varphi_2 = \mathrm{atan2}(x, y)$$

$x^2 + y^2 = 1$
~~$x \in ($~~
$y \in (-1, 1]$
ditto

- Affine Group

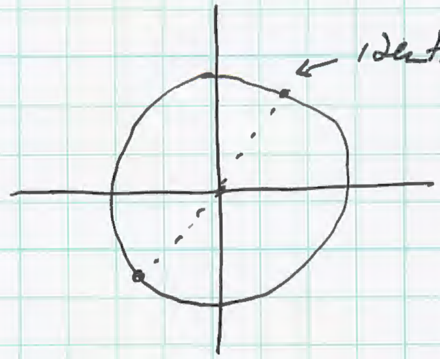
$$GL(n) \times \mathbb{R}^n$$

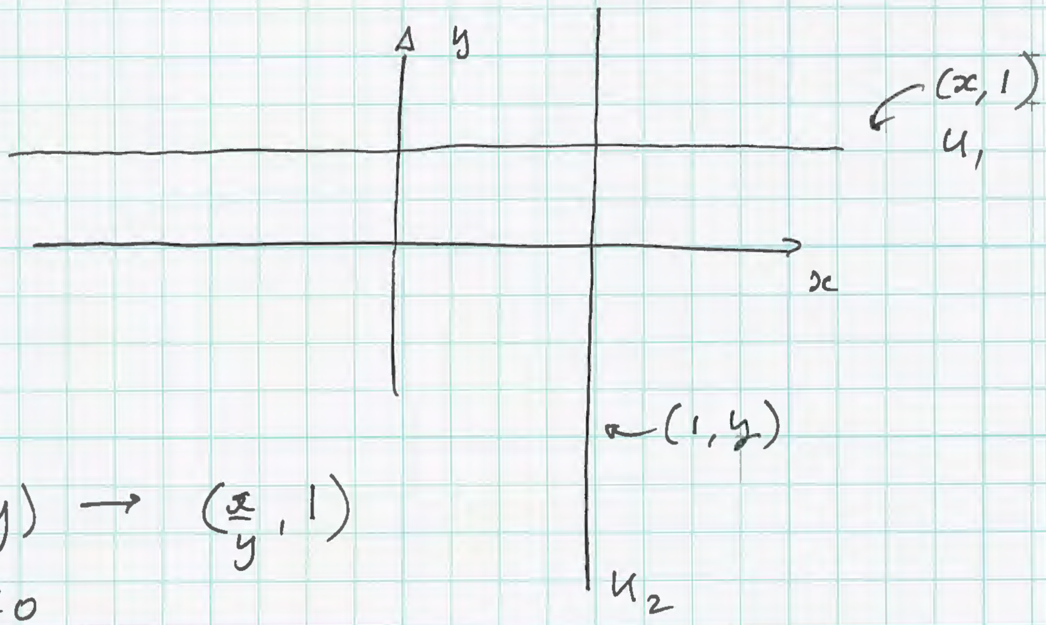
<u>EX</u>: u_i, φ_i ?

- <u>Real projective line</u> : $\mathbb{R}P^1$
  - this is important, it encodes lines thru origin
  - Take $(x, y) \in \mathbb{R}^2 - (0,0)$ ~~where~~
  - identify (x, y) and $\lambda(x, y)$ for $\lambda \neq 0$

equivalent

identify opposite sides of circle





$\phi_1 : (x,y) \longrightarrow \left(\frac{x}{y}, 1\right)$

for $y \neq 0$

$\phi_2 : (x,y) \longrightarrow \left(1, \frac{y}{x}\right)$

for $x \neq 0$

now theres only 1 coordinate in u_1, write z_1

" u_2 " z_2

$\phi_2^{-1} : z_2 \longrightarrow \underbrace{t(1, z_2)}_{\text{ray through pts in } U_2}$

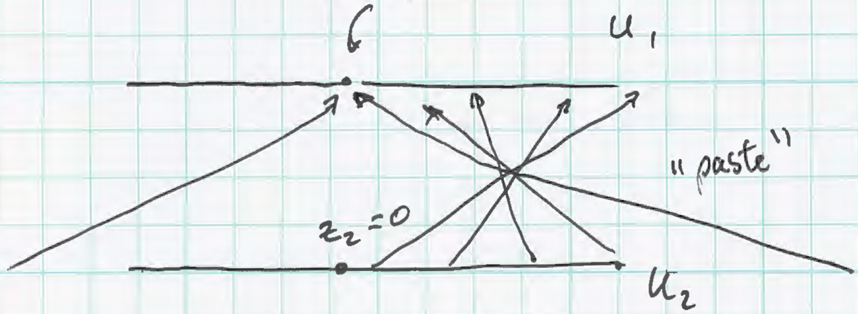
$\phi_1 \circ \phi_2^{-1} = \left(\frac{1}{z_2}\right) \qquad z_2 \neq 0$

$\phi_2 \circ \phi_1^{-1} = \frac{1}{z_1} \qquad z_1 \neq 0$

So U_1 = line pasted to U_2 everywhere except at 1 pt

U_2 = line " U_1

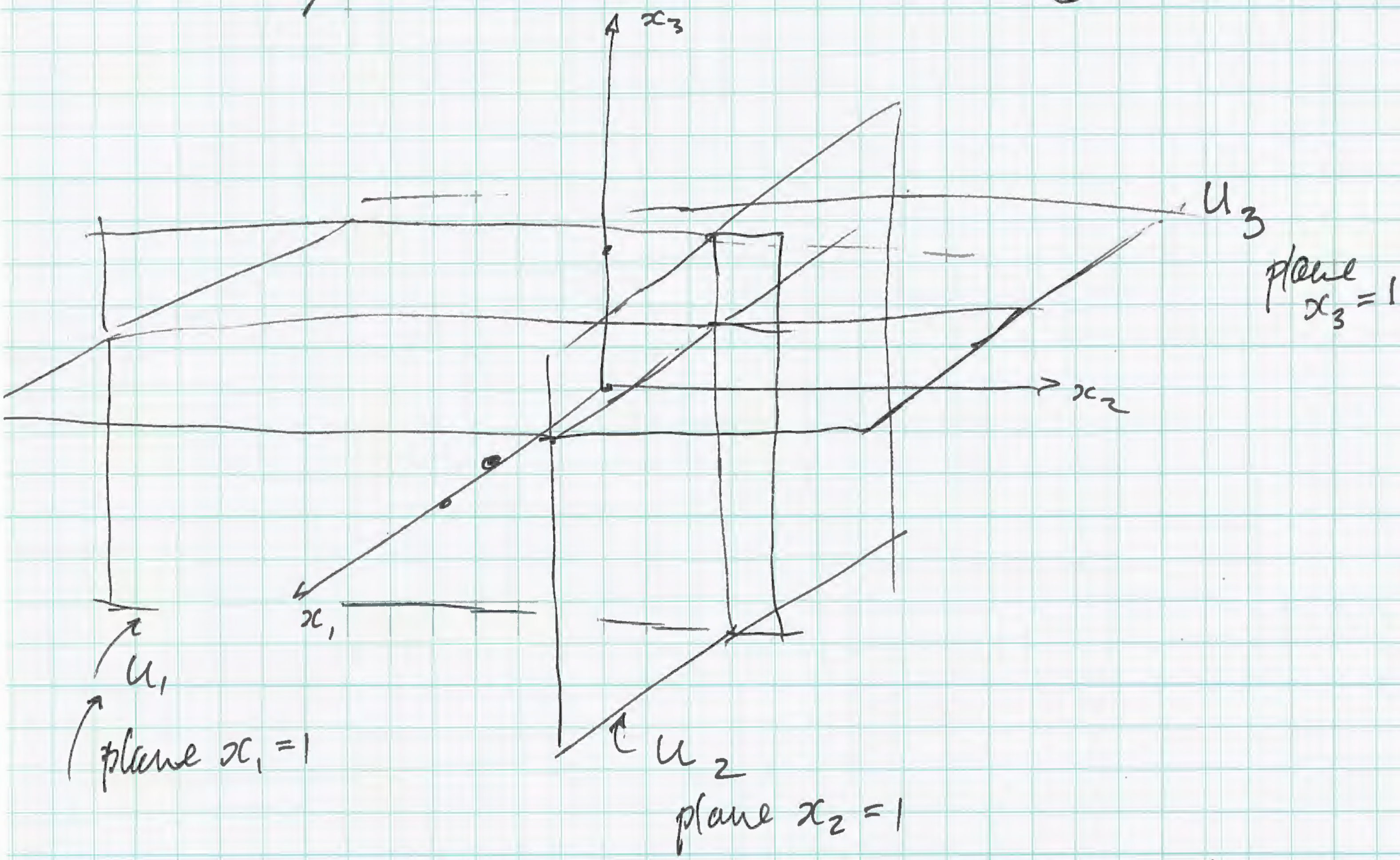
$z_1 = 0$ not in image of paste, but arbitrarily close



Another way to think of this – we've attached a pt at ∞ to line.

Real projective plane: $\mathbb{R}P^2$

- important in camera, multiview geom
- $(x_1, x_2, x_3) \in \mathbb{R}^3 - (0,0,0)$
- identify (x_1, x_2, x_3) with $\lambda(x_1, x_2, x_3)$, $\lambda \neq 0$
- equivalently, all lines through origin



- each line pierces at least one of these planes

$$\varphi_3 : (x_1, x_2, x_3) \longrightarrow \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right) \quad \text{for } x_3 \neq 0$$

φ_1, φ_2 follow

leads us to:

$$U_1 = (a, b) \in \mathbb{R}^2$$
$$U_2 = (c, d)$$
$$U_3 = (e, f)$$

$$\varphi_1 \circ \varphi_2^{-1} = \left(\frac{1}{c}, \frac{d}{c}\right)$$

: this pastes U_1, U_2 together along $\left\{U_2 - \binom{0}{d}\right\}$

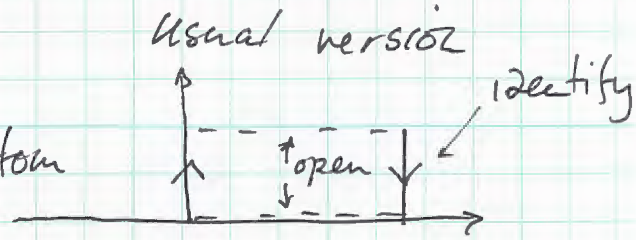
etc

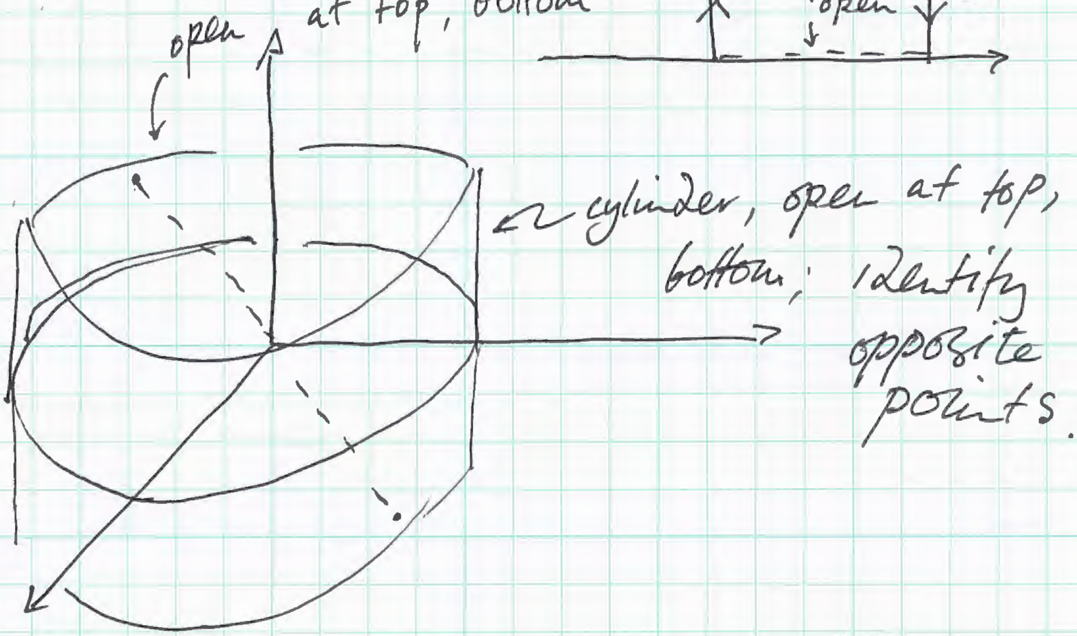
Notice I could build the whole thing with U's and transitions.

real projective space, dimension n:

$\mathbb{RP}^n$ – all lines through origin in $\mathbb{R}^{n+1}$

Moebius band:





$U_1 = (1, \alpha, \beta)$ $\quad \beta \in (0,1)$

$U_2 = (\gamma, 1, \delta)$ $\quad \delta \in (0,1)$

Notice similarity to $\mathbb{RP}^1$

$\phi_2 \circ \phi_1^{-1}: \left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right)$ $\quad \alpha \neq 0$

$\phi_1 \circ \phi_2^{-1} \; \left(\frac{1}{\gamma}, \frac{\delta}{\gamma}\right)$ $\quad \delta \neq 0$

EX: $\mathbb{R}P^2$ can be obtained by pasting edge of disk to Moebius band (takes some thought).

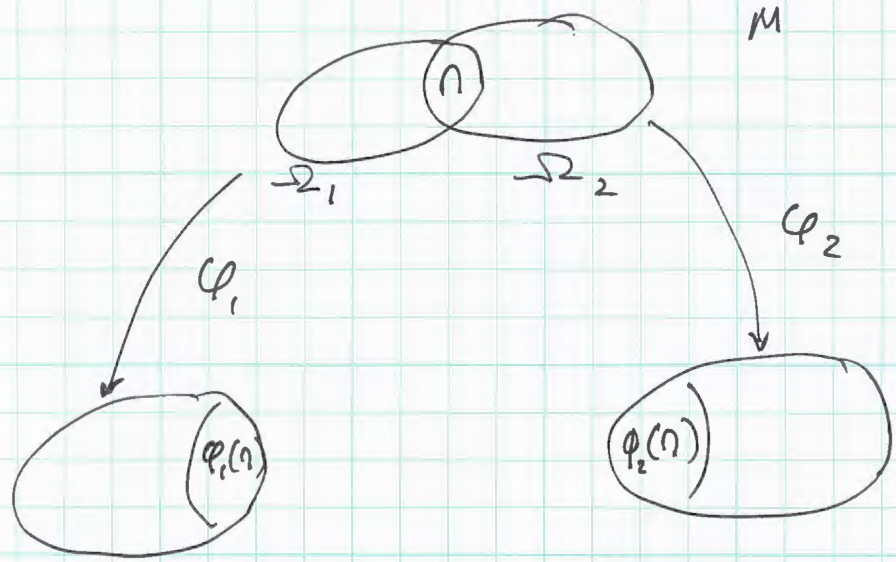
notice $J_{\varphi_2 \circ \varphi_1^{-1}} = \begin{bmatrix} \frac{-1}{\alpha^2} & -\beta/\alpha^2 \\ 0 & \frac{1}{\alpha} \end{bmatrix}$ so $\det J \le 0$ for $\alpha \le 0$

similar for $\varphi_2 \circ \varphi_1^{-1}$; we'll use this

~~The tangent space to a manifold~~. :

- ~~at each point p in U_i, there is~~ .
  ~~a tangent plan space~~
- ~~take a Vector in~~

• We can place functions on these objects straightforward, but <u>must</u> be consistent on $\cap$'s



$f_2 : U_2 \to$ whatever

U_1 $\quad f_1 : U_1 \to$ whatever

must have

$$\cancel{f_1 \text{ on } \phi_1(\cap) = f_2 \circ \left(\varphi_2 \circ \varphi_1^{-1}\right) \text{ on } \varphi}$$

$$f_1 \text{ on } \phi_1(\cap) = f_2 \circ \left(\varphi_2 \circ \varphi_1^{-1}\right) \text{ on } \varphi_1(\cap)$$

Notice M is rather disappearing here — the real issue is the U's, and $\varphi_i \circ \varphi_j^{-1}$

- We can place tangent vectors on M, too
  - at each point p, of ~~φ_i~~ U_i there is a tangent space (spanned by $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\right)$)
  - a vector $\underline{u}$ at p is given by

$$u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} \ldots$$

  - interpret this as $J_{\varphi_i^{-1}}(\underline{u})$ on M
- More interesting is a <u>tangent vector field</u>.
- n smooth fns on U_i, consistent as above

<u>EX</u>: Check that this gives a unique vector on $\cap$'s.

More examples

· The tangent vector space of a manifold

$T(M)$ $\leftarrow$ all tangent vectors at every pt of M

~~for $T(M)$ the~~

M has $\mathcal{U}_i$, φ_i

$T(M)$ ·· V_i, Ψ_i

$$V_i = \mathcal{U}_i \times \mathbb{R}^n$$

$$\Psi_i = \varphi_i \times \mathrm{Id}.$$

·

. The space of directed lines in 2D

· circle at origin gives dirn
· point on tangent line gives translation

$T(S^1)$

. The space of undirected lines in 2D

$T(P^1)$

. " ~~undir~~ directed " 3D

$T(S^2)$ (4 D.!)

. " undirected " 3D

$T(P^2)$

Now we can (a) place a metric tensor on these objects (b) do geometry with it.

geometry, example ~~#~~ 2.

$$g_{ij} = \begin{pmatrix} \frac{1}{x_2^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{pmatrix} \qquad x_2 > 0$$

geodesics? NOT all straight lines

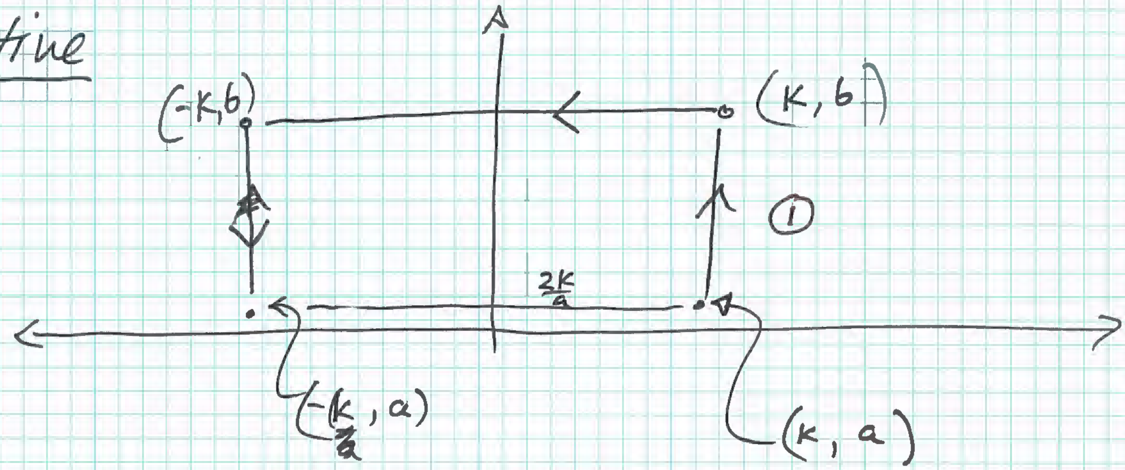
notice that to get from (k, ε) to $(-k, \varepsilon)$ it's probably a good idea to go up a bit

- In this metric, distance from (k, a) to $(-k, a)$ along a horizontal path $\begin{matrix} x(t) \\ y(t) \end{matrix} = \begin{pmatrix} t \\ a \end{pmatrix}$

$$\int_{-k}^{k} \sqrt{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}^T \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}} \, dt = \frac{2k}{a}$$

big if a is small

Alternative



path length: $2 \times$ length of segment 1

$+ \; 2\frac{K}{b}$

because translating a vertical vector doesn't change its length

length of 1 $=$ length of $\left[(K,t) \text{ for } t \in (a,b)\right]$

$$= \int_a^b \sqrt{\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}^T \begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}}\, dt = \int_a^b \frac{1}{t}\, dt = \log b - \log a$$

So path length is $2\log b - 2\log a + 2\frac{K}{b}$

this is minimized when $b = K$.

So: to go across, you should head in!

But by how much?

Geodesics are solutions to

$$y\ddot{x} - 2\dot{x}\dot{y} = 0$$
$$y\ddot{y} + \dot{x}^2 - \dot{y}^2 = 0$$

Should look nasty to you; EX check my calculation

EX: $\begin{pmatrix} -r\tanh t \\ r\,\mathrm{sech}\, t \end{pmatrix}$ is a geodesic

EX: $\begin{pmatrix} 0 \\ e^t \end{pmatrix}$ is a geodesic

EX: if $(x(t), y(t))$ is a geodesic, then so is $(x(t)+c, y(t))$

We now have all geodesics (notice each pt, dirn, identifies a geodesic)