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Metric equivalence of surfaces

It is evident that any finite part whatever of the curved surface will retain the same integral curvature after development upon another surface.

C. F. Gauss (8 October 1827)

The problem of constructing useful flat maps of the Earth led to the mathematical question of the existence of an ideal map projection (Chapter 8^{bis}), that is, a mapping $Y\colon (R \subset S^2) \to \mathbb{R}^2$ preserving lengths, angles, and area. The archaic terminology for such a map projection is a *development* of the sphere onto the plane. Euler showed that no such development can exist (Theorem 8^{bis}.1). A natural generalization of this problem is to determine whether there is a development of a given surface onto another.

A diffeomorphism between two surfaces, $\phi\colon S_1 \to S_2$, identifies the analytic structure of the surfaces, such as coordinate charts, at corresponding points. However, in order to identify the geometric structures of two surfaces, concepts such as arc length and area must also correspond. These data are encoded by the first fundamental form.

Definition 10.1. *A diffeomorphism $\phi\colon S_1 \to S_2$ is an* **isometry** *if for all $p \in S_1$ and $\vec{w}_1, \vec{w}_2 \in T_p(S_1)$ we have*

$$\mathrm{I}_p(\vec{w}_1, \vec{w}_2) = \mathrm{I}_{\phi(p)}(d\phi_p(\vec{w}_1), d\phi_p(\vec{w}_2)).$$

Two surfaces are **isometric** *if there is an isometry between them.*

The simplest examples of isometries are provided by the rigid motions of $\mathbb{R}^3$. In the appendix to Chapter 7 we showed that any rigid motion $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by an invertible linear mapping $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $A^{-1} = A^t$ plus a translation, $F = A + \vec{v}_0$. If $S \subset \mathbb{R}^3$ is a regular surface, then so is $F(S) = S'$ and $F|_S\colon S \to S'$ is easily seen to be an isometry of surfaces.

If we further restrict our attention to isometries of $\mathbb{R}^3$ that fix the origin, the linear isometries, then restriction of any such mapping to the sphere $S^2 = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} \cdot \vec{v} = 1\}$ is an isometry $A\colon S^2 \to S^2$. In fact, every isometry of S^2 to itself arises in this manner. This class of mappings includes rotations of the sphere around a line through its center and reflection across a plane containing the center of the sphere.

Not every isometry between surfaces is the restriction of a rigid motion of $\mathbb{R}^3$. For example, the cylinder $(\cos\theta, \sin\theta, z)$ with $0 < \theta < 2\pi$ and $-\infty < z < \infty$ may be "opened up" to an infinite open strip $(\theta, z, 0)$ by the isometry $\phi(\cos\theta, \sin\theta, z) = (\theta, z, 0)$. This mapping is not the restriction of a linear mapping. The isometries that do not arise from rigid motions of $\mathbb{R}^3$ are the most interesting.

Furthermore, diffeomorphic surfaces need not be isometric. For example, two spheres of different radii are diffeomorphic, but not isometric – the arc length of a complete great circle depends on the radius of the sphere. The notion of isometry determines an equivalence relation on surfaces.

The notion of isometry makes precise the ideas of rigid motions and congruence. We define a **congruence** to be a self-isometry of a surface, $\phi\colon S \to S$. Two *figures*, that is, subsets of S, are **congruent** if there is an isometry with $\phi(\text{figure}_1) = \text{figure}_2$. A figure made up of segments of curves on a surface may be thought of as rods in a configuration and the term *rigid motion* is synonymous with congruence.

It follows from the definition that the inverse of an isometry is also an isometry and so the set of congruences of a surface forms a group. The importance of this observation cannot be overestimated. It is the basis of another approach to geometry via so-called *transformation groups*, initiated by Felix Klein (1849–1925) and Sophus Lie (1847–99). (An introduction to this approach is found in Ryan (1986).)

Properties that are preserved under isometries are the most important to the geometry of surfaces. We call such properties **intrinsic**. Properties that depend on the particular description of a surface are called **extrinsic**. For example, the fact that the z-axis is asymptotically close to the surface of revolution of the tractrix is an extrinsic property of the surface. We will see later that this surface intrinsically looks the same from almost every point on it.

To investigate intrinsic properties, we often argue locally with the apparatus of functions associated to a coordinate patch.

Proposition 10.2. *If $\phi\colon S_1 \to S_2$ is an isometry, and $p \in S_1$, then there are coordinate charts $x\colon (U \subset \mathbb{R}^2) \to S_1$ around p and $\bar{x}\colon (U \subset \mathbb{R}^2) \to S_2$ around $\phi(p)$ such that the component functions of the metric associated to x and $\bar{x}$, respectively, satisfy $E = \bar{E}$, $F = \bar{F}$, and $G = \bar{G}$.*

PROOF. Let $x\colon (U \subset \mathbb{R}^2) \to S_1$ be any coordinate chart around p, and define $\bar{x}\colon U \to S_2$ to be the coordinate chart given by $\bar{x} = \phi \circ x$. This satisfies the properties of a coordinate patch by virtue of the properties of a diffeomorphism. Now $\bar{x}_u = d\phi_p(x_u)$ and $\bar{x}_v = d\phi_p(x_v)$. Since ϕ is an isometry, $E(u, v) = \bar{E}(u, v)$, $F(u, v) = \bar{F}(u, v)$, and $G(u, v) = \bar{G}(u, v)$ by direct calculation.

The proposition gives an important pointwise property of an isometry. An immediate corollary is that length, angle, and area are preserved by an isometry. For properties sufficiently local, it is enough to have a local version of isometry to compare surfaces geometrically.

Definition 10.3. *A mapping $\phi\colon (V \subset S_1) \to S_2$ of a neighborhood V of a point p in S_1 is a* **local isometry at** *p if there are neighborhoods $W \subset V$ of p and $\bar{W}$ of $\phi(p)$ with $\phi|_W\colon W \to \bar{W}$ an isometry. Two surfaces are* **locally isometric** *if there is a local isometry at every point for each of the surfaces.*

A local isometry may fail to be an isometry. For example, a cylinder is locally isometric to the plane (roll it out). However, because the plane and the cylinder have different topological properties, no single isometry identifies the cylinder with part of the plane.