## Applications of exp and geodesics :

- The mean shift algorithm

let $\underline{x}_i$ be n points in $\mathbb{R}^d$ we wish to cluster. We do so by finding the peak of a kernel density estimate

$$f(y) = \frac{1}{n} \sum_{i=1}^{n} \cancel{f}\, k\left( \frac{\|y - \underline{x}_i\|^2}{h^2} \right)$$

where $\int k(u)\, dA = 1$

(which means that $f$ is a density)

Now notice that

$$\nabla_y f = \frac{1}{n} \sum_{i=1}^{n} \left[ k'(\,) \cdot 2 \frac{(y - \underline{x}_i)}{h^2} \right]$$

So that:

$$\frac{h^2}{2} \frac{\nabla_y f}{\sum_{i=1}^{n} k'\left(\frac{\|y-x_i\|^2}{h^2}\right)} = y - \frac{\sum k'() x_i}{\sum k'()}$$

Now we know that at the peak, $\nabla_y f = 0$

So, AT THE PEAK.

$$\hat{y} = \frac{\sum k'() x_i}{\sum k'()}$$

This suggests the iteration

recall this is a fn of $y_K$!

$$y_{K+1} = y_K + \underbrace{\left[ \frac{\sum x_i k'()}{\sum k'()} - y_K \right]}_{\text{mean shift vector}}$$

This iteration is effective, and leads to a very useful clusterer.

BUT what if the objects we wish to cluster are points on some Riemannian manifold?

· Example estimates of pose from different sources

$$\left. \in \underbrace{SO(3) \times R^3}_{\substack{\downarrow \\ = SE(3)}} \right] \text{Euclidean group}$$

**Example** · Chromatic noise

**noise model:** intensity is correctly estimated, **direction** of RGB vector is not

- eg (I think) Color demosaicing

mean shift on 2-sphere

(in fact, hemisphere - paper says $G_{3,1}$, but this is **nonsense**)

**Example:** 3D Motion clustering.

- A set of 3D pts in F orthographic views gives a rank 4 Data matrix, when motion is rigid

- Equivalently, the data matrix represents a point in

$$G_{2F,4}$$

← the point ≡ subspace yields the camera motion and 3D geometry.

- Now, if we have several independently moving rigid objects, IF we can segment the objects, above applies to each separately

- Traditional problem:

Segment AND obtain recons, motion

- <u>One</u> strategy
  - each group of 4 points yields an <u>estimate</u> of a subspac
  - (Hopefully) inliers dominate outliers – careful choice of groups can help here
  - We then <u>cluster</u> to find segments
- Clustering is on $G_{2F,4}$

Q: how can we do mean shift on a Riemannian manifold?

A: Core algorithm is quite straightforward, but computations can get nasty.

Algorithm: $\underline{x}_i \in M$

(represented in some set of coords, etc)

define the density estimate

$$f(\underline{y}) = \frac{c}{n} \sum_i k\left( \frac{d(x_i, \underline{y})^2}{h^2} \right)$$

Where:

c normalizes

$d(\underline{x}_i, \underline{y})$ is distance on manifold

Notice there is a minor technical difficulty here — the <u>integral</u> of a function depends on where you put it, — it's quite hard to simply shift fn's around - which is why the normalization const is at the front.

We need to define the <u>gradient</u> of a function on a Riemannian manifold

$$\nabla f \; :\equiv \text{ vector such that}$$

$$Xf = \langle X, \nabla f \rangle \quad \text{for}$$

<u>all</u> $X$

## In coordinates

repn of X in local coords

$$Xf = \sum_i x_i \frac{\partial f}{\partial u_i}$$

local coords

$$\langle X, Y \rangle = \sum_{i,j} g_{ij} x_i y_j$$

so define $g^{ij}$ st

$$\sum_j g^{ij} g_{jk} = \delta^i_k \leftarrow = \begin{cases} 1 & i = k \\ 0 & \text{otherwise} \end{cases}$$

then

$$\nabla f = \left( \sum_j g^{ij} \frac{\partial f}{\partial u_j} \right) e_i$$

unit basis vector for i'th coord.

$$\nabla f = \frac{c}{n} \sum_i k'(). \left[ \frac{\nabla d^2(x_i, y)}{h^2} \right]$$

- notice that this is a Tangent vector
- if we can evaluate $\nabla d^2(x_i, y)$ we're in business.

- We now need some facts about geodesics and the exponential map.
- Consider the geodesic leaving $y$, in dirn $T$.

~~length~~ $\exp_y(T)$ $\equiv$ ~~travels along~~ section of this geodesic from $u=0$ to $u=1$

we assume $T$ is small enough that this exists length [ exp (T) ]

$$= \int_{u=0}^{1} \langle T, T \rangle^{1/2} \, du .$$

(where the curve is given by ODE $\nabla_T T = 0$

<u>Now</u> :

$$T \langle T, T \rangle = \text{rate of change of } \langle T,T \rangle \text{ as we move along geodesic}$$

by properties of connections $\longrightarrow$ $= 2 \langle \nabla_T T, T \rangle$

$$= 0$$

$\nearrow$

by properties of geodesics.

Notice

$$\text{length}(\exp(\alpha T)) = \alpha \,\text{length}(\exp(T))$$

$(\alpha \geq 0)$

AND

$$\|T\| = 1 \Rightarrow \text{length}(\exp(T)) = \int_0^1 1 \, du$$

$$= 1$$

So $\text{length}(\exp(T)) = \|T\|$.

Now consider the geodesic from $y$ to $x_i$

- **define** $V$ by $x_i = \exp_y(V)$

- **notice**: a small positive step along $V$ moves me **towards** $x_i$

- $$d^2(x_i, y) = \underbrace{\|V\|^2 = \left[\langle V, V\rangle^{1/2}\right]^2}_{\text{there is a sign issue here.}}$$

$$\nabla d^2(x_i, y) = 2\, d(x_i, y) \,.\, \nabla d(x_i, y)$$

$$= 2\|V\| \,.\left[\frac{-V}{\|V\|}\right] = -V$$

↑ drive along geodesic away from $x_i$

The mean shift update now becomes

$$\underline{m}_{y_k} = - \frac{\sum_i k' \cdot v_i}{\sum_i k'}$$

This is a tangent vector at $y_k$

and then we form

$$y_{k+1} = \exp_{y_k}\left(\underline{m}_{y_k}\right).$$

Notice

- in $\mathbb{R}^n$, this is just what we're used to
- for anything else, we need to be good at exp.

Some examples:

- Lie groups.

(a Lie group is a group that is a differentiable manifold, where the group's action on itself $G \times G \to G$ is differentiable; they have lots of interesting properties; all our usual matrix groups are Lie groups)

- assume a matrix representation
- define two MATRIX operators

$$e^{M} = \sum_{i=0}^{\infty} \frac{1}{i!} M^{i}$$

$$\log(M) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (M - Id)^{i}$$

You can verify that $\log(e^{M}) = M$

$e^{\log M} = M$ etc.

These series may come with requirements on M for convergence!

One can now show (O'Neill, Ch 11) that

matrix tangent to group

$$\exp_{X}(M) = X e^{[X^{-1}.M]}$$

Matrix; a point on group

if $\exp_X (V) = Y$ then

$$V = X \log_{\partial} \left[ X^{-1} Y \right]$$

None of this is particularly attractive; but in special cases, we can do things

eg: $SO(3)$ (3D rotations)

<u>notice</u>: if $M(\phi)$ is $\in SO(3)$

then $M^T M = Id$

so $M^T \nabla_{\phi} M + \cancel{M} \left( \nabla_{\phi} M^T \right) M = 0$

now consider tangents at origin, $M = Id$

$$\nabla_{\phi} M + \nabla_{\phi} M^T = 0$$

We always work with the natural inner product on the matrix space

so if $u, v$ are tangent vectors represented as matrices we use

$$\langle u, v \rangle = \sum_{ij} u_{ij} v_{ij}$$

$$= \mathrm{Trace}(u^T v)$$

So tangent space at origin consists of $N$ st $N + N^T = 0$

write

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

(here $\omega = [\omega_x, \omega_y, \omega_z]$ is axis of rotation, $\|\omega\|$ is magnitude)

then

$$e^{[\omega]_\times} = Id + \frac{\sin\|\omega\|}{\|\omega\|}[\omega]_\times \cancel{+ \frac{(1-\cos\|\omega\|)}{\|\omega\|^2}}$$

$$+ \frac{(1-\cos\|\omega\|)}{\|\omega\|^2} . [\omega_\times]^2$$

(called the Rodriguez formula - expand, and match terms w/ Taylor series)

It's also straightforward to extract axis angle from a rotation. Consider $M \in SO(3)$

$$Mv = v \qquad \text{if } v \text{ is } \underline{\text{axis}}$$

other two eigenvalues reveal angle; cf eigenvalues of 2D rotation.

$$\left( \lambda_1 , \lambda_2 = e^{\pm i\theta} \right)$$

This seems to work rather well for clustering pose. (Subbarao + Meer paper)

Working on Grassmanians

(Earlier notes were a little sloppy; better here)

$G_{np}$ = { ~~Set~~ space of p-dimensional flats through the origin in $\mathbb{R}^n$ }

{ which is the same as p-1 d flats in $\mathbb{R}^{n-1}$ }

Multiple representations available. We will parametrize with

$$[Y] = n \times p \text{ matrix}$$

$$\text{st} \quad Y^T Y = \text{Id}.$$

Notice that multiple $Y$'s can represent the same point on the Grassmanian

- If $M_{p \times p}$ is $p \times p$ rotation

  $M^T M = Id$

  then $YM$ represents the same point

· Interpretation

- p orthogonal basis ~~for~~ vectors for the space we wish to rep'n
- $M_{p \times p}$, right action is a rotation of that basis w/in that space.

## Tangent vectors:

- Easy argument (There's a harder argument in Edelman).

want $\Delta$ st.

$$(Y+\varepsilon\Delta)^T(Y+\varepsilon\Delta) = \mathrm{Id}$$

to first order.

so

$$\underset{\substack{\uparrow\\ n\times p}}{Y^T\Delta} = 0$$

Tangent space is $p(n-p)$ dimensional, as it should be.

## Metric

$$\langle \Delta, \Gamma \rangle = \mathrm{tr}(\Delta^T \Gamma) \qquad \text{as before.}$$

Then. there is a parametric form for exp

$$\exp_Y(H) = \begin{bmatrix} YV . & U \end{bmatrix} \begin{pmatrix} \cos \Sigma \\ \sin \Sigma \end{pmatrix} V^T$$

where $U \Sigma V^T$ is SVD of $H$.

**Proof**: Edelman, p15, easy!

**AND** if

$$M = \exp_Y(H)$$

$$H = A \sin^{-1}(S) B^T$$

where

$$A S D^T = M - Y Y^T M$$

$$B C D^T = Y^T M$$

$$C^T C + S^T S = Id$$

$S, C$ diagonal.

(Subbarao + Meer — Havent checked this!)