Some Riemannian Geometry

Recall I claimed that one could compute Gaussian curvature from metric alone; we can work in terms of Christoffel symbols, etc.

→ recall also I gave an expression for C.S.'s in terms of metric (I) and it's derivative

→ we will generalize slightly.

- write g_{ij} for the ijth component of the metric tensor

- symmetric, P.D. matrix, dimension d, a function of coordinates $x_1 \ldots x_d$ (in the case of surfaces, (x_1, x_2) are the parameters we discussed, $[g_{ij}]$ is I (first ff))

- at each point of domain $U \subset \mathbb{R}^d$, we have a tangent vector space,

spanned by. $\left\{ \frac{\partial}{\partial x_1} \quad \ldots \quad \frac{\partial}{\partial x_d} \right\}$

We interpret g_{ij} by.

consider $\underline{a}$ ~~a~~, $\underline{b} \in T_p(U)$

$\sum_i a_i \frac{\partial}{\partial x_i}$ $\sum_j b_j \frac{\partial}{\partial x_j}$ ← tangent space at p of U.

then $\langle \underline{a}, \underline{b} \rangle = \sum_{ij} a_i b_j g_{ij}$

is dot · product of $\underline{a}$ and $\underline{b}$

So $[g_{ij}] = g$ is like I — a metric.

$$\int_{e}^{f} \left\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial t} \right\rangle^{1/2} dt$$ is length of $c(t)$, from $t = e$ to $t = f$

$$\int_{V \subset U} \det(g)\, dx_1\, dx_2 \cdot dx_d$$ is volume (area) of V

~~Christo~~

Examples: (2D, for now)

① $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ② $g_{ij} = \begin{pmatrix} \frac{1}{x_2^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{pmatrix}$ for $x_2 > 0$

③ $g_{ij} = \begin{pmatrix} \frac{4}{1+x_1^2+x_2^2} & 0 \\ 0 & \frac{4}{1+x_1^2+x_2^2} \end{pmatrix}$ $x_1, x_2 \in \mathbb{R}^2$

④ $g_{ij} = \begin{pmatrix} \cos^2 x_2 & 0 \\ 0 & 1 \end{pmatrix}$ $x_1 \in [0, 2\pi)$ closed, open. $x_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

→ notice these are symmetric, PD; but what are they?

→ We can compute Gaussian curvature.

$$K = -\frac{1}{g_{11}}\left[\frac{\partial}{\partial x_1}\Gamma^2_{12} - \frac{\partial}{\partial x_2}\Gamma^2_{11} + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\right]$$

(you could get this from derivations earlier in notes — but notice (a) something of this sort must be true for surfaces because K is intrinsic AND I gave expression for C.S.'s in terms of I (b) easily looked up) recall I gave expressions for Christoffel symbols earlier.

case ① all $\Gamma_{ij}^{k} = 0$; $K = 0$

case ② $\Gamma_{11}^{1} = 0$

$\Gamma_{11}^{2} = \frac{1}{x_2}$

$\Gamma_{12}^{1} = -\frac{1}{x_2}$; $K = -1$

$\Gamma_{12}^{2} = 0$

$\Gamma_{22}^{1} = 0$

$\Gamma_{22}^{2} = \frac{1}{x_2}$

case ③ should yield $K = 1$

case ④ should also yield $K = 1$

<u>Now</u>: cases 3, 4 suggest we have a <u>problem</u>.

they appear to be spheres, <u>but</u> there are topological issues — case 1 is missing a point, case 2 at least 2 points.

This motivates the notion of a <u>manifold</u>

let M be a <u>Hausdorff</u> topological space

necessary, but not really an issue in our lives.

a C^p atlas on M is given by:

- an open cover U_i $i \in I$ of M
- a family of homeomorphisms

$$\phi_i : U_i \to \Omega_i \subset \mathbb{R}^n \quad , \Omega_i \text{ open.}$$

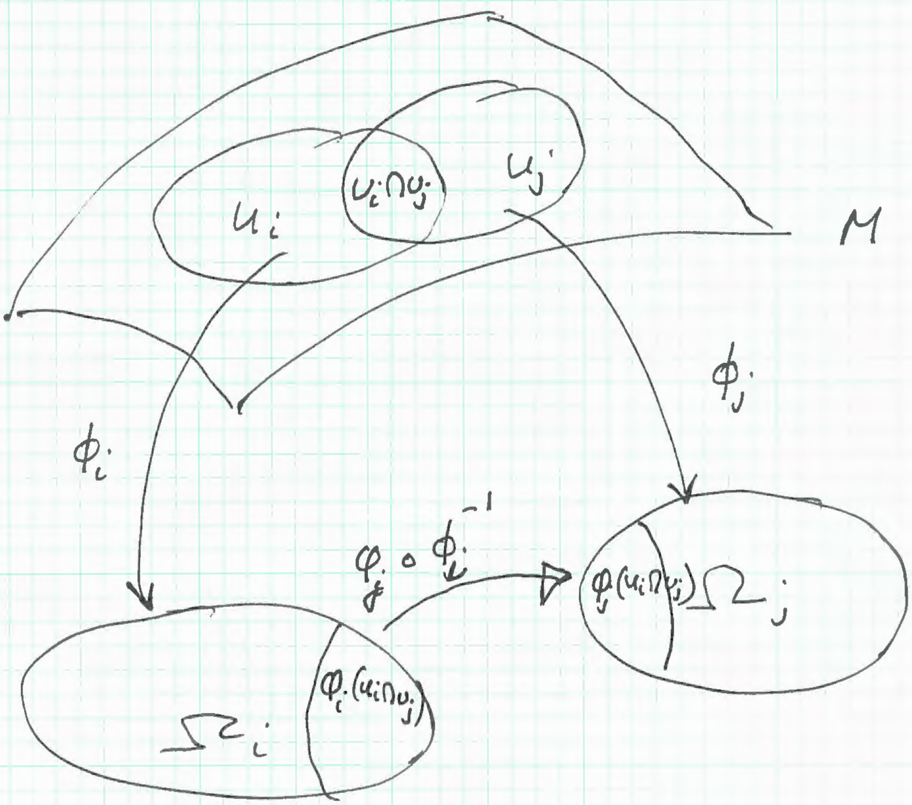
same n.

such that for $i, j \in I$

$\phi_j \circ \phi_i^{-1}$ is C^p diffeomorphism from

$\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$

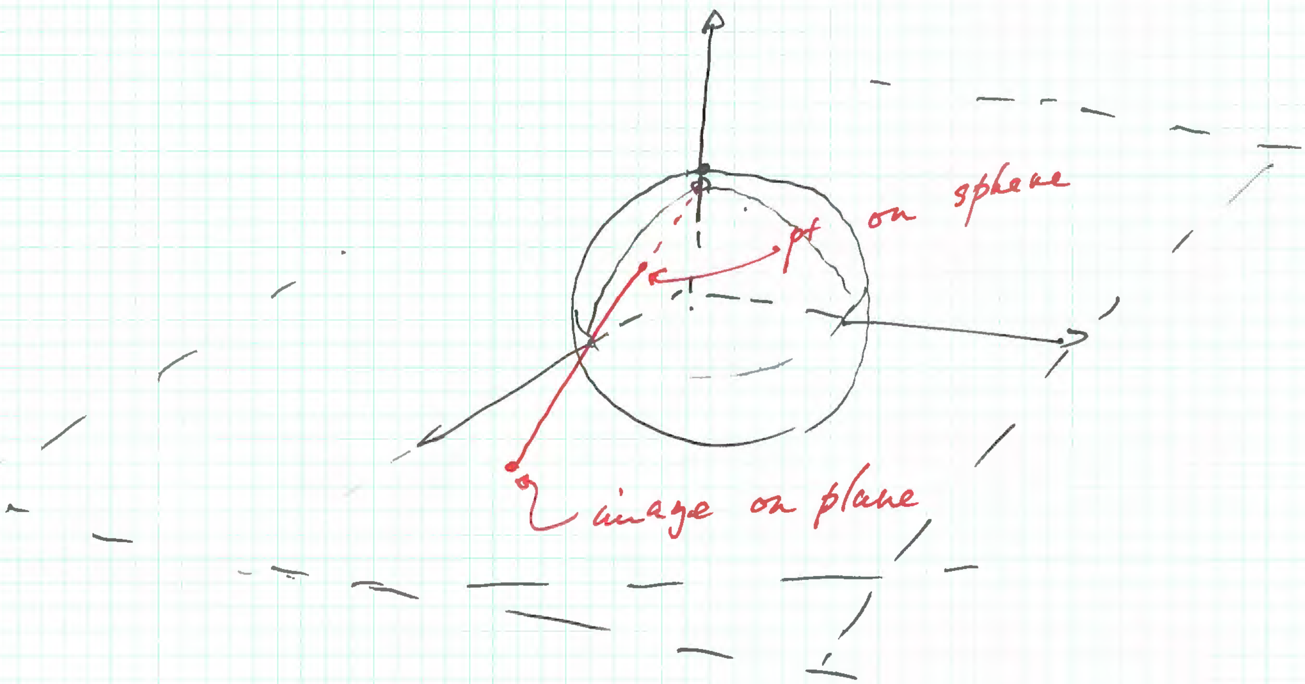
- we call U_i, ϕ_i <u>charts</u>



Notice: it doesn't really matter which way ϕ_i go — there are defn's the other way (i.e. $\Omega_i \to U_i$) which we can think of as coordinates on M

Notice: the big issue here is getting overlapping charts to be consistent

- in overlap, we must be able to label points consistently eg <u>sphere</u>



construction (stereographic proj'n) projects every point on sphere (<u>except</u> $(0,0,1)$) to a pt on plane.

<u>Exercise</u> · what is this in coords?

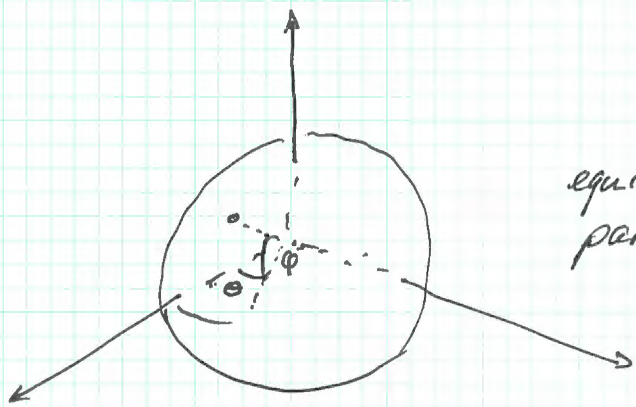
· show case 3 is the metric in this proj?

this is one chart; the U is ~~the~~ $\mathbb{R}^2$ and the φ is map as drawn.

<u>Exercise</u>: · how do we build a second chart?

· Show they're consistent.

Case 4 is more interesting:



equiv to param by. $\begin{pmatrix} \cos\varphi \cos\theta \\ \cos\varphi \sin\theta \\ \sin\varphi \end{pmatrix}$

notice $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ are bad, because

map to $U \subset \mathbb{R}^2$ looks like

$$\theta = \text{atan2}(y, x)$$

$$\varphi = \text{asin } z$$

but when $y = x = 0$, any θ is OK.

$\therefore$ not a continuous map.

notice also atan2 is only 1-1 if we restrict range, eg $[0, 2\pi)$

but closed at this end

so $U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$

ϕ is as above

and we've omitted a "seam" of points

Ex: • supply 2 other U_i, φ_i to fix

Now all this ~~a~~ defn works for $n > 2$ as well.

Some examples

• all matrices ~~a~~, $d \times d$, s.t. $\det M \neq 0$

$GL(d)$, general linear group.

Ex what are good U, φ?

• all $d \times d$ matrices ~~a~~ M st

$\det M = 1$

$M^T M = Id$

$\leftarrow SO(d)$ proper rotations; special orthogonal group

Ex: for $d = 2$, what are good U, φ?

$d = 3$, . . .

- The circle, on the plane

$\varphi_1 = \mathrm{atan2}(y, x)$ $\quad$ $x^2 + y^2 = 1$, $x \in (-1, 1]$ ← open at this end

1-1 with constraints as above

$\varphi_2 = \mathrm{atan2}(x, y)$ $\quad$ $x^2 + y^2 = 1$, ~~$x \in ($~~ $y \in (-1, 1]$ ← ditto

- Affine Group

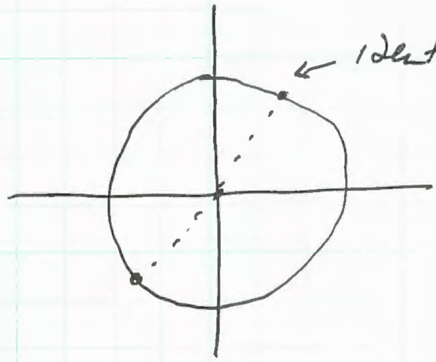
$$GL(n) \times \mathbb{R}^n$$

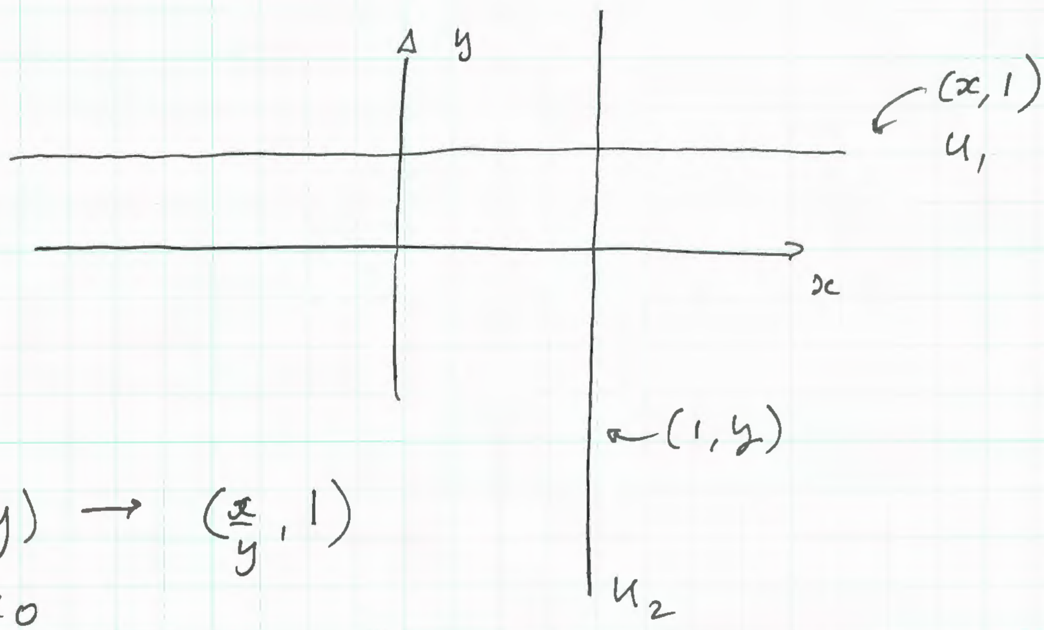
<u>EX:</u> $\quad u_i \ , \varphi_i \quad ?$

- <u>Real projective line</u> : $\mathbb{R}P^1$

  - this is important, it encodes lines thru origin
  - Take $(x, y) \in \mathbb{R}^2 - (0,0)$ ~~there~~
  - identify (x, y) and $\lambda(x, y)$ for $\lambda \neq 0$

equivalent





$\phi_1 : (x, y) \longrightarrow \left(\frac{x}{y}, 1\right)$

for $y \neq 0$

$\phi_2 : (x, y) \longrightarrow \left(1, \frac{y}{x}\right)$

for $x \neq 0$

now theres only 1 coordinate in u_1, write z_1

" u_2 " z_2

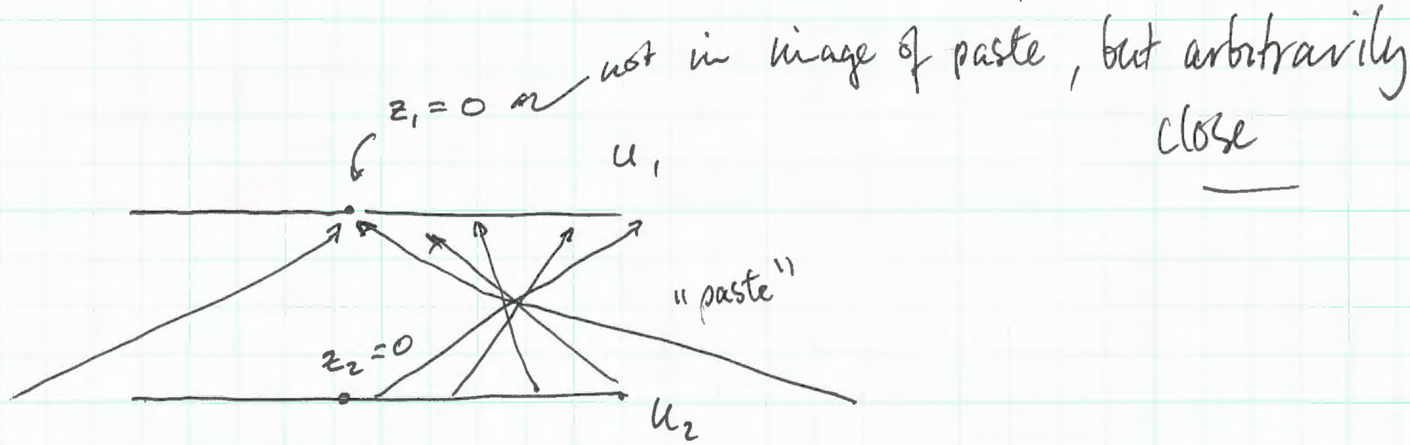
$$\phi_2^{-1} : z_2 \longrightarrow \underbrace{t(1, z_2)}_{\text{ray through pts in } U_2}$$

$$\phi_1 \circ \phi_2^{-1} = \left(\frac{1}{z_2}\right) \qquad z_2 \neq 0$$

$$\phi_2 \circ \phi_1^{-1} = \frac{1}{z_1} \qquad z_1 \neq 0$$

So U_1 = line pasted to U_2 everywhere except at 1 pt

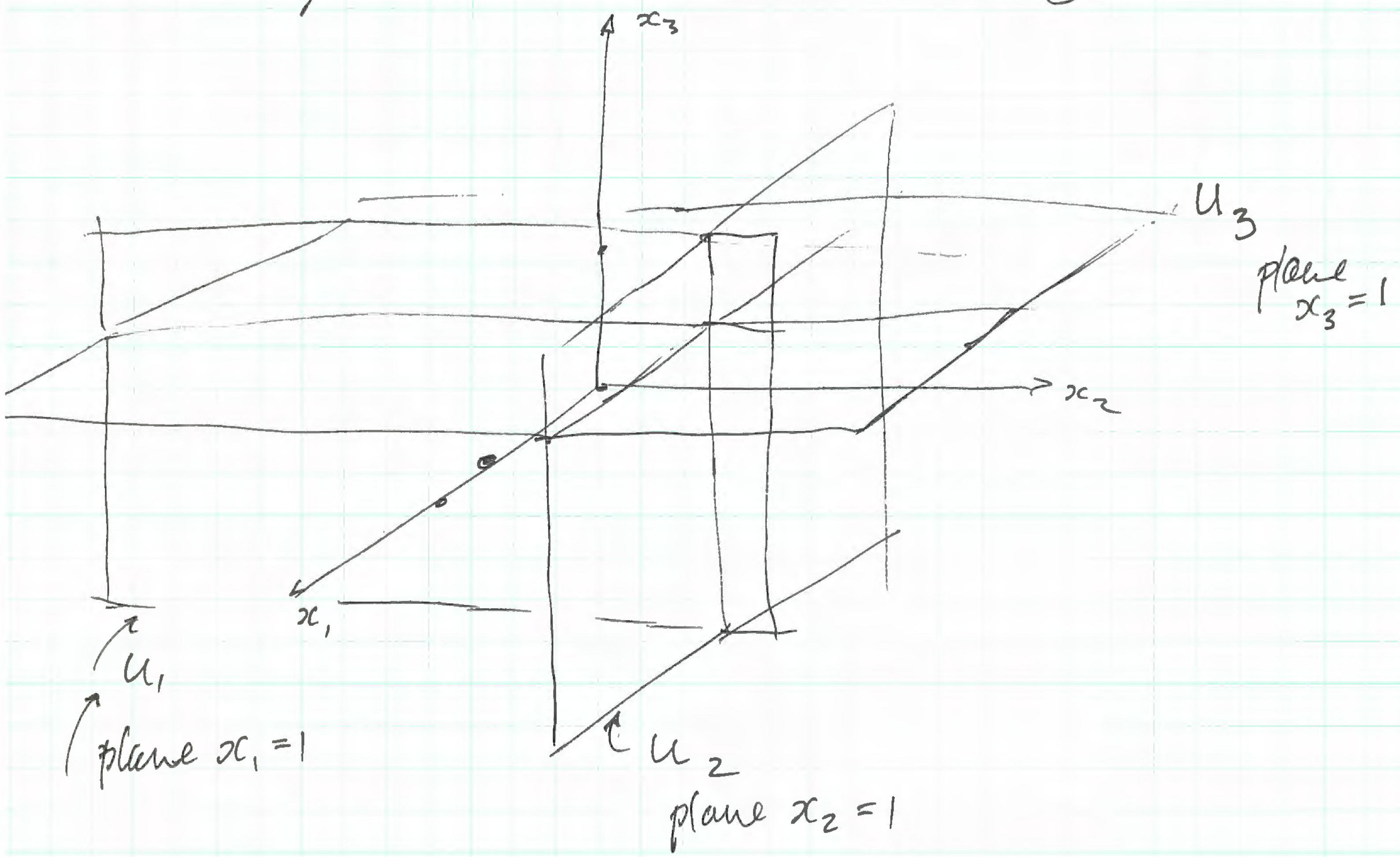
U_2 = line " U_1



Another way to think of this – we've attached a pt at ∞ to line.

Real projective plane: $\mathbb{RP}^2$

- important in camera, multiview geom
- $(x_1, x_2, x_3) \in \mathbb{R}^3 - (0,0,0)$
- identify (x_1, x_2, x_3) with $\lambda(x_1, x_2, x_3)$, $\lambda \neq 0$
- equivalently, all lines through origin



- each line pierces at least one of these planes

$\varphi_3 : (x_1, x_2, x_3) \longrightarrow \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)$ for $x_3 \neq 0$

φ_1, φ_2 follow

leads us to:

$$\begin{aligned} U_1 &= (a, b) \in \mathbb{R}^2 \\ U_2 &= (c, d) \\ U_3 &= (e, f) \end{aligned}$$

$\varphi_1 \circ \varphi_2^{-1} = \left(\frac{1}{c}, \frac{d}{c}\right)$: this pastes U_1, U_2 together along $\left\{U_2 - \begin{pmatrix} 0 \\ d \end{pmatrix}\right\}$

etc

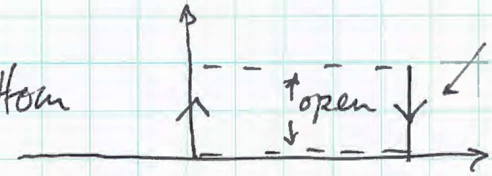
Notice I could build the whole thing with U's and transitions.

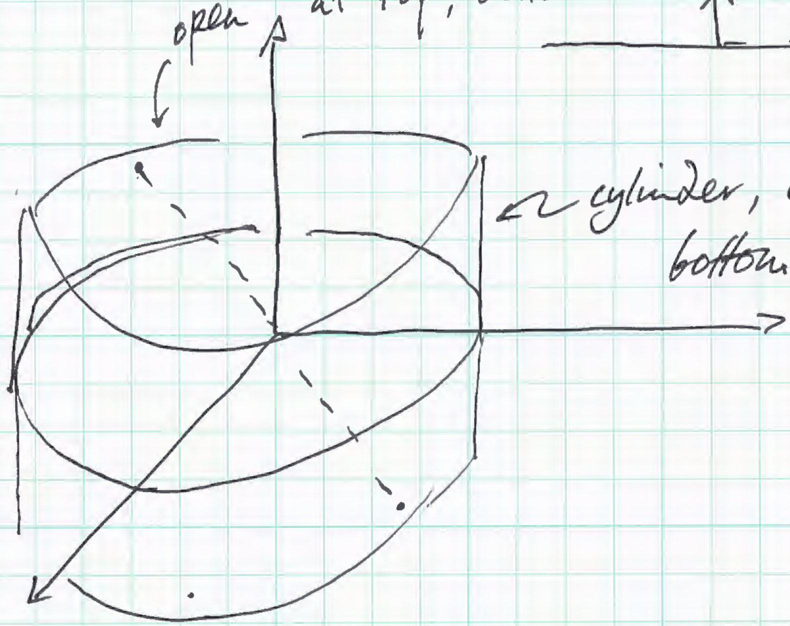
10e

real projective space, dimension n:

$\mathbb{R}P^n$ — all lines through origin in $\mathbb{R}^{n+1}$

Moebius band:





$u_1 = (1, \alpha, \beta)$ $\quad \beta \in (0,1)$

$u_2 = (\gamma, 1, \delta)$ $\quad \delta \in (0,1)$

Notice similarity to $\mathbb{R}P^1$

$\phi_2 \circ \phi_1^{-1}: \left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right)$ $\quad \alpha \neq 0$

$\phi_1 \circ \phi_2^{-1} \quad \left(\frac{1}{\gamma}, \frac{\delta}{\gamma}\right)$ $\quad \delta \neq 0$

EX: $\mathbb{R}P^2$ can be obtained by pasting edge of disk to Moebius band (takes some thought).

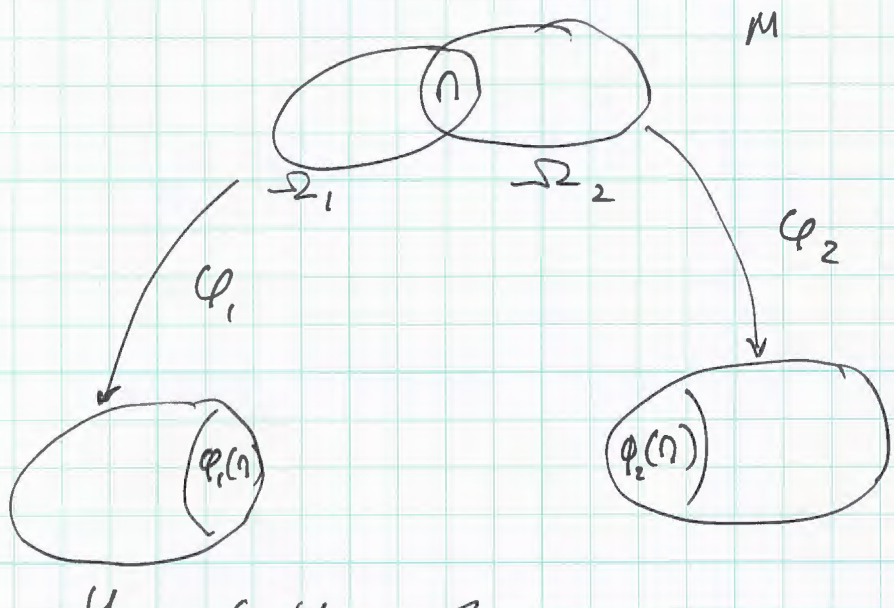
notice $J_{\varphi_2 \circ \varphi_1^{-1}} = \begin{bmatrix} \frac{-1}{\alpha^2} & -\beta/\alpha^2 \\ 0 & \frac{1}{\alpha} \end{bmatrix}$ so $\det J < 0$ for $\alpha < 0$

similar for $\varphi_2 \circ \varphi_1^{-1}$; we'll use this

~~The tangent space to a manifold~~ :

- ~~at each point p in U_i, there is~~
  ~~a tangent plan space~~
- ~~take a vector in~~

- We can place functions on these objects straightforward, but <u>must</u> be consistent on $\cap$'s



$f_2 : U_2 \to$ whatever

$f_1 : U_1 \to$ whatever

must have

~~f_1 on $\phi_1(\cap) = f_2 \circ (\varphi_2 \circ \varphi_1^{-1})$ on ϕ~~

$$f_1 \text{ on } \phi_1(\cap) \quad = \quad f_2 \circ \left(\varphi_2 \circ \varphi_1^{-1}\right) \quad \text{on } \varphi_1(\cap)$$

Notice M is rather disappearing here — the real issue is the U's, and $\varphi_i \circ \varphi_j^{-1}$

- We can place tangent vectors on M, too
    - at each point p, of U_i there is a tangent space (spanned by $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots\right)$)
    - a vector $\underline{u}$ at p is given by
$$u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} \dots$$
    - interpret this as $J_{\varphi_i^{-1}}(\underline{u})$ on M
- More interesting is a <u>tangent vector field</u>.
- n smooth fns on U_i, consistent as above

<u>EX</u>: check that this gives a unique vector on $\cap$'s.

More examples

· The tangent vector space of a manifold

$T(M)$ <- all tangent vectors at every pt of M

~~for $T(M)$ the~~

M has U_i , φ_i

$T(M)$ ·· V_i , ψ_i

$$V_i = U_i \times \mathbb{R}^n$$

$$\psi_i = \varphi_i \times Id.$$

·

. The space of <u>directed</u> lines in 2D

- circle at origin gives dirn
- point on tangent line gives translation

$T(S^1)$

. The space of undirected lines in 2D

$T(P^1)$

. " ~~undir~~ directed " 3D

$T(S^2)$ (4 D!)

. " undirected " 3D

$T(P^2)$

Grassmanians:

- Space of k-dimensional flats through the origin in n-dimensional space (equiv: $(k-1)D$ flats in $(n-1)D$ proj space)

- example:
  - 1 D flats (lines) through origin in 3D

  - a flat is given by an equivalence class of pairs of eqns.

$$\begin{matrix} ax + by + cz & & 0 \\ & = & \\ dx + ey + fz & & 0 \end{matrix}$$

AND any non-degenerate linear comb of these

represent as all 2×3 matrices of rank 2

where 2 matrices are equivalent

if there is a left action of $GL(2)$

linking them i.e

$$M_{2\times 2} \, G_{2\times 3} = H_{2\times 3}$$

then $G \equiv H$

Ω_i on Grassmanian

- for many $M_{2\times 3}$ the left 2×2 block has full rank

so $\begin{pmatrix} 1 & 0 & u_0 \\ 0 & 1 & v_0 \end{pmatrix}$ are unique coords for these lines

- for others, can use

$$\begin{pmatrix} 1 & u_1 & 0 \\ 0 & v_1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u_2 & 1 & 0 \\ v_2 & 0 & 1 \end{pmatrix}$$

<u>transition functions</u>

$$\begin{pmatrix} 1 & -u_0/v_0 \\ 0 & \frac{1}{v_0} \end{pmatrix} \begin{pmatrix} 1 & 0 & u_0 \\ 0 & 1 & v_0 \end{pmatrix} = \begin{pmatrix} 1 & -u_0/v_0 & 0 \\ 0 & \frac{1}{v_0} & 1 \end{pmatrix}$$

and the rest is obvious

NOTE all this works for k, n, $k < n$

<u>Ex</u>: what is dim $\underline{G(k, n)}$?

<u>notation</u> Grassmanian of k-flats thru origin in $\mathbb{R}^n$

Now we can (a) place a metric tensor on these objects (b) do geometry with it.

geometry, example ~~#~~ 2.

$$g_{ij} = \begin{pmatrix} \frac{1}{x_2^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{pmatrix} \qquad x_2 > 0$$

geodesics? NOT all straight lines

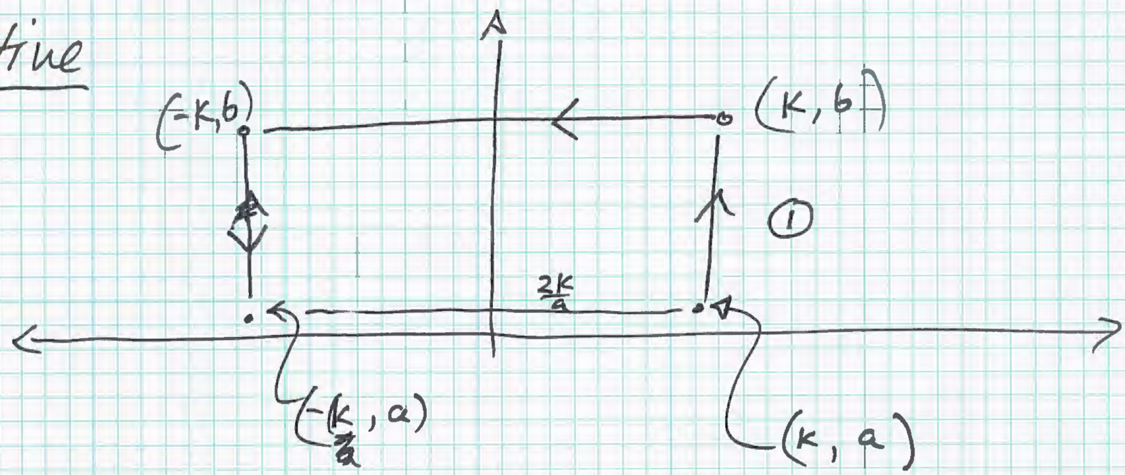
notice that to get from (k, ε) to $(-k, \varepsilon)$ it's probably a good idea to go up a bit

- In this metric, distance from (k, a) to $(-k, a)$ along a horizontal path $\begin{matrix} x(t) \\ y(t) \end{matrix} = \begin{pmatrix} t \\ a \end{pmatrix}$

$$\int_{-k}^{k} \sqrt{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}^T \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}} \, dt = \frac{2k}{a}$$

big if a is small

Alternative



path length: 2 * length of segment 1 + $2\frac{K}{b}$

because translating a vertical vector doesn't change its length

length of 1 = length of [(K, t) for $t \in (a, b)$]

$$= \int_a^b \sqrt{\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}^T \begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}}\, dt = \int_a^b \frac{1}{t}\, dt = \log b - \log a$$

So path length is $2\log b - 2\log a + 2\frac{K}{b}$

this is minimized when $b = K$.

So: to go across, you should head in!

But by how much?

Geodesics are solutions to

$$y\ddot{x} - 2\dot{x}\dot{y} = 0$$
$$y\ddot{y} + \dot{x}^2 - \dot{y}^2 = 0$$

Should look nasty to you;

EX check my calculation

EX: $\begin{pmatrix} -r\tanh t \\ r\,\mathrm{sech}\, t \end{pmatrix}$ is a geodesic

EX: $\begin{pmatrix} 0 \\ e^t \end{pmatrix}$ is a geodesic

EX: if $(x(t), y(t))$ is a geodesic, then so is $(x(t)+c, y(t))$

We now have all geodesics (notice each pt, dirn, identifies a geodesic)

We must now deal with a significant, subtle difficulty

- The tangent vector space at p and at q are not "the same"

  (Same dim, ~~same~~ COPIES of same vector space)

- given $V \in T_p(M)$, we do not know which vector in $T_q(M)$ corresponds

- this means we can't differentiate!

- We encountered this problem with surfaces, but in an apparently different form.

  - Differentiate vector field on $S \subset \mathbb{R}^n$,
    ~~you end up w/ vectors~~
    using familiar arrangements,
    you end up w/ vectors that aren't tangent – we projected off normal component, to get covariant derivative

- Can't do this for manifolds — don't have an embedding!

Define a connection

$\nabla_X Y$ map from $(\Gamma(TM) \times \Gamma(TM))$

$\rightarrow \Gamma(TM)$

Notation: smooth vector fields on M

Γ = smooth sections.

with properties:

0) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

i) $\nabla_{fX+gZ} Y = f \nabla_X Y + g \nabla_Z Y$

ii) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$

directional derivative of f in X direction — a scalar

. There are MANY connections. We care most about a special connection.

. Define

$$[X, Y]$$

to be the vector such that

$$[X,Y]f = X(Yf) - Y(Xf)$$

EX: this is a vector   EX: (XY) is not!

Then the connection we want has

iii) $\nabla_X Y - \nabla_Y X = [X, Y]$

AND is compatible with the metric (next page).

Write $\langle X, Y \rangle$ for the local dot-product of X, Y given by the g_{ij} (equivalently, by $I - fff$)

then we want

$$\text{IV)} \qquad X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

it turns out that

a) such a connection exists

b) it's unique

b) is easy:

notice that

$$X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle$$
$$+ \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle = 2 \cdot \langle \nabla_X Y, Z\rangle$$

So we can define in terms of metric.

$$X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle + \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle$$

$$\langle \nabla_X Y, Z\rangle + \cancel{\langle Y, \nabla_X Z\rangle}^{\,a} +$$

$$\cancel{\langle \nabla_Y X, Z\rangle}^{\,b} + \cancel{\langle X, \nabla_Y Z\rangle}^{\,c} +$$

$$-\cancel{\langle \nabla_Z X, Y\rangle}^{\,d} - \cancel{\langle X, \nabla_Z Y\rangle}^{\,e} +$$

$$\langle \nabla_X Y - \cancel{\nabla_Y X}^{\,b}, Z\rangle$$

$$-\langle [\cancel{\nabla_X Z}^{\,a} - \cancel{\nabla_Z X}^{\,d}], Y\rangle$$

$$-\langle [\cancel{\nabla_Y Z}^{\,c} - \cancel{\nabla_Z Y}^{\,e}], X\rangle$$

Notice:

$$\langle \nabla_X Y - \nabla_Y X, Z\rangle = \langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle \quad \text{etc.}$$

$$\langle Y, \nabla_X Z\rangle = \langle \nabla_X Z, Y\rangle \quad \text{etc.}$$

Now match terms and knock out as marked

YIELDS $\quad 2\langle \nabla_X Y, Z\rangle \quad$ as claimed

We can compute the connection in coordinates

$$\nabla_X Y = \sum_i \left[X_i \left[\nabla_{\underline{e}_i} Y \right] \right]$$

(property 1; $\underline{e}_i$ are basis vectors)

$$= \sum_i \left[X_i \left[\sum_j \frac{\partial Y_j}{\partial x_i} \cdot \underline{e}_j + \sum_j Y_j \nabla_{\underline{e}_i} \underline{e}_j \right] \right]$$

Now $\nabla_{\underline{e}_i} \underline{e}_j$ is a vector, so there must exist Γ_{ij}^{K} s.t. $\nabla_{\underline{e}_i} \underline{e}_j = \Gamma_{ij}^{K} \underline{e}_K$

our old buddies the Christoffel symbols.

Now we can use the algebra from p17 to evaluate:

$$2 \langle \nabla_{e_i} e_j, e_K \rangle = \frac{\partial}{\partial x_i} \langle e_j, e_K \rangle + \frac{\partial}{\partial x_j} \langle e_i, e_K \rangle - \frac{\partial}{\partial x_K} \langle e_i, e_j \rangle$$

(Noticing that $[e_i, e_j] = 0$).

Now $\langle e_i, e_j \rangle = g_{ij}$. so

$$\sum_{\ell} \Gamma_{ij}^{\ell} g_{\ell k} = \frac{1}{2}\left[\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}\right]$$

Notice also that iii) means $\Gamma_{ji}^{k} = \Gamma_{ij}^{k}$

Curvature :

Define

$$R(X,Y)Z = \nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) + \nabla_{[X,Y]} Z$$

a map

~~takes~~ two vectors X, Y Determine a map from $\Gamma(T(M)) \longrightarrow \Gamma(T(M))$

$$Z \longrightarrow R(X,Y)Z.$$

Technical properties of R

i) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(Proof: bash the defn, and use the Jacobi identity $[[X,Y]Z] + [[Y,Z],X] + [[Z,X],Y] = 0$; or look up, under the name Bianchi identity)

Now write

$$\langle R(X,Y)Z, T\rangle \quad \text{as} \quad (X,Y,Z,T)$$

scalar

ii) $(X,Y,Z,T) = -(Y,X,Z,T)$ (defn)

iii) $(X,Y,Z,T) = -(X,Y,T,Z)$ (Less obvious; show by showing $(X,Y,Z,Z) = 0$) or look up

iv) $(X,Y,Z,T) = (Z,T,X,Y)$ (Moderate work, Do Carmo, p 91-92)

These are ~~iv) is~~ important; in effect, they ~~it~~ says that (X,Y,Z,T) describes a relationship between 2 2D spaces, one spanned by X,Y other by Z,T - iv) says swapping the spaces does nothing.

Curvature in this form is rather mysterious, but we can observe its effects via geodesics.

Recall:

I showed on a surface that a geodesic satisfies a 2nd order ODE

Write $\underline{c}(t) = \sum_i c_i(t)\, e_i$ for geodesic

then

$$\ddot{c}_k + \sum_{ij} \Gamma_{ij}^{k} \dot{c}_i \dot{c}_j = 0$$

for each k.

This works in any dim (steps in proof don't depend on dim).

Another, very useful, form of the eqn.

$$\nabla_{\dot{c}} \dot{c} = 0$$

[$\dot{c}$ is tangent vector
Proof: subst into exp. for ~~geode~~ ∇, recall
$\dfrac{\partial \frac{\partial c_j}{\partial t}}{\partial x_i} = \dfrac{1}{c_i} \dfrac{\partial^2 c_j}{\partial t^2}$]

Now we want to think about families of geodesics leaving a pt.

Consider pt $\underline{p}$.

Choosing dirn V yields a unique geodesic leaving that point in dirn V (2 order ODE)

So at a point p we have a map (called exp)

exp: $V \longrightarrow$ fixed distance along geodesic given by V (choose 1)

Notice

$\exp(tV)$ moves $\begin{pmatrix}\text{further}\\ \text{closer}\end{pmatrix}$ along that geodesic

exp is 1:1 for some U.

where $0 \supset U \subset \mathbb{R}^n = T_p(M)$

but might not be for V big enough (eg sphere)

let $V(s)$ be a family of vectors

$\exp t\, V(s)$

gives a family of geodesics

fix s, change t; $\equiv$ move along a geodesic

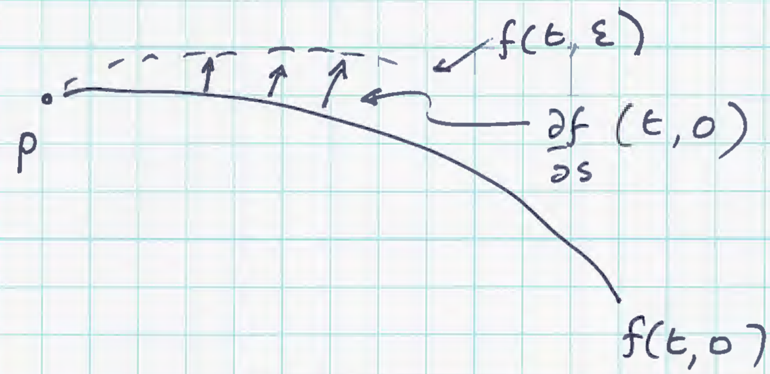
fix t, change s; $\equiv$ hop from geodesic to nearby geodesic smoothly.

Write $f(t,s) = \exp t\, v(s)$

$$\left(\frac{\partial f}{\partial s}\right)(t, 0)$$ is particularly interesting

this is a vector field along the geodesic $\exp t\, v(0)$ which "points to the closest geodesic"

i.e. $f(t,\varepsilon) \approx f(t,0) + \varepsilon \frac{\partial f}{\partial s}(t,0)$



Now $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial t}$ are vectors on M.

We can show (do Carmo, p 98-99)

that

$$\left[\nabla_{\frac{\partial f}{\partial t}} \left[\nabla_{\frac{\partial f}{\partial s}} V \right] \right] - \left[\nabla_{\frac{\partial f}{\partial s}} \left[\nabla_{\frac{\partial f}{\partial t}} V \right] \right]$$

$$= R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) V$$

for any V a vector field. Proof isn't particularly instructive – one shows that the $\left[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right]$ term is zero.

This means that

$$\nabla_{\frac{\partial f}{\partial t}} \left[\nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} \right] - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}$$

$$= \nabla_{\frac{\partial f}{\partial s}} \left[\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} \right] = \nabla_{\frac{\partial f}{\partial s}} \left[0 \right] = 0$$

tangent of geodesic

Now consider

$$\nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}$$

← this is a mixed second partial.

$$\cancel{= \sum \left[\left(\frac{\partial f_i}{\partial s} \right) \right.}$$

expand defn to get

$$\underbrace{\nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}}_{\text{a vector}} = \sum_k \Big[\underbrace{\sum_i e_i \Big(\overbrace{\frac{\partial f_k}{\partial t}}^{\text{k'th component of } f} \Big) \frac{\partial f_i}{\partial s} + \sum_{i,j} \frac{\partial f_i}{\partial s} \frac{\partial f_j}{\partial t} \Gamma_{ij}^k}_{\text{k'th component}} \Big] \underbrace{e_k}_{\text{k'th basis vector}}$$

Now consider $\sum_i \frac{\partial f_i}{\partial s} e_i$

e_i ← i'th basis vector – $\frac{\partial}{\partial x_i}$

but x_i is given by f_i

So this is $\frac{\partial}{\partial s}$.

$$\text{So } \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} = \sum_k \Big[\frac{\partial^2 f_k}{\partial s \, \partial t} + \sum_{i,j} \frac{\partial f_i}{\partial s} \frac{\partial f_j}{\partial t} \Gamma_{ij}^k \Big] e_k$$

$$= \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s}$$

Yields

$$\nabla_{\frac{\partial f}{\partial t}} \left[\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} \right] + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial t} = 0$$

traditionally, written Y

traditionally, $\dot{c}$ or T = tangent to geodesic

Y is a vector field along a geodesic that satisfies

$$\nabla_T \left[\nabla_T Y \right] + R(Y, T)\, T = 0$$

second order ODE, linear in Y

anything that solves this is a <u>Jacobi field</u>

Particularly interesting are Jacobi fields with $Y(0) = 0$

– these give "nearby" geodesics

- ~~C~~urvature
  - pulls geodesics together
  - pushes them apart

depending on sign.

to go further, we need to know more about R.

SECTIONAL CURVATURE

- choose X, Y not parallel at p.
- they define a 2D space in T_p
- define ~~R~~

recall: $\langle R(X,Y)X, Y\rangle$

$$K(X,Y) = \frac{(X,Y,X,Y)}{\sqrt{\langle X,X\rangle \langle Y,Y\rangle - \langle X,Y\rangle^2}}$$

NOTICE

$K(X,Y)$ depends on span of X, Y

but not on X, Y

[Proof: check $K(X,Y) = K(Y,X)$;
$K(X,Y) = K(\lambda X, Y)$
$K(X+\lambda Y, Y) = K(X,Y)$ EX worth doing]

This is the SECTIONAL CURVATURE of the 2D subspace $\sigma = \text{span}(X,Y)$

a) If you know $K(\sigma)$ for all σ, you can recover R.
(do Carmo, p 94 or Gallot–Hulin–Lafontaine p 104)

b) Sectional curvature of surface = Gaussian curvature.

(Proof: bash expressions)

We can now compute sectional curvatures for some objects

a) $\mathbb{R}^n$ — all sectional curvatures are zero

(Actually, by noticing $\Gamma_{ij}^{k} = 0$, you can get $\langle R(X,Y)Z, T\rangle = 0$ for all)

b) n-sphere in $\mathbb{R}^{n+1}$, using metric inherited from that embedding (= canonical metric)

– the group $SO(n+1)$ acts on $\mathbb{R}^{n+1}$, ~~moving an~~ can move any one point to any other, and is an isometry

$\Rightarrow$ R is the same at each point now consider $(1, 0, 0 \cdot \cdot)$ on the sphere

there is a subgroup of $SO(n+1)$ that fixes this point

$$\begin{pmatrix} 1 & 0 & \cdots \\ 0 & & \\ \vdots & A & \\ \vdots & & \end{pmatrix} \quad \text{where } A \in SO(n)$$

but you can rotate <u>any</u> 2 space to any other w/A

— So sectional curvatures are all equal

- no great leap required to get $K(\sigma) = 1$

(Gaussian curvature)

Back to geodesics and ~~sectional cu~~ Jacobi fields

Assume

a) M has constant sectional curvature $K \leftarrow$ scalar

b) Geodesic is unit speed

$$\langle T, T \rangle = 1$$

c) Y is Jacobi, $\langle Y, T \rangle = 0$

Then $R(T, Y)T = KY$

scalar

(do Carmo, p 112, p 96 – manipulation)

Then

$$\nabla_{T}\left[\nabla_{T} Y\right] + KY = 0$$

now choose some $\underline{w}(t)$ so that

vector

a) $\langle w, T \rangle = 0$

b) $\langle w, w \rangle = 1$

it's perp to geodesic

c) $\nabla_T w = 0 \leftarrow$ doesn't rotate – parallel transport along g

Then for what φ is

$\varphi(t)\, w(t)$ a Jacobi field ?

$$\phi'' + K\varphi = 0$$

So

$K > 0$ $\qquad \phi = \dfrac{\sin(t\sqrt{K})}{\sqrt{K}}$

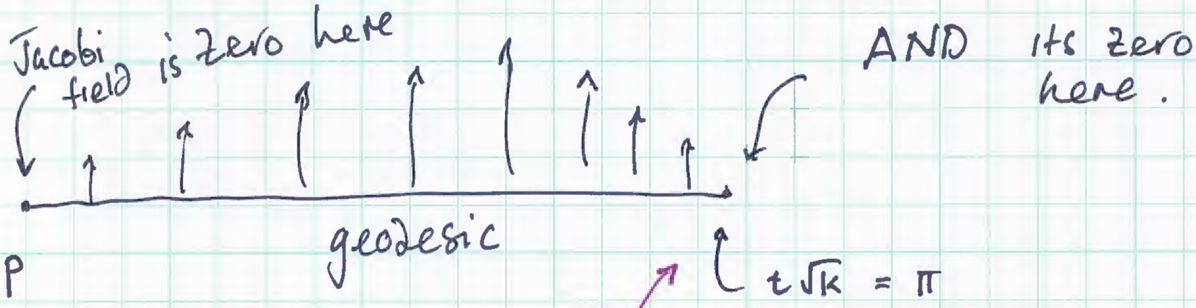
$K = 0$ $\qquad \phi = t$

$K < 0$ $\qquad \phi = \dfrac{\sinh(t\sqrt{-K})}{\sqrt{-K}}$

Initial conditions: $\quad \phi(0) = 0 \qquad \left.\dfrac{d}{dt}\left[\phi(t)\, w(t)\right]\right|_{t=0} := w(0)$

$\cancel{\phi'(0) =}$

consider $K > 0$



$t\sqrt{K} = \pi$

$\therefore \; t = \dfrac{\pi}{\sqrt{K}}$

Referred to as a <u>conjugate point</u>

(at least) two geodesics collide here

<u>EXAMPLE</u>