Coordinate transformations, vectors and tensors

We have two representations X and $\hat{X}$ of THE SAME vector, in 2 different coordinate systems

how are they related?

- a) We can tell two vectors are the same, by observing their action on functions

$$X f = \hat{X} \hat{f} \quad \left(\text{at csp. points}\right)$$

f: represented in first c-sys

$\hat{f}$: second c sys

- b) a vector is a differential operator

so

$$X = \sum_i x_i \frac{\partial}{\partial u_{1,i}}$$

$\frac{\partial}{\partial u_{1,i}}$: basis elements in first csys and $$\hat{X} = \sum_j \hat{x}_j \frac{\partial}{\partial u_{2,j}}$$

basis elements in second csys

Now:

$$\frac{\partial}{\partial u_{2j}} = \sum_i \frac{\partial u_{1i}}{\partial u_{2j}} \frac{\partial}{\partial u_{1i}}$$

$$\sum_j \hat{x}_j \frac{\partial}{\partial u_{2j}} = \sum_j \hat{x}_j \left[\sum_i \frac{\partial u_{1i}}{\partial u_{2j}} \cdot \frac{\partial}{\partial u_{1i}} \right]$$

$$x_i = \sum_j \frac{\partial u_{1i}}{\partial u_{2j}} \cdot \hat{x}_j$$

Now consider a bilinear function that accepts two vectors, and produces a scalar.

eg $\langle X, Y \rangle$ inner product

We want $\langle X, Y \rangle = \langle \hat{X}, \hat{Y} \rangle$

for representations of the SAME vectors at the same point in different coordinate systems.

Write

$$\langle X, Y \rangle = \sum_{ij} g_{ij}\, x_i\, y_j$$

$$\langle \hat{X}, \hat{Y} \rangle = \sum_{ij} \hat{g}_{ij}\, \hat{x}_i\, \hat{y}_j$$

from above, we can recover how g transforms

$$\sum_{ij} g_{ij} \left(\sum_r \frac{\partial u_{1i}}{\partial u_{2r}} \hat{x}_r \right) \left(\sum_s \frac{\partial u_{1j}}{\partial u_{2s}} \hat{y}_s \right)$$

this yields <u>transformation</u>

Notice my procedure works for objects that are more complicated

eg accept two vectors, produce a vector

accept three vectors, "

etc.