Tensor product surfaces

- Natural way to think of a surface:
  - curve is swept, and (possibly) deformed.
  - Examples:
    - ruled surface (line is swept),
    - surface of revolution (circle is swept along line, grows and shrinks).
- Surface form:

$$\mathbf{x}(u, v) = (x_1(u)x_2(v), y_1(u)y_2(v), z_1(u)z_2(v))$$

Tensor product surfaces

- Usually, domain is rectangular;
  - until further notice, all domains are rectangular.
- Classical tensor product interpolate:
  - Gouraud shading on a rectangle
    - this gives a bilinear interpolate of the rectangles vertex values.
- Continuity constraints for surfaces are more interesting than for curves
- Our curves have form:

$$\sum_i \mathbf{X}_i f_i(u)$$

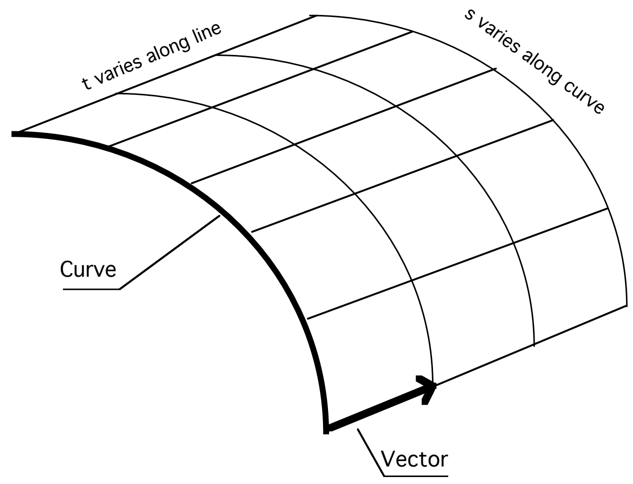
Tensor product surfaces

- Suggests form for surfaces:

$$\sum_{ij} \mathbf{X}_{ij} f_i(u) f_j(v)$$

Extruded surfaces

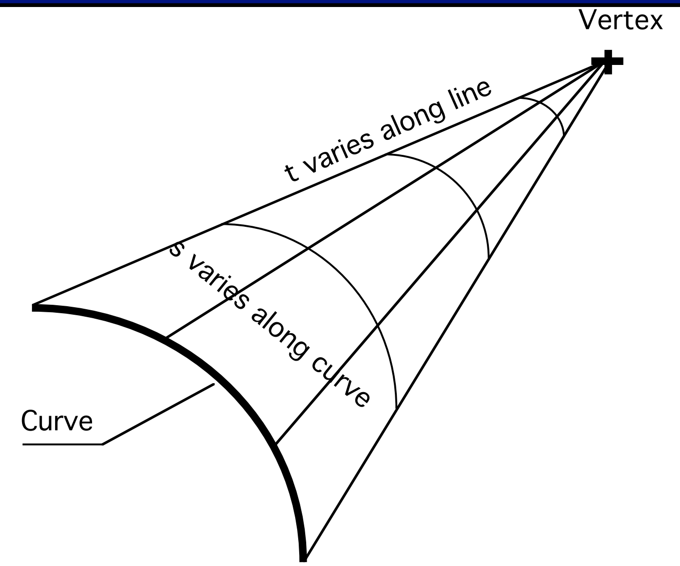
- Geometrical model - Pasta machine
- Take curve and “extrude” surface along vector
- Many human artifacts have this form - rolled steel, etc.



$$(x(s,t), y(s,t), z(s,t)) = (x_c(s), y_c(s), z_c(s)) + t(v_0, v_1, v_2)$$

Cones

- From every point on a curve, construct a line segment through a single fixed point in space - the vertex
- Curve can be space or plane curve, but shouldn't pass through the vertex



$$(x(s,t), y(s,t), z(s,t)) = (1-t)(x_c(s), y_c(s), z_c(s)) + t(v_0, v_1, v_2)$$

Commercial particle systems (wondertouch)

Surfaces of revolution

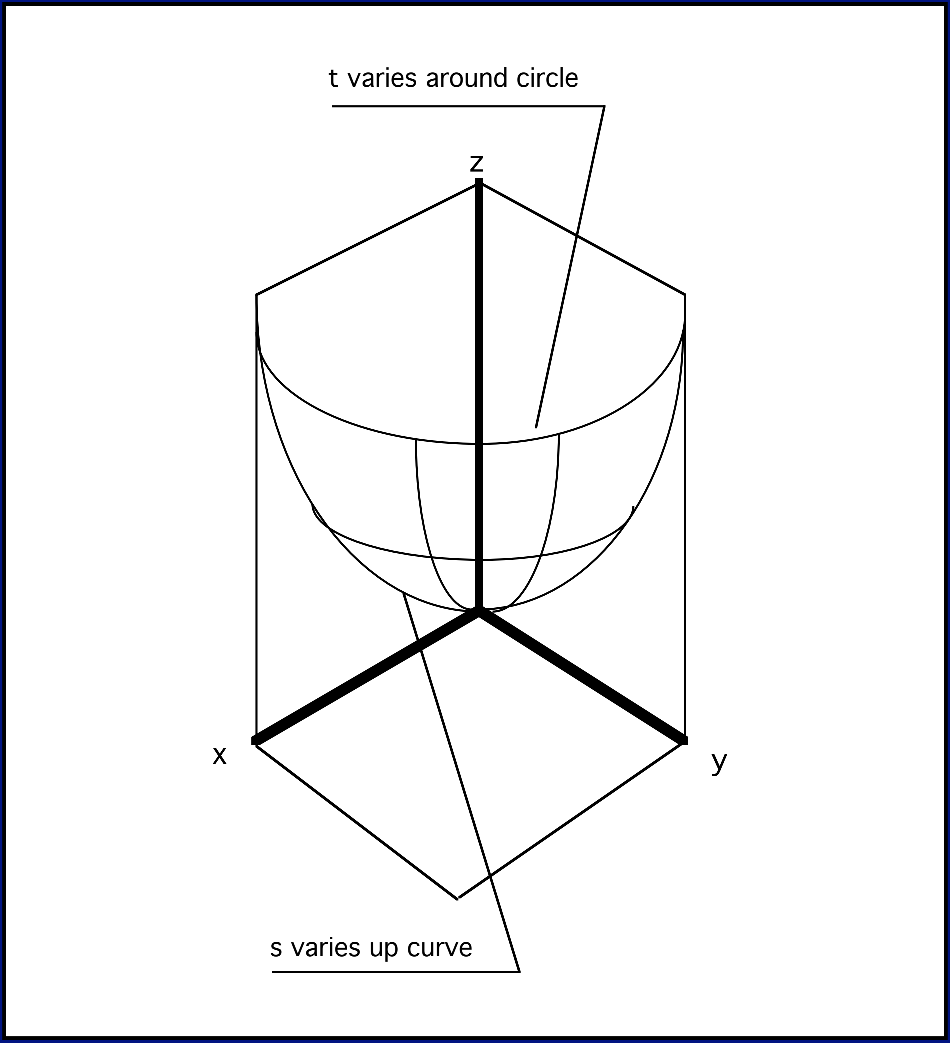
- Plane curve + axis
- "spin" plane curve around axis to get surface
- Choice of plane is arbitrary, choice of axis affects surface
- In this case, curve is on x-z plane, axis is z axis.

$$(x(s,t), y(s,t), z(s,t)) =$$

$$(x_c(s)\cos(t), x_c(s)\sin(t), z_c(s))$$

SOR-2

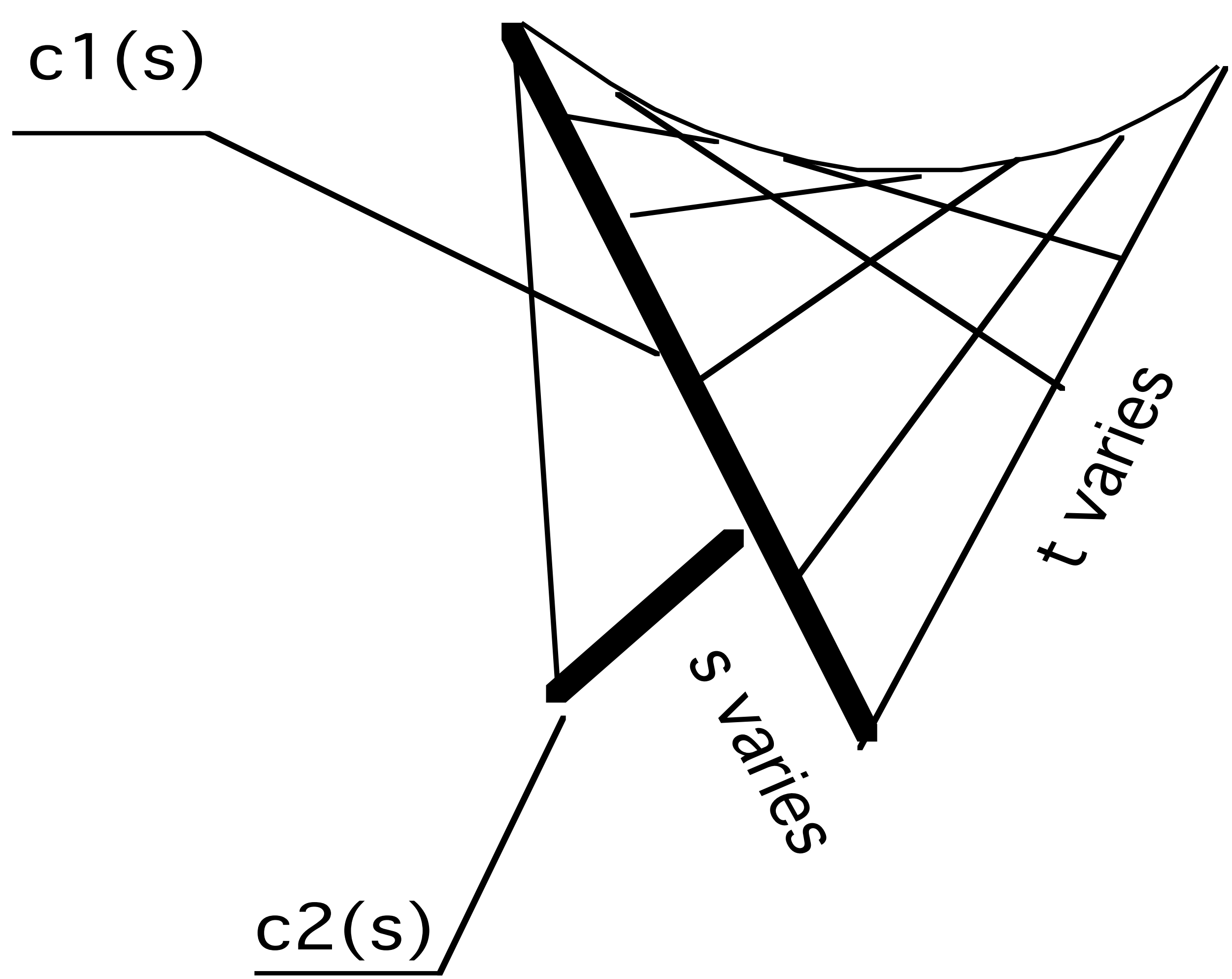
- Many artifacts are SOR’s, as they’re easy to make on a lathe.
- Controlling is quite easy - concentrate on the cross section.
- Axis crossing cross-section leads to ugly geometry.



Ruled surfaces

- Popular, because it's easy to build a curved surface out of straight segments - eg pavilions, etc.
- Take two space curves, and join corresponding points - same s - with line segment.
- Even if space curves are lines, the surface is usually curved.

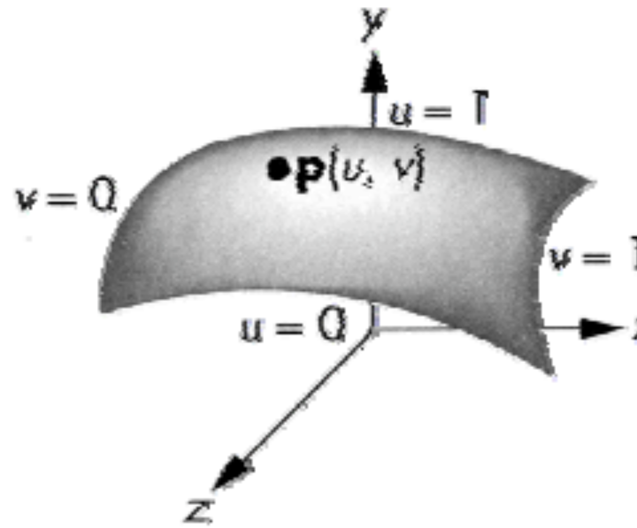
$$
\begin{aligned}
(x(s,t), y(s,t), z(s,t)) = \\
(1-t)(x_1(s), y_1(s), z_1(s)) + \\
t(x_2(s), y_2(s), z_2(s))
\end{aligned}
$$


c1(s)
t varies
s varies
c2(s)

Parametric Surface Patches

As with parametric curves, define a vector-valued function

$$\mathbf{p}(u,v) = \begin{bmatrix} x(u,v) & y(u,v) & z(u,v) \end{bmatrix}$$



Derivatives are tangent to surface

- *not* necessarily orthogonal

$$\mathbf{p}_u(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{bmatrix}$$

$$\mathbf{p}_v(u,v) = \begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$$

And the unit normal can be computed as

$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\left\| \mathbf{p}_u \times \mathbf{p}_v \right\|}$$

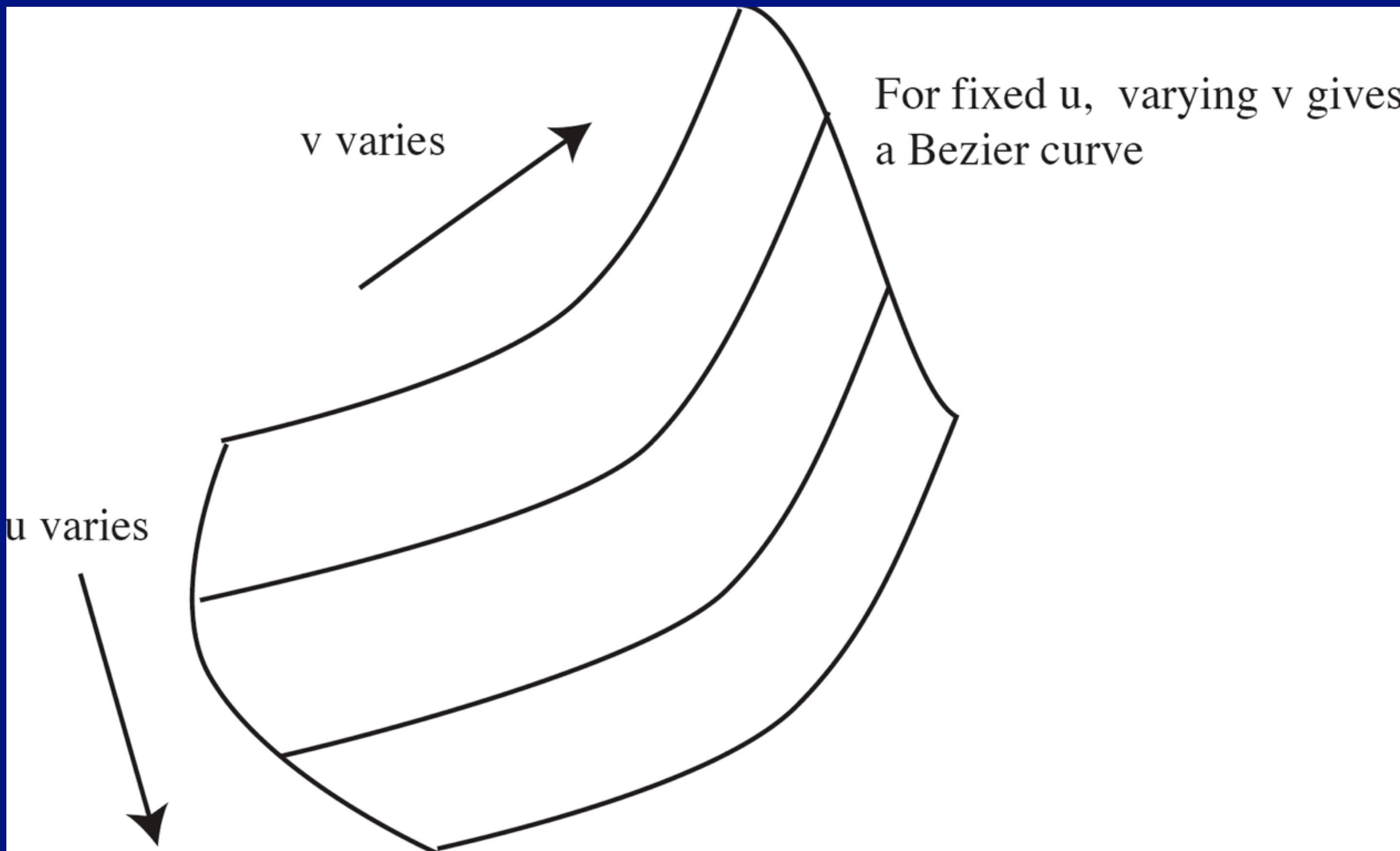
Tensor Product Bezier Patches

- By the same process as with Bezier curves, we can derive the form:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\mathbf{b}_{ij}B_i^n(u)B_j^m(v)$$

  - here B(u) are the Bezier-Bernstein polynomials, as before
  - these are blending functions
- It follows from the tensor product form that:
  - interpolates four vertex points
  - surface is in convex hull of control points
  - tangent plane at each vertex is given by three points at that vertex
  - repeated de Casteljau (one direction, then the other) gives a point on the surface, and tangent plane to surface

Tensor product Bezier patches



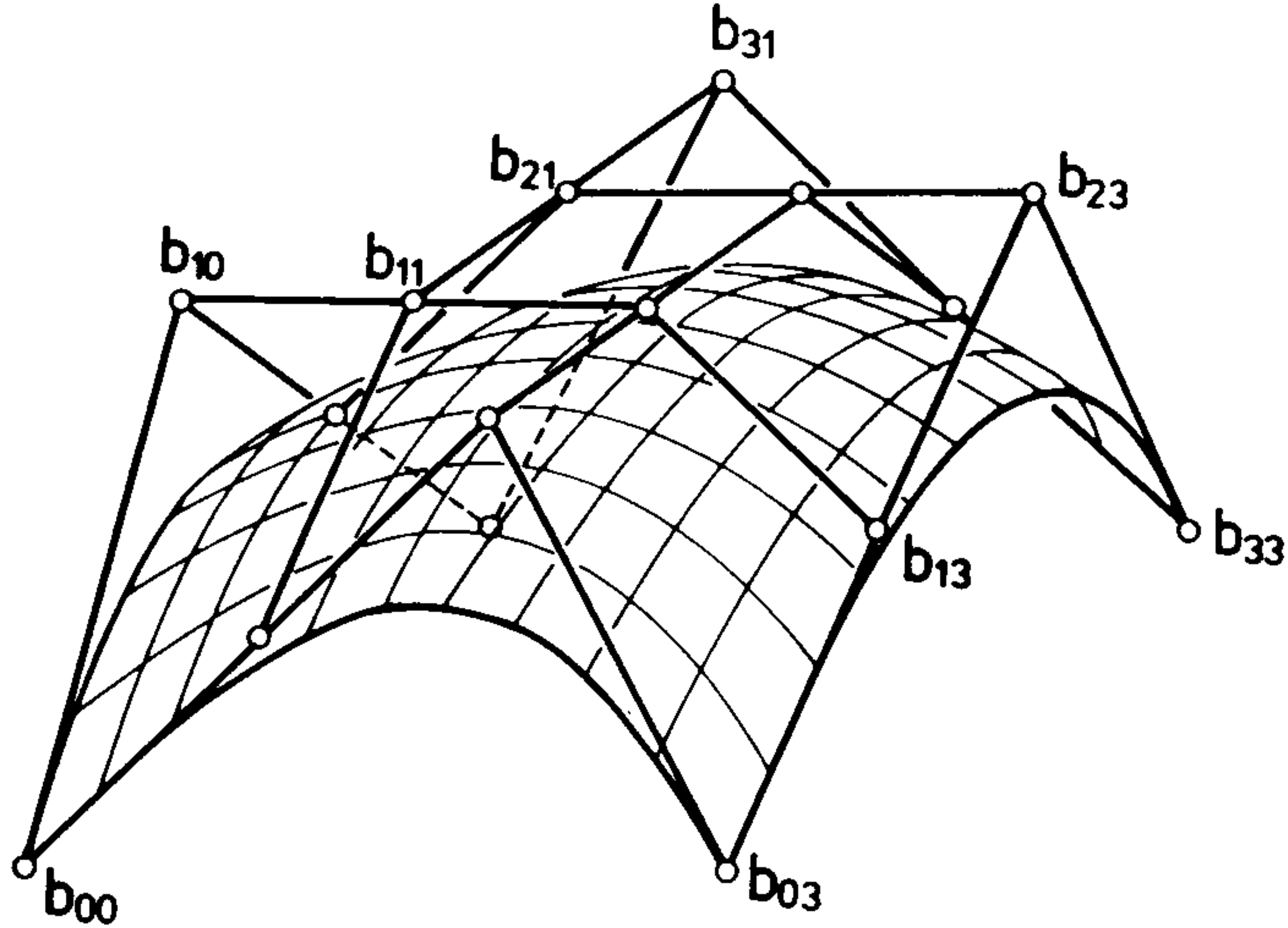


Fig. 6.3. Bézier surface of degree (3,3) and its Bézier net.

Tensor Product Bezier patches

- Construct by de Casteljau algorithm
  - repeated linear interpolation one way
    - now go the other way
  - OR
    - repeated bilinear interpolation

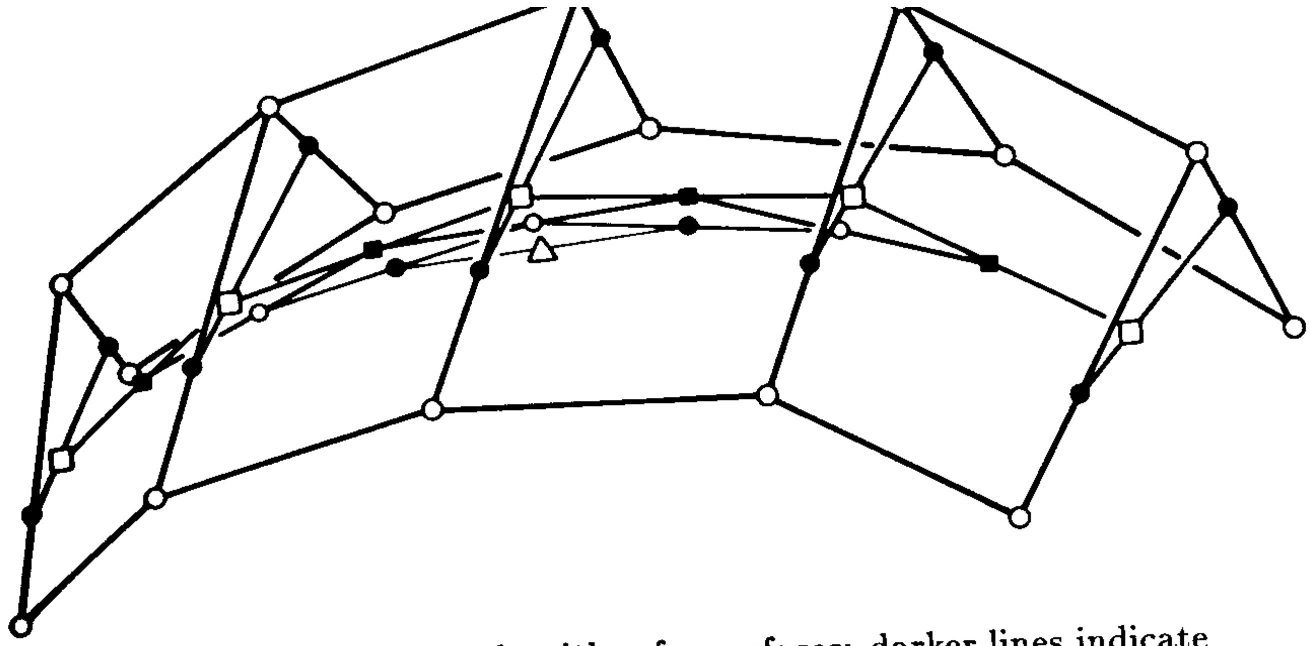
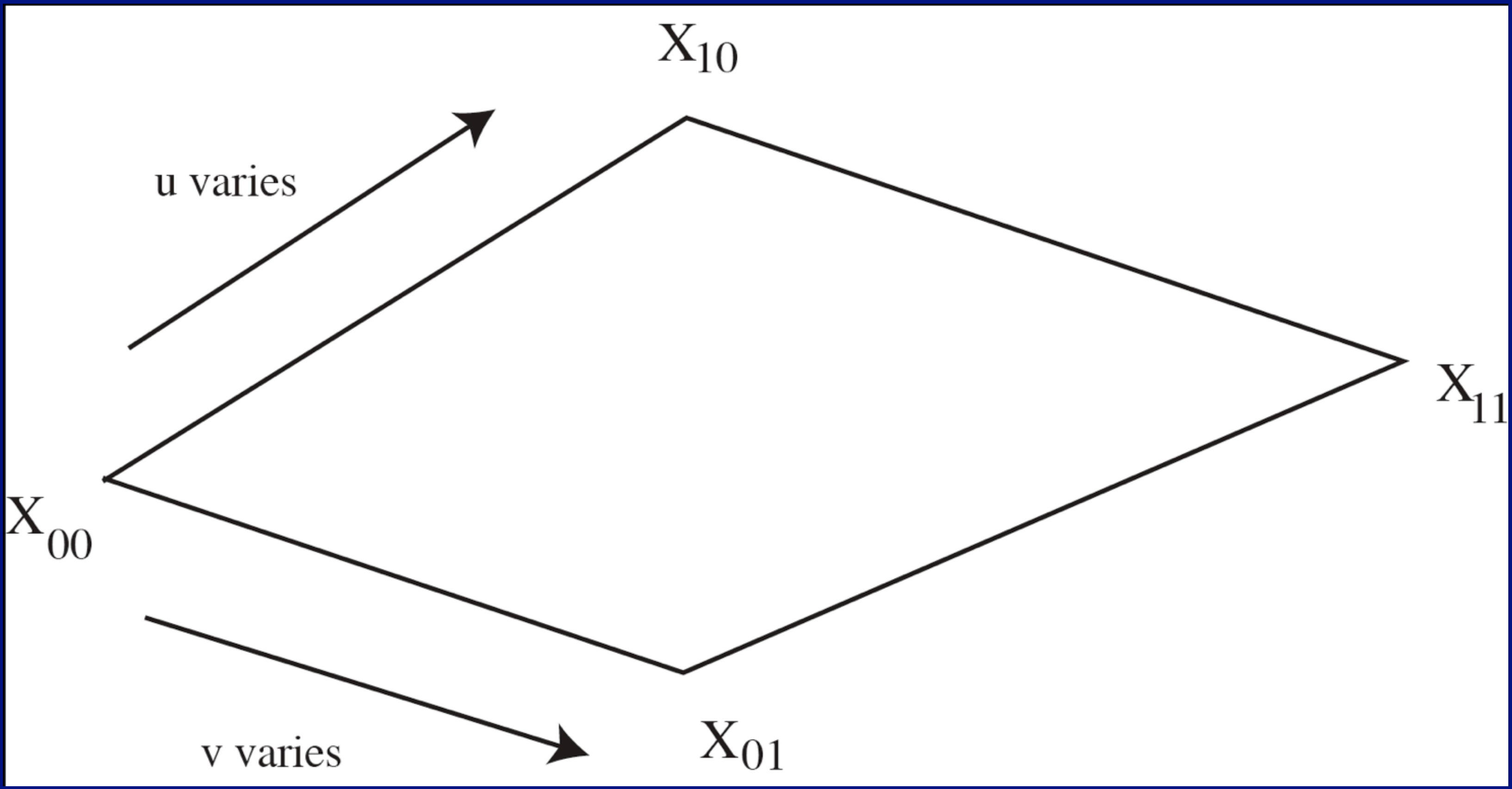


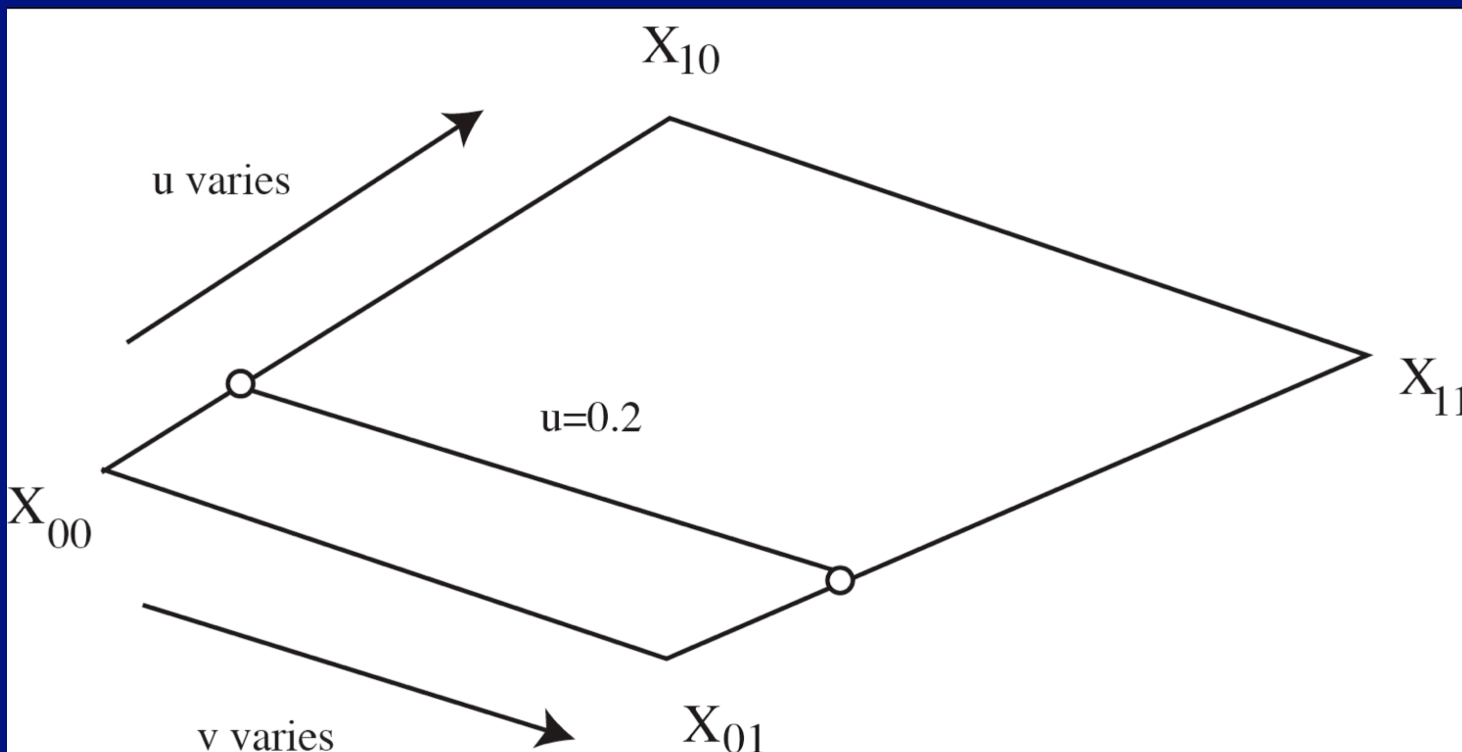
Fig. 6.7a. de Casteljau algorithm for surfaces: darker lines indicate interpolation in the v-direction.

Repeated linear interpolation one way yields control points; use these points for repeated linear interpolation in the other direction

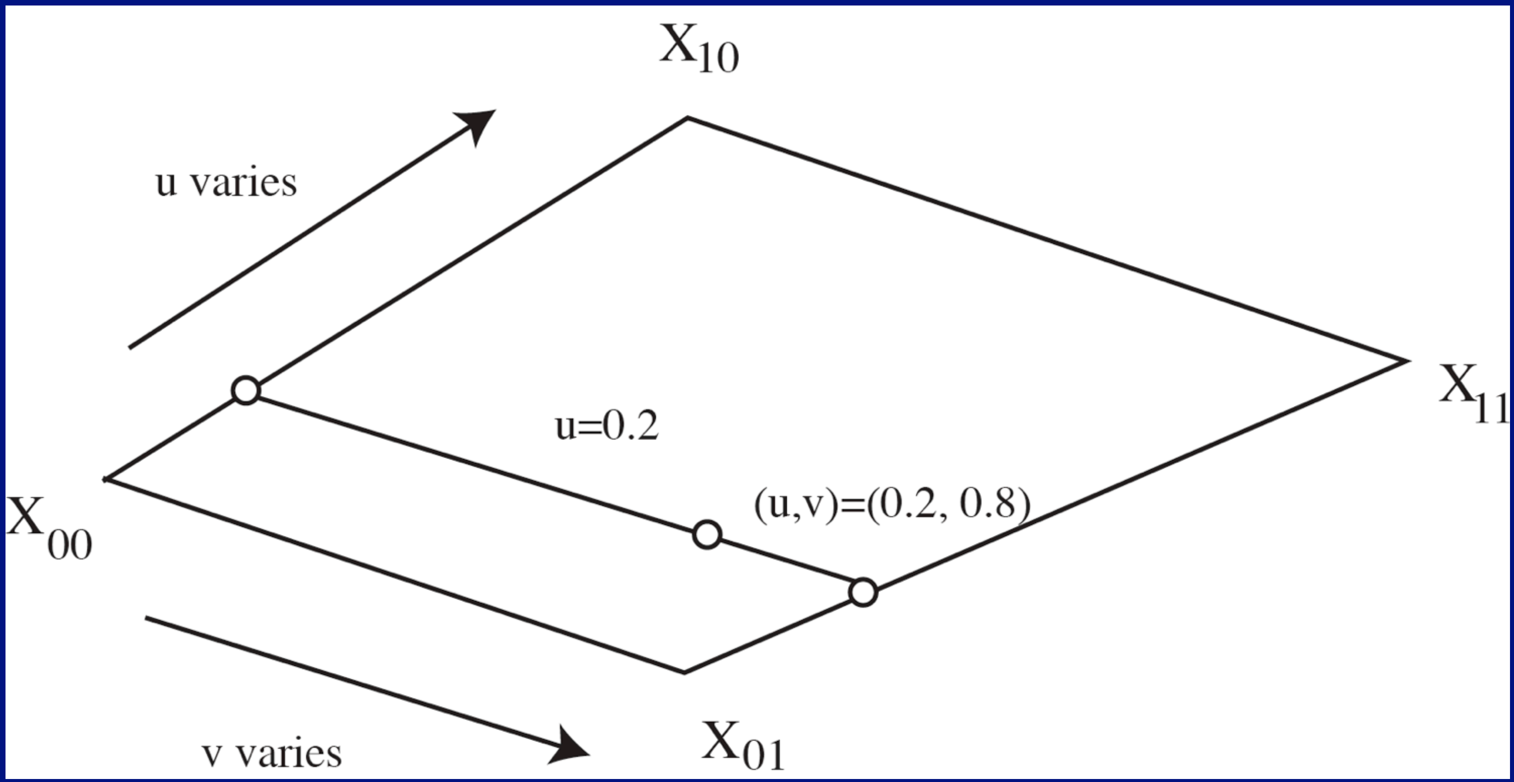
Bilinear interpolation



Bilinear interpolation



Bilinear Interpolation



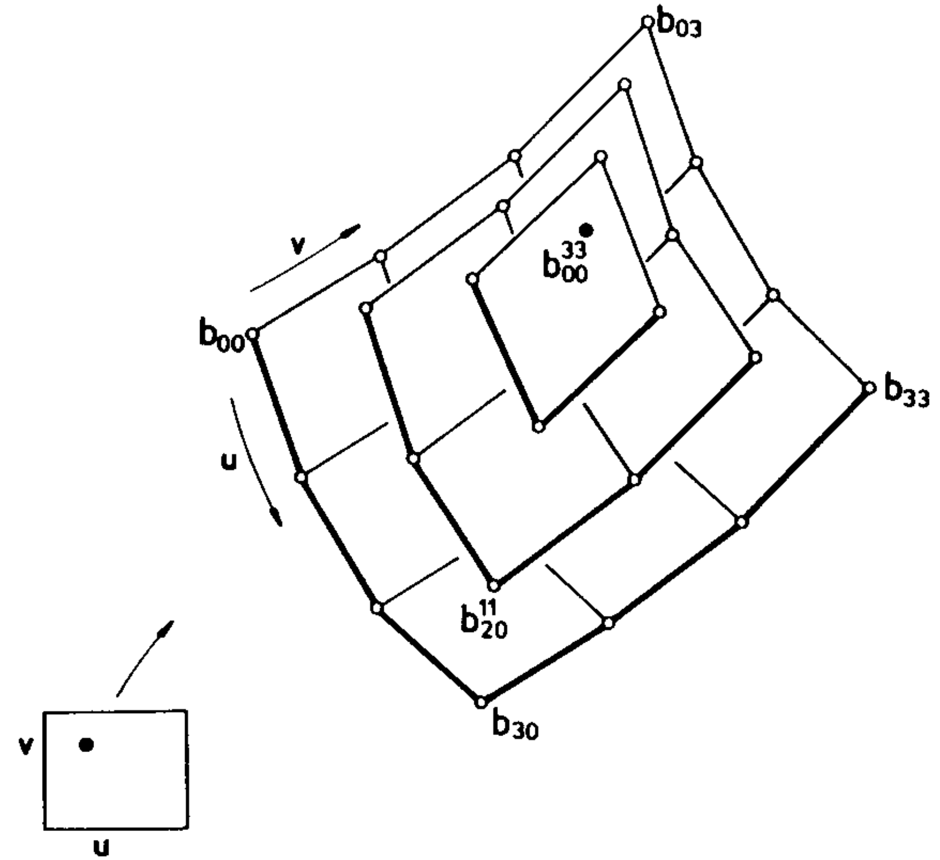


Fig. 6.7b. The de Casteljau algorithm viewed as bilinear interpolation.

Repeated bilinear interpolation yields a surface too

Tensor Product Bezier Patches

- It follows from the tensor product form that surface:
  - interpolates four vertex points
  - tangent plane at each vertex is given by three points at that vertex
  - repeated de Casteljau (one direction, then the other) gives a point on the surface, tangent plane to surface

Tensor product Bezier patches

- Recall we wrote curves as:

$$\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathcal{M} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

- We can write surface as:

$$\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathcal{M} \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} \end{bmatrix} \mathcal{M}^T \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

Bezier curve subdivision

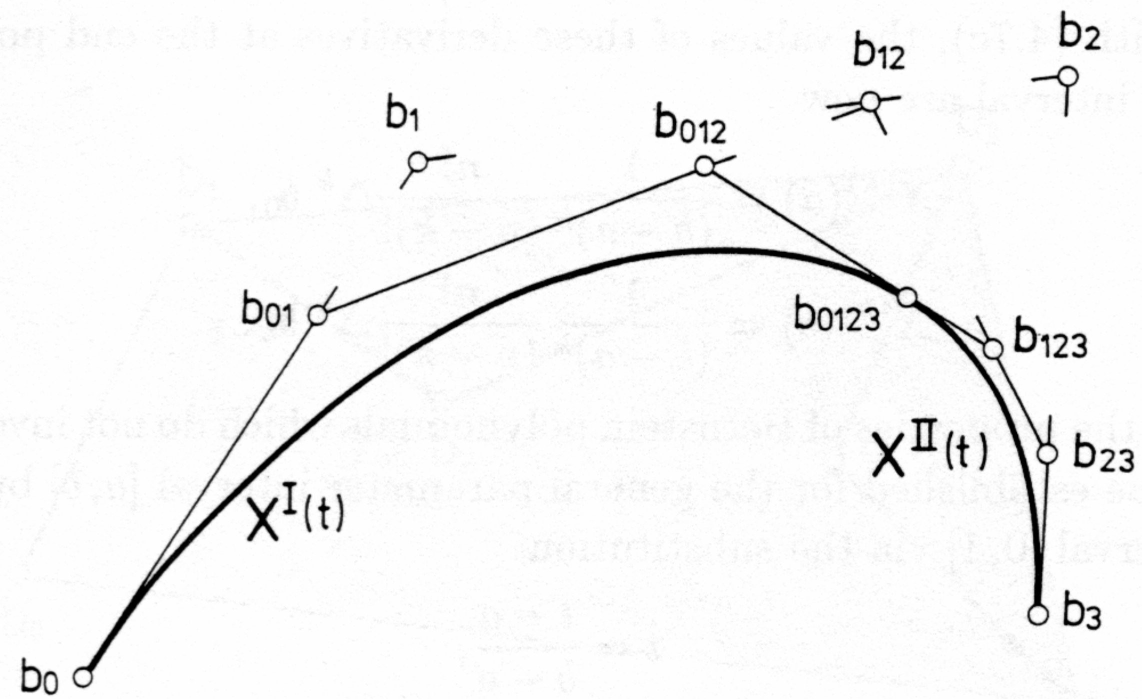


Fig. 4.5. Decomposition of a Bézier curve into two C^3 continuous curve segments (cf. Fig. 4.4).

Applies also to surfaces

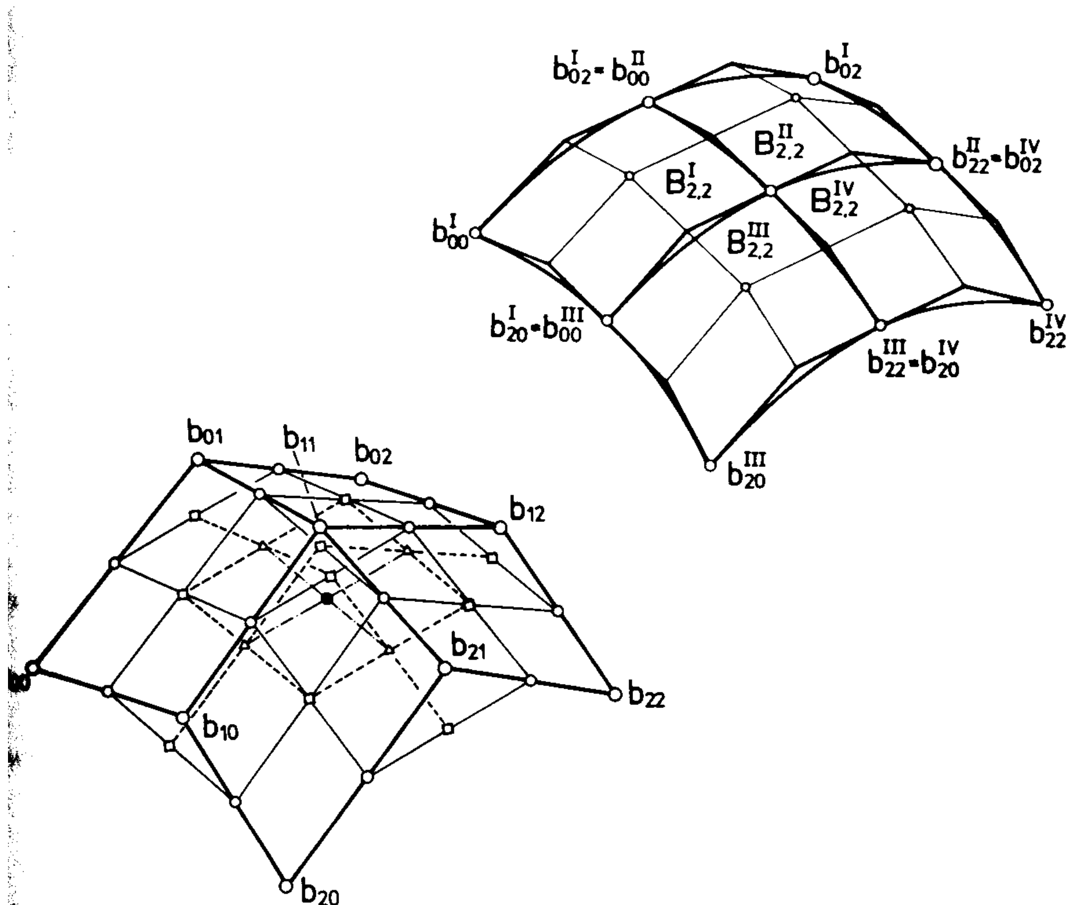


Fig. 6.8. Subdivision of a Bézier surface using the de Casteljau algorithm.

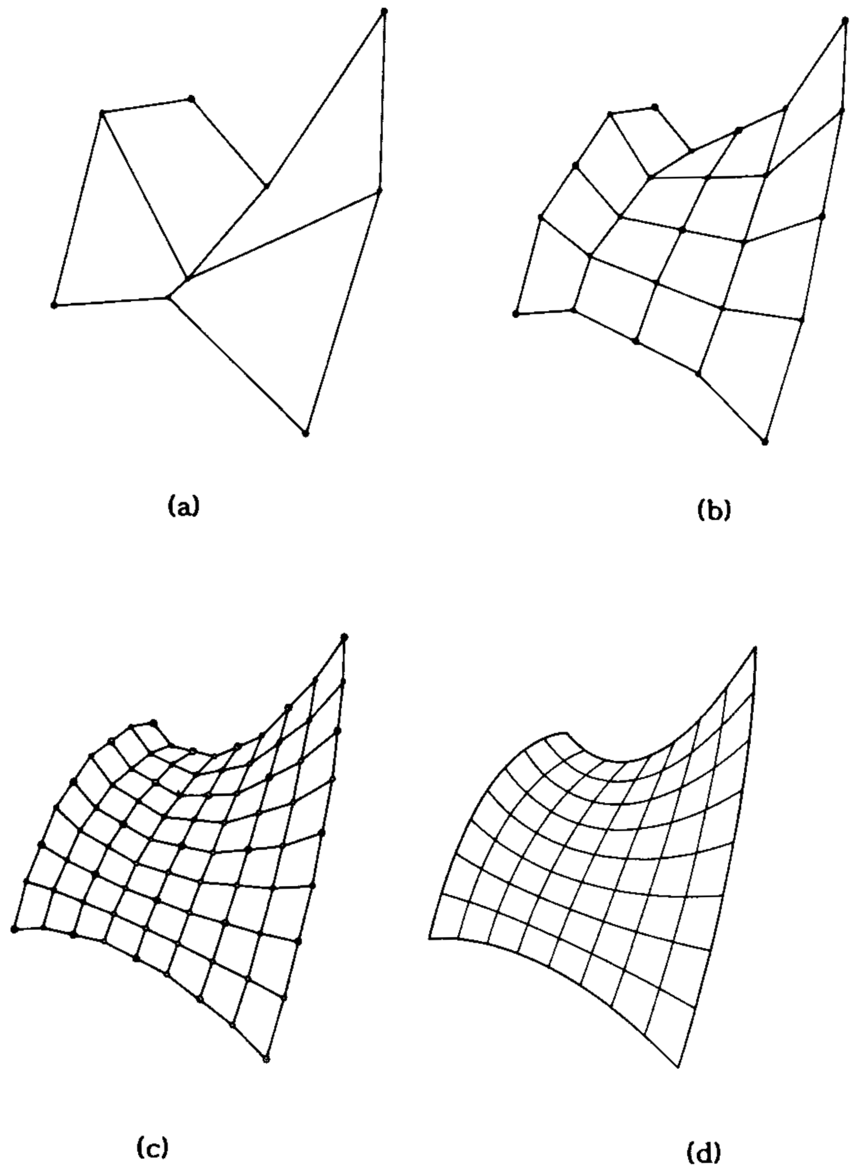


Fig. 6.9. Repeated refinement of a Bézier net (a) leads to approximations (b), (c) of the Bézier surface (d).

Tensor product splines

- Simplest
  - bicubic interpolating surface
- B-splines

Bicubic interpolating surfaces

- We have a grid in u, v of parameter values
  - know height at each grid point
    - and anything else we need to know (continuity means this is very little)
  - must build an interpolating surface
- Form in each grid block:

$$Z_{ij}(s,t) = \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} s^i t^j$$

Constructing a spline

- Notation:

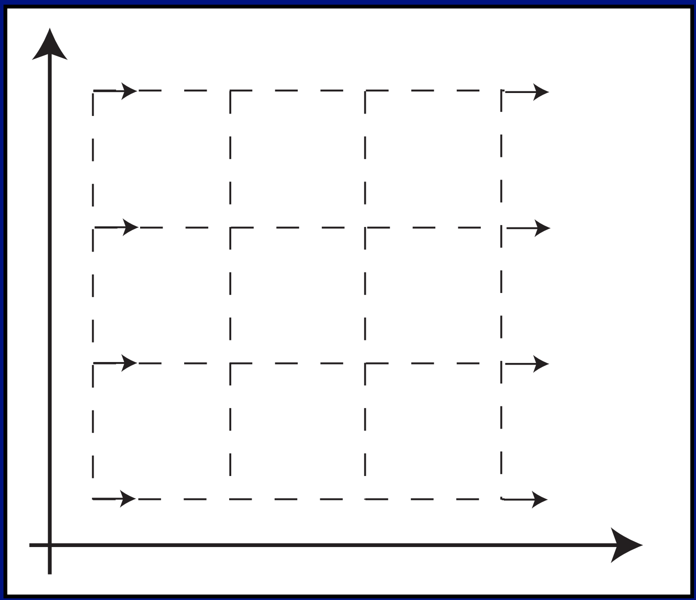
$$
\begin{aligned}
p &= \frac{\partial Z}{\partial x} \\
q &= \frac{\partial Z}{\partial y} \\
r &= \frac{\partial^2 Z}{\partial x \partial y}
\end{aligned}
$$

-

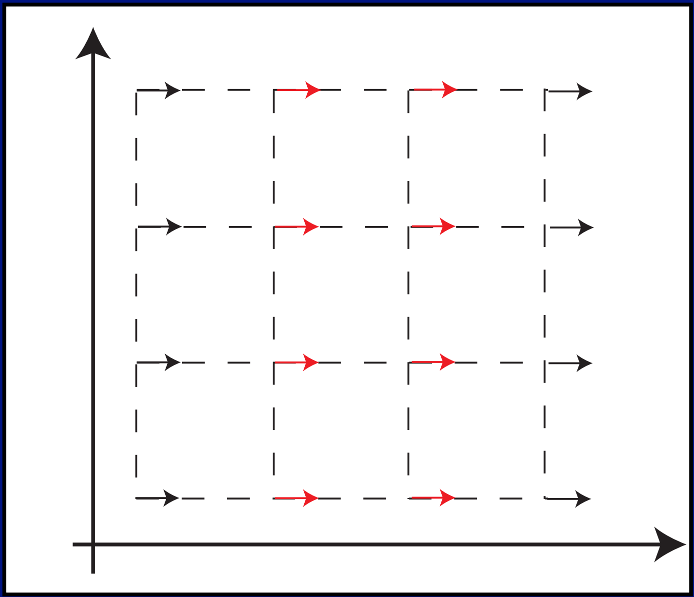
Step 1: Patches from info

- Z, p, q, r at each corner yields the patch
- Linear algebra in monomials (exercise)

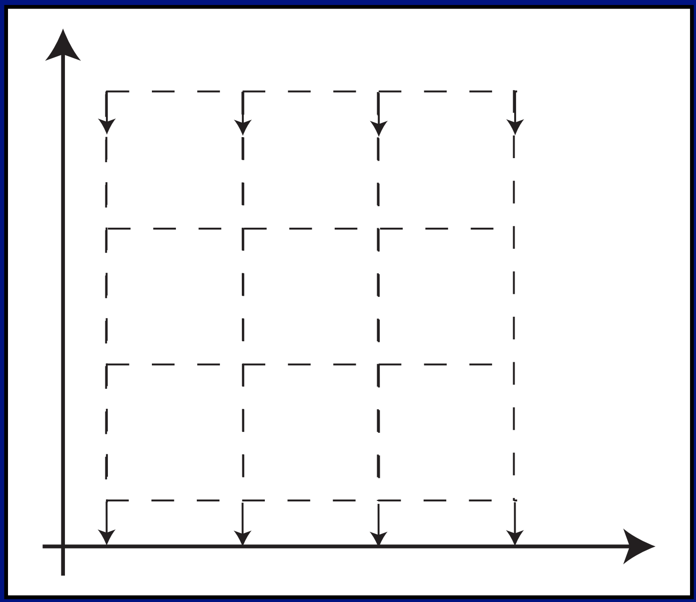
Step 2: Continuity reveals p's



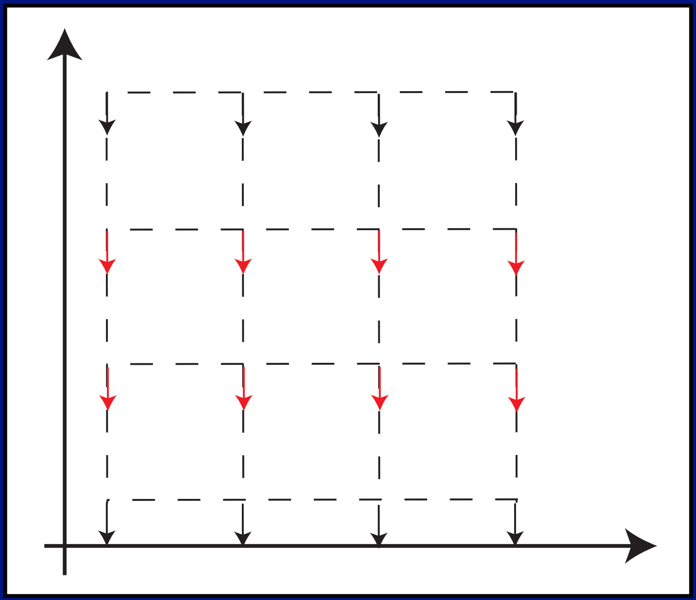
p's follow from continuity, spline in x dir



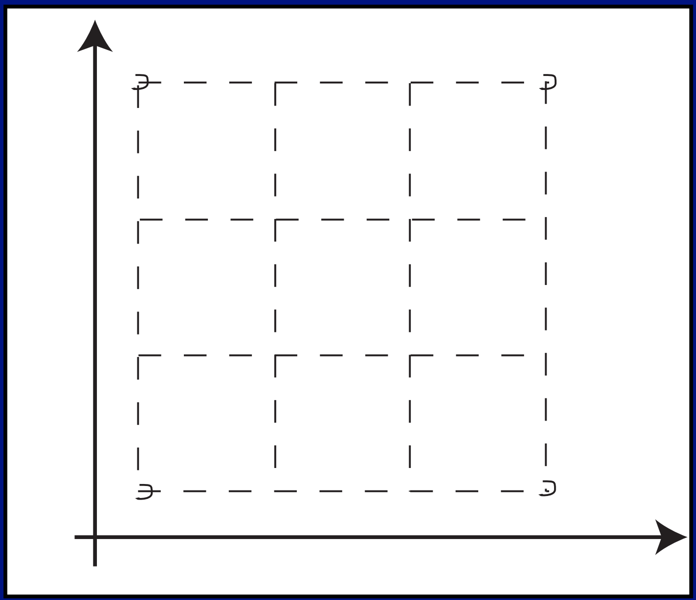
Step 3: Continuity reveals q's



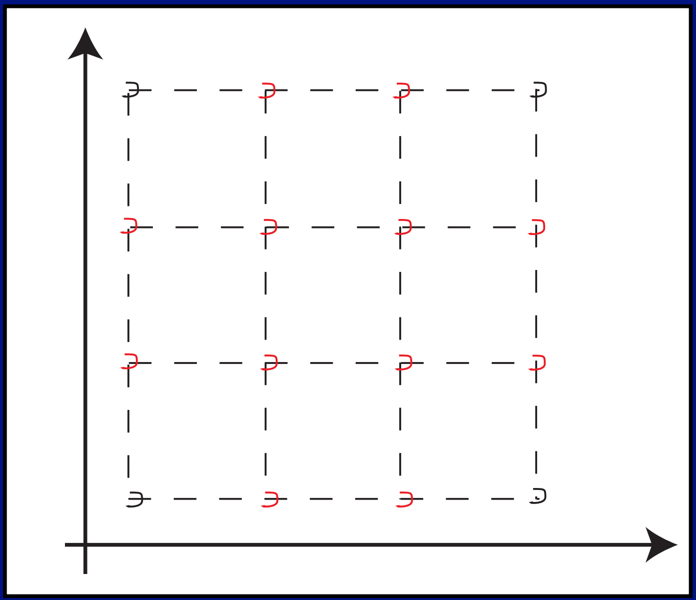
q's follow from continuity, spline in y dir



Step 4: Continuity reveals r's



Because p is a cubic spline in y,
resp q is a cubic spline in x,
we can "fill in" r from corners



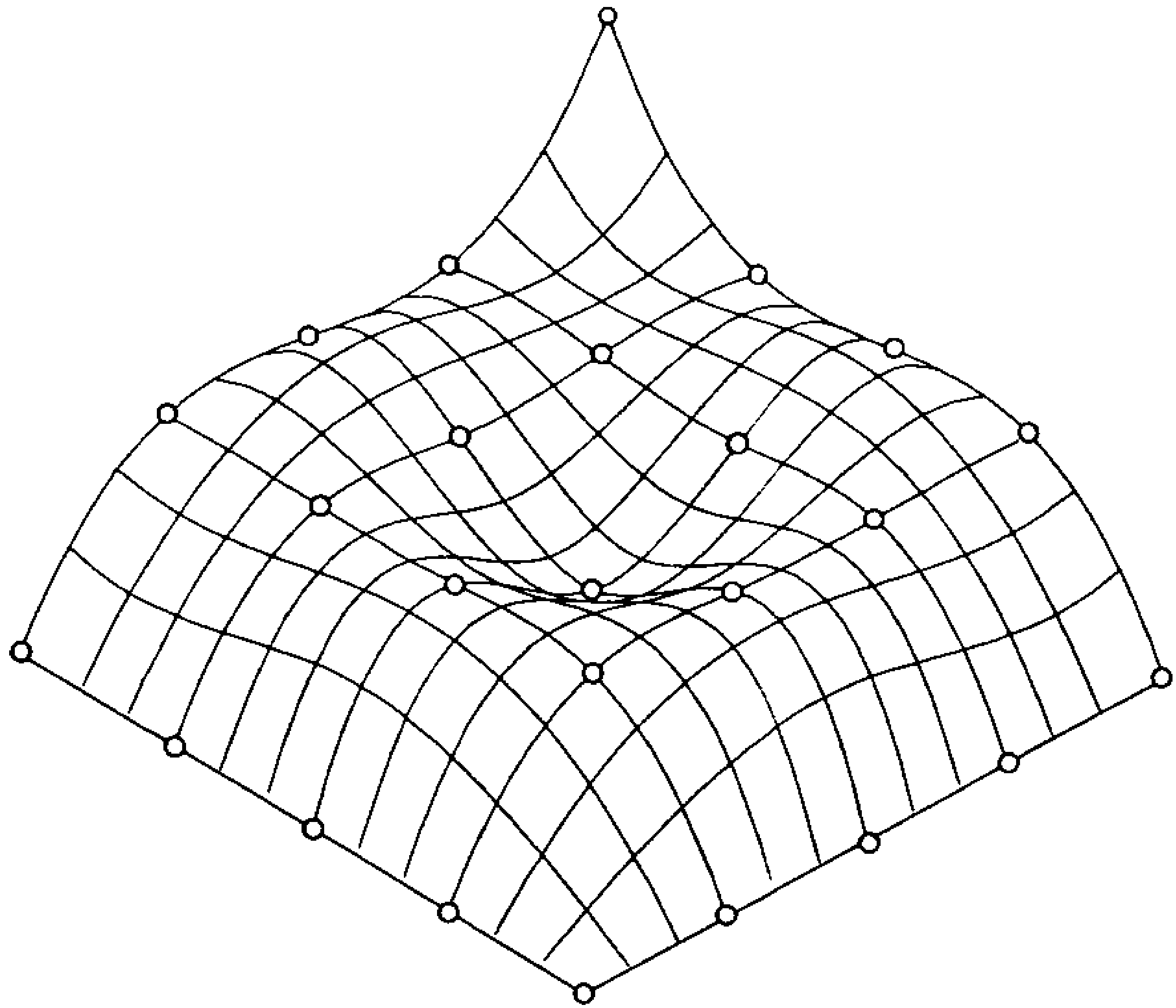


Fig. 6.2. Bicubic spline surface interpolating 5×5 points (circles).

Tensor product bsplines

- (As you'd expect) form is

$$\sum_{ij} \mathbf{P}_{ij} N_{id}(s) N_{jk}(t)$$

- Two cases are most important
  - periodic (surface is a torus)
  - patches
    - we repeat knots at start and finish of s, t

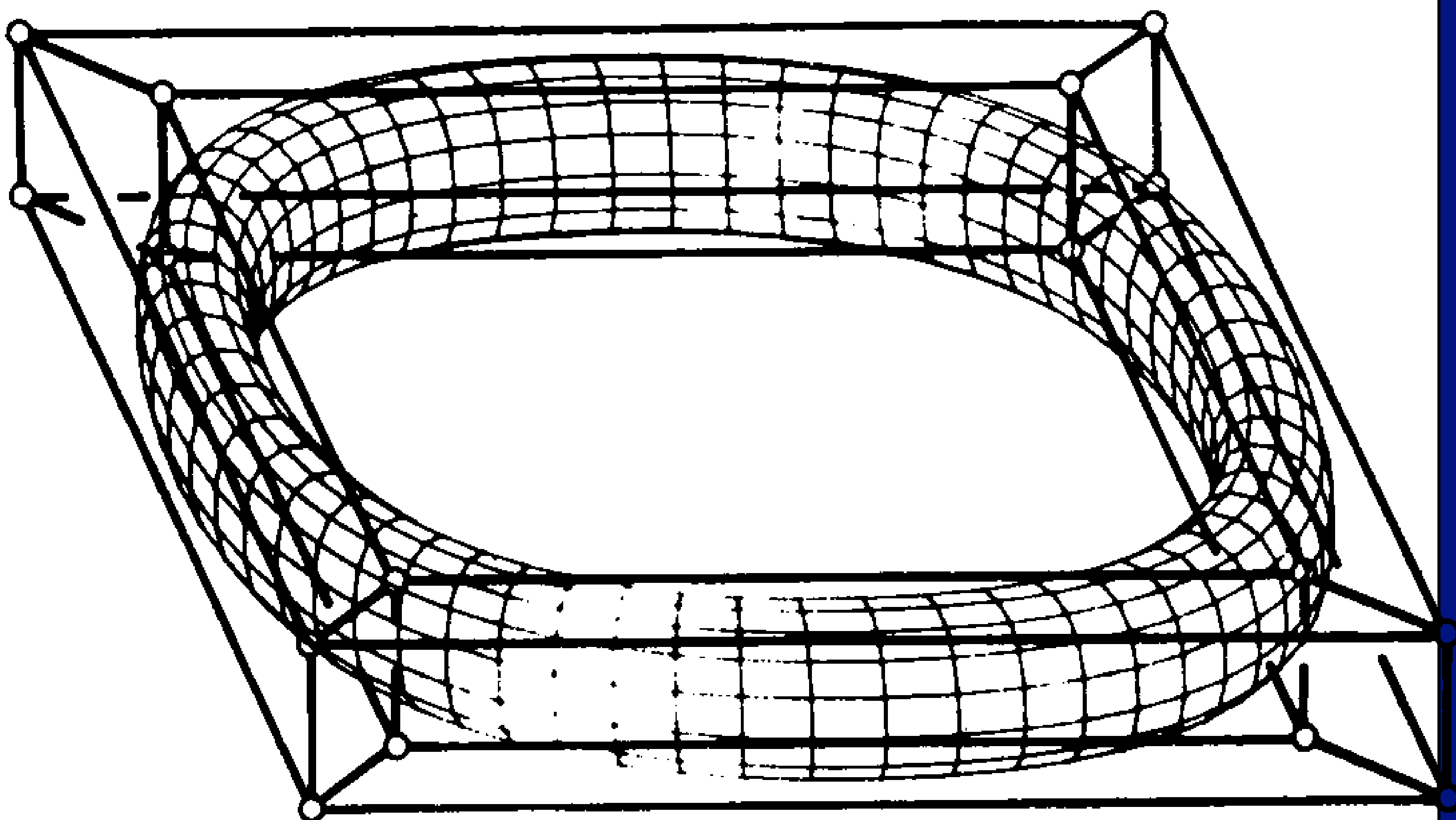


Fig. 6.13c. B-spline surface of order $k = 3$ with periodic basis functions in both the u- and v-directions.

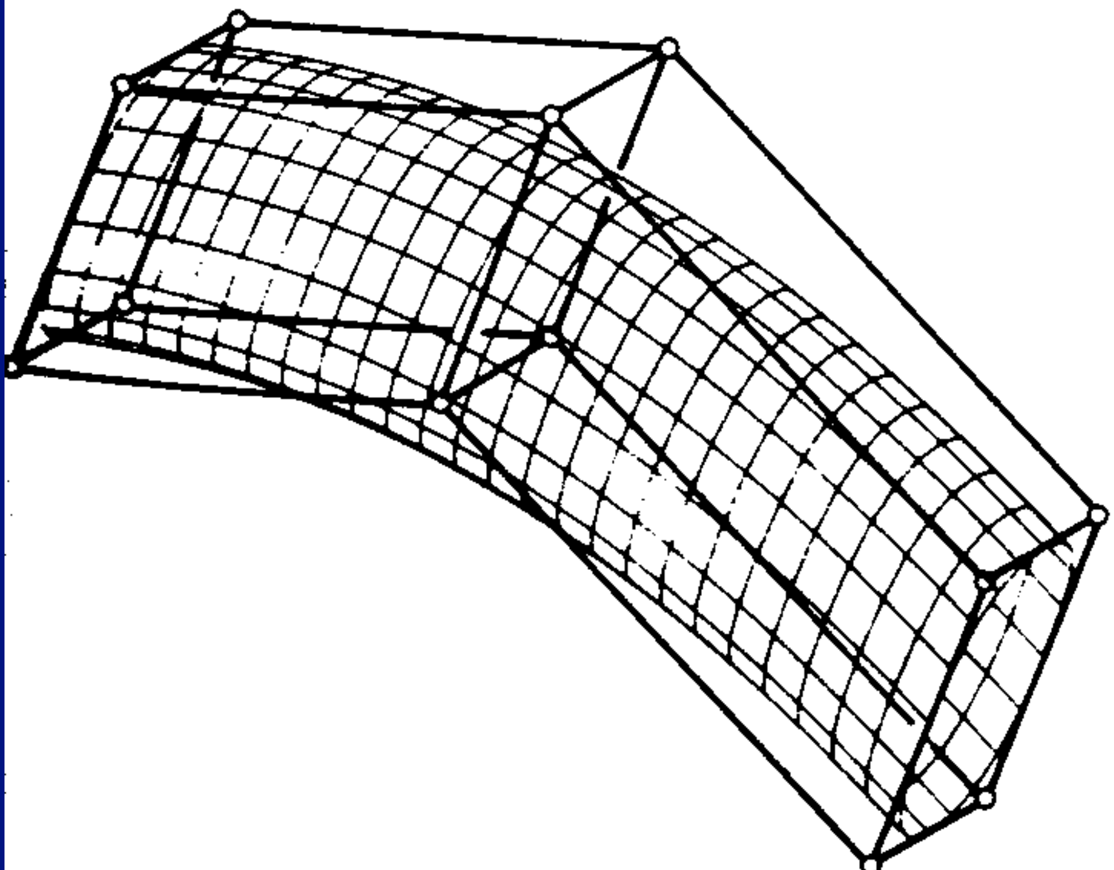
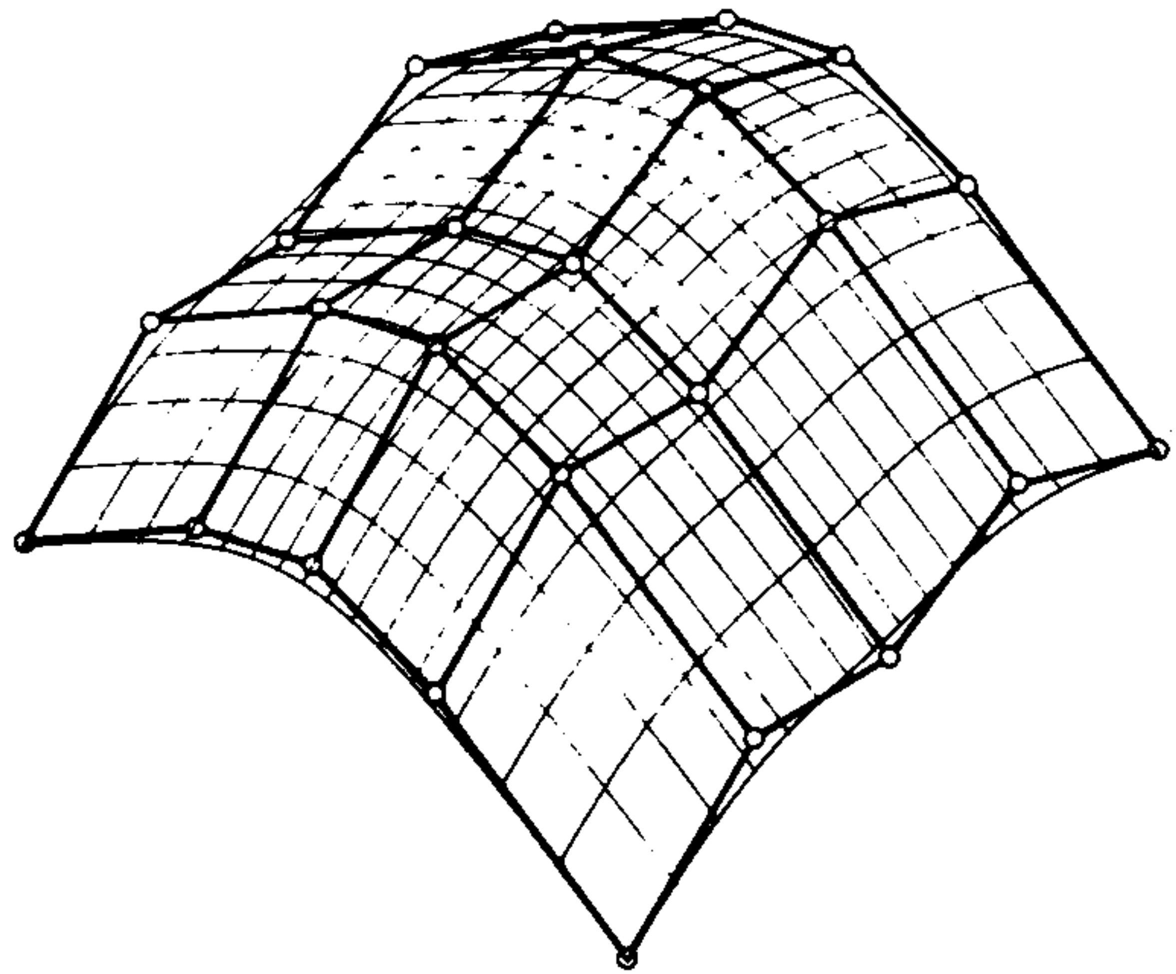


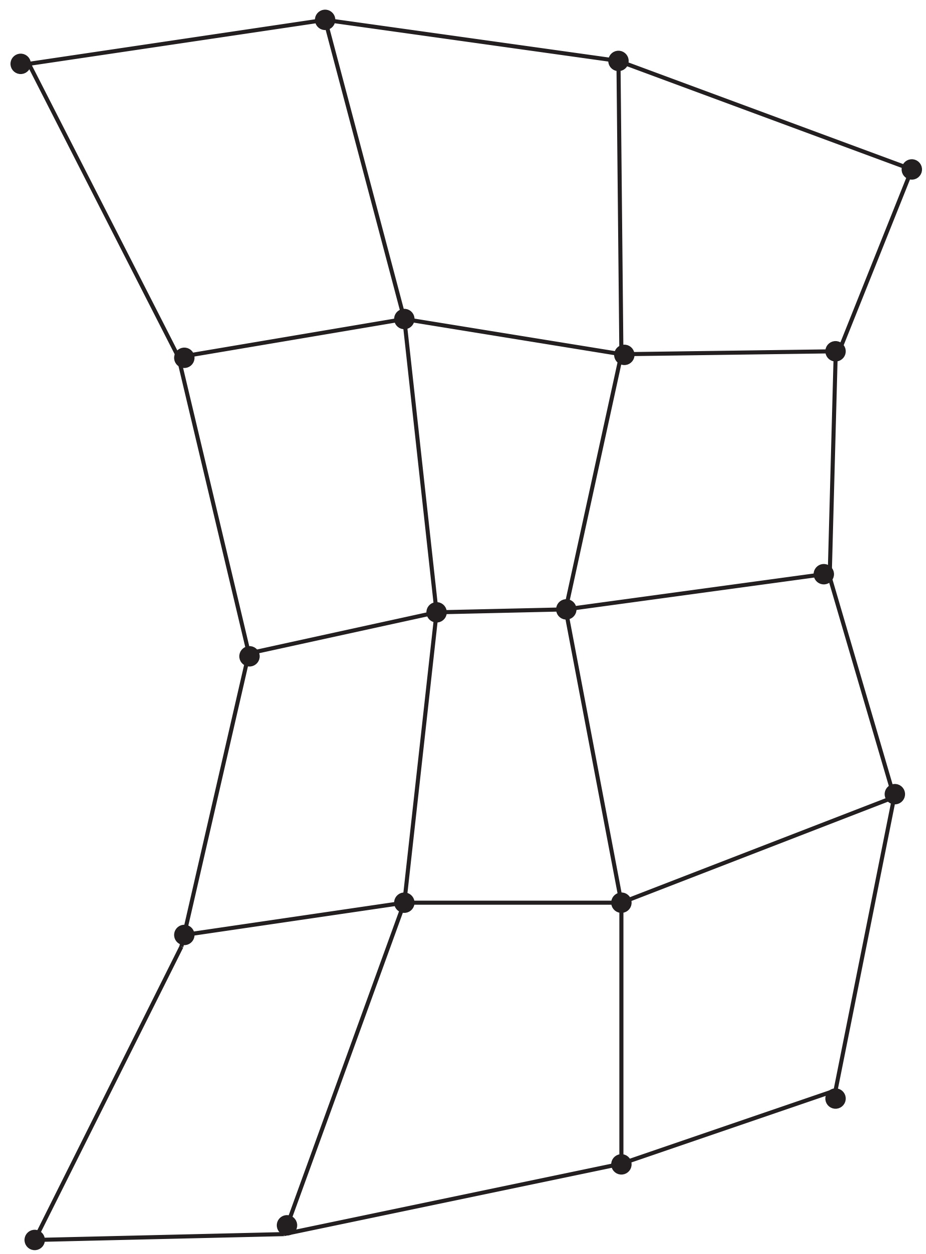
Fig. 6.13b. B-spline surface of order $k = 3$ with basis functions periodic in the u-direction.

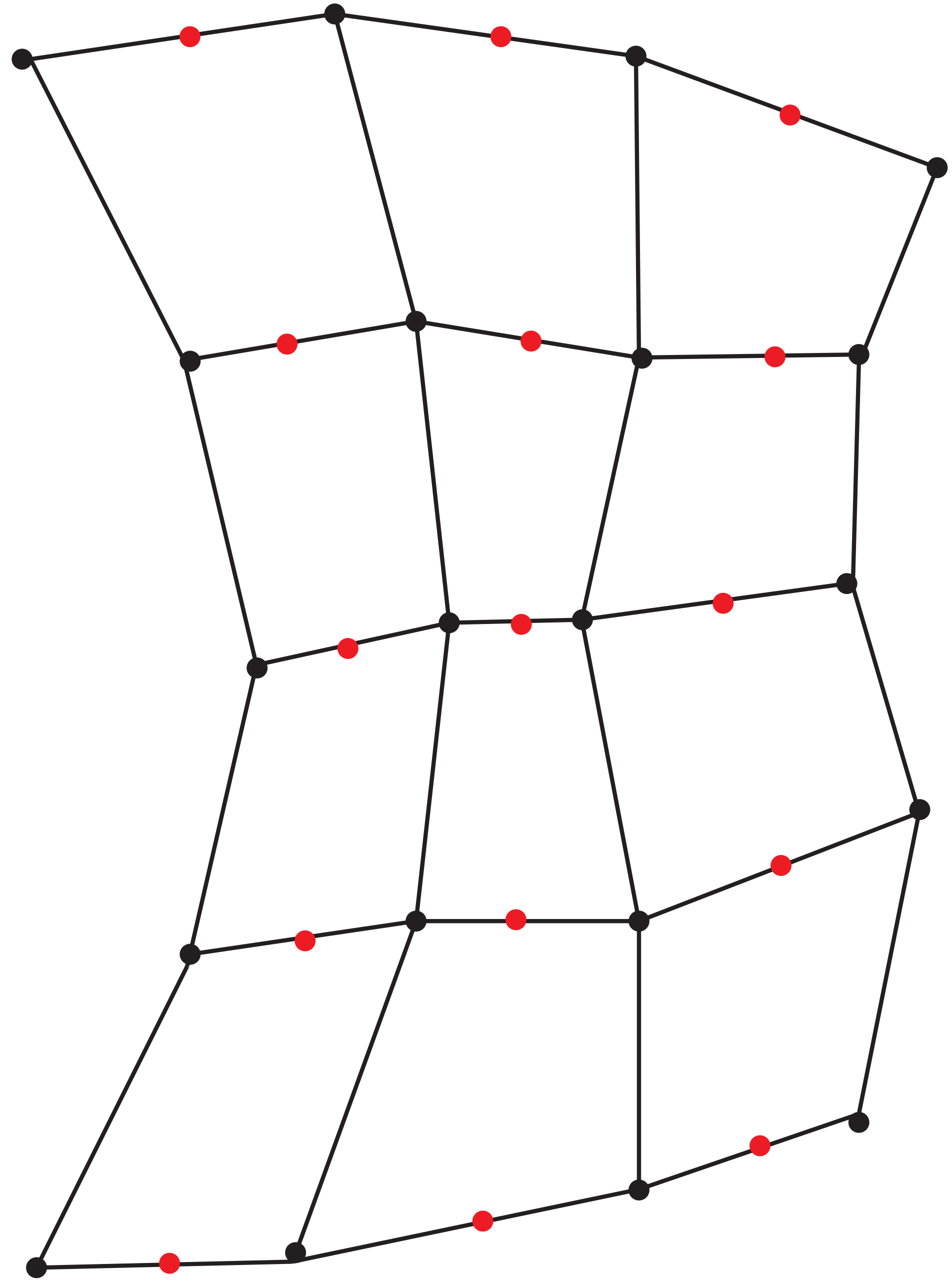
Fig. 6.13a. B-spline surface of order $k = 4$ and its de Boor net (nonperiodic basis functions).

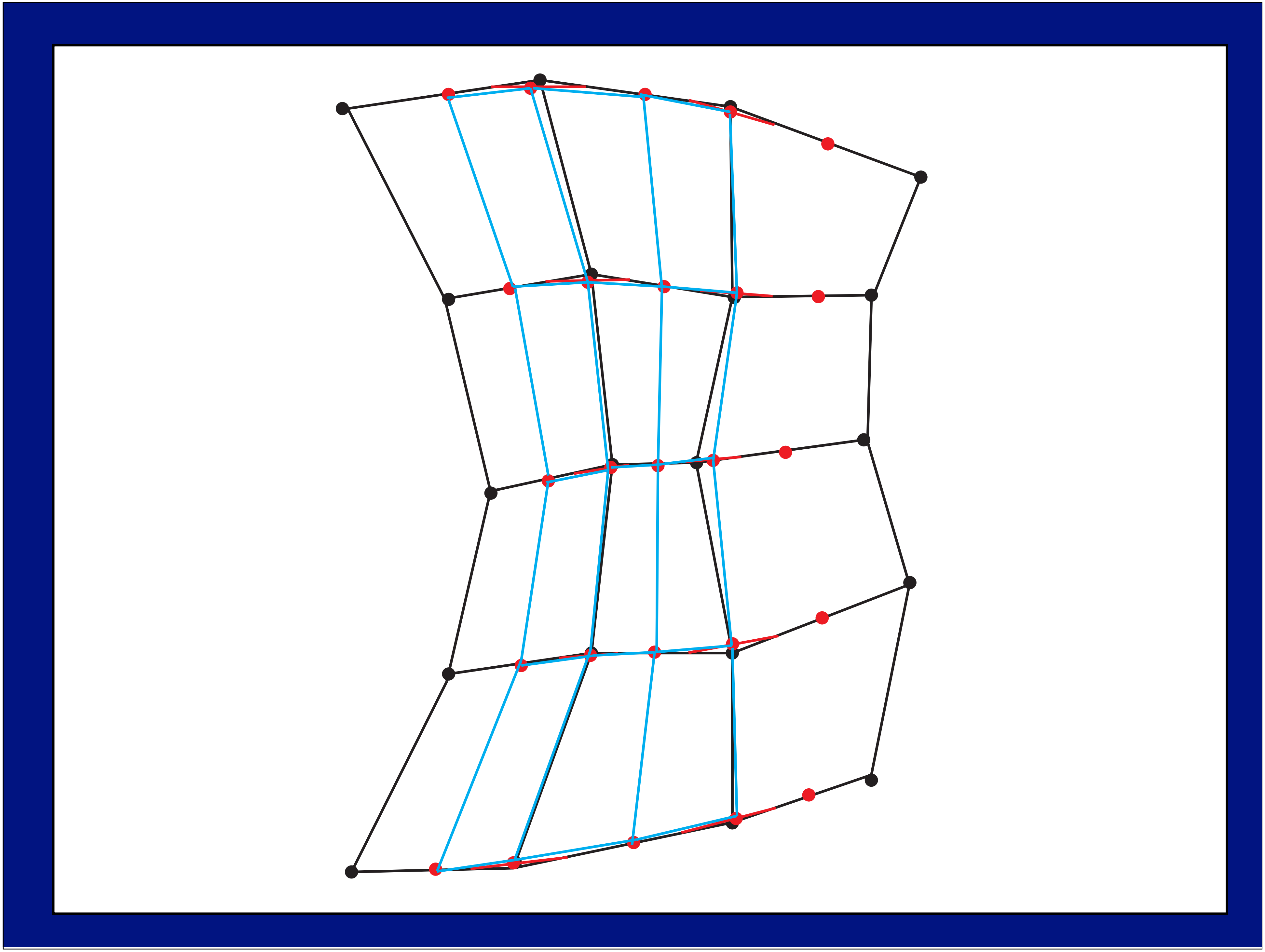


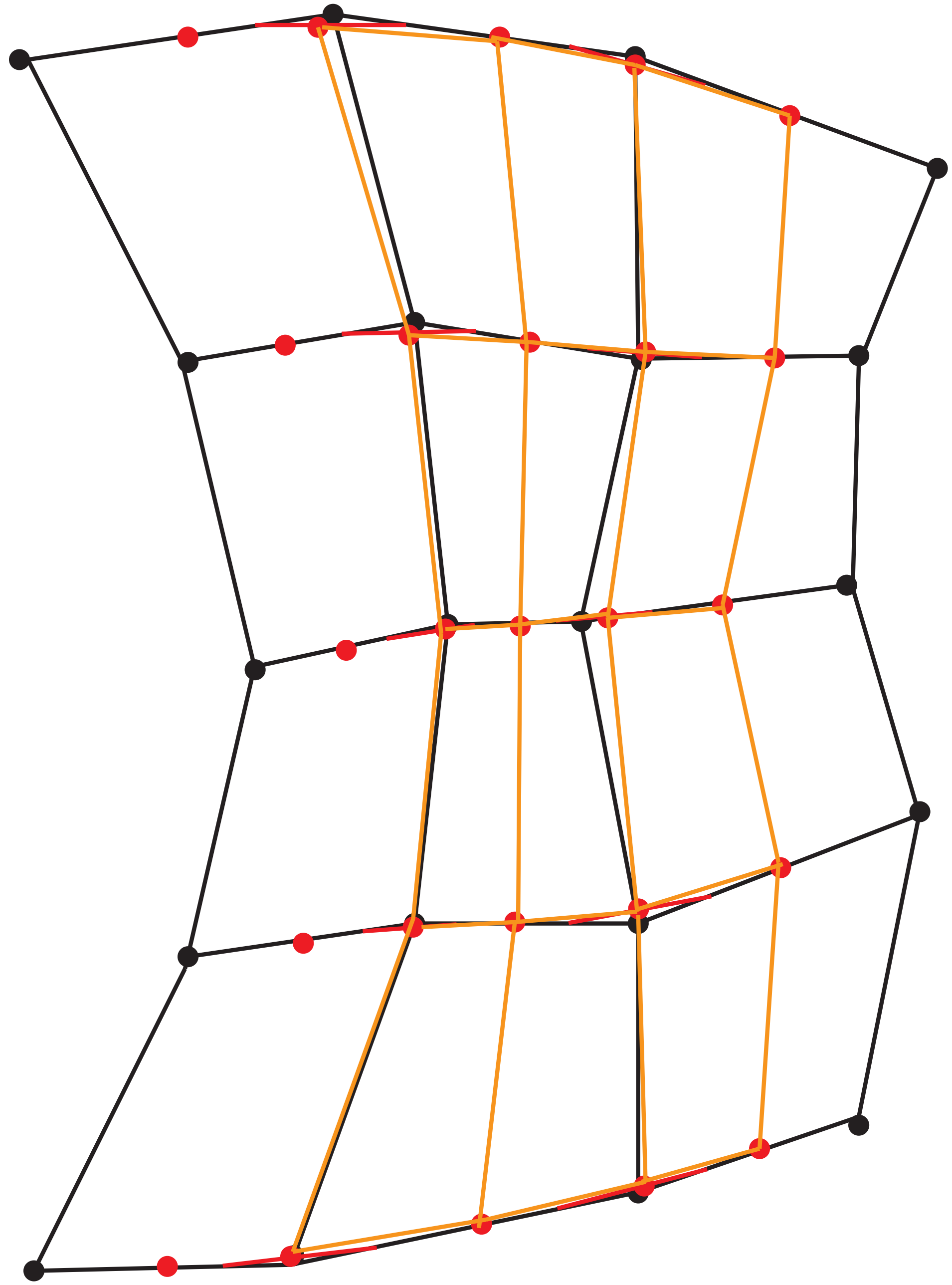
B-Spline subdivision

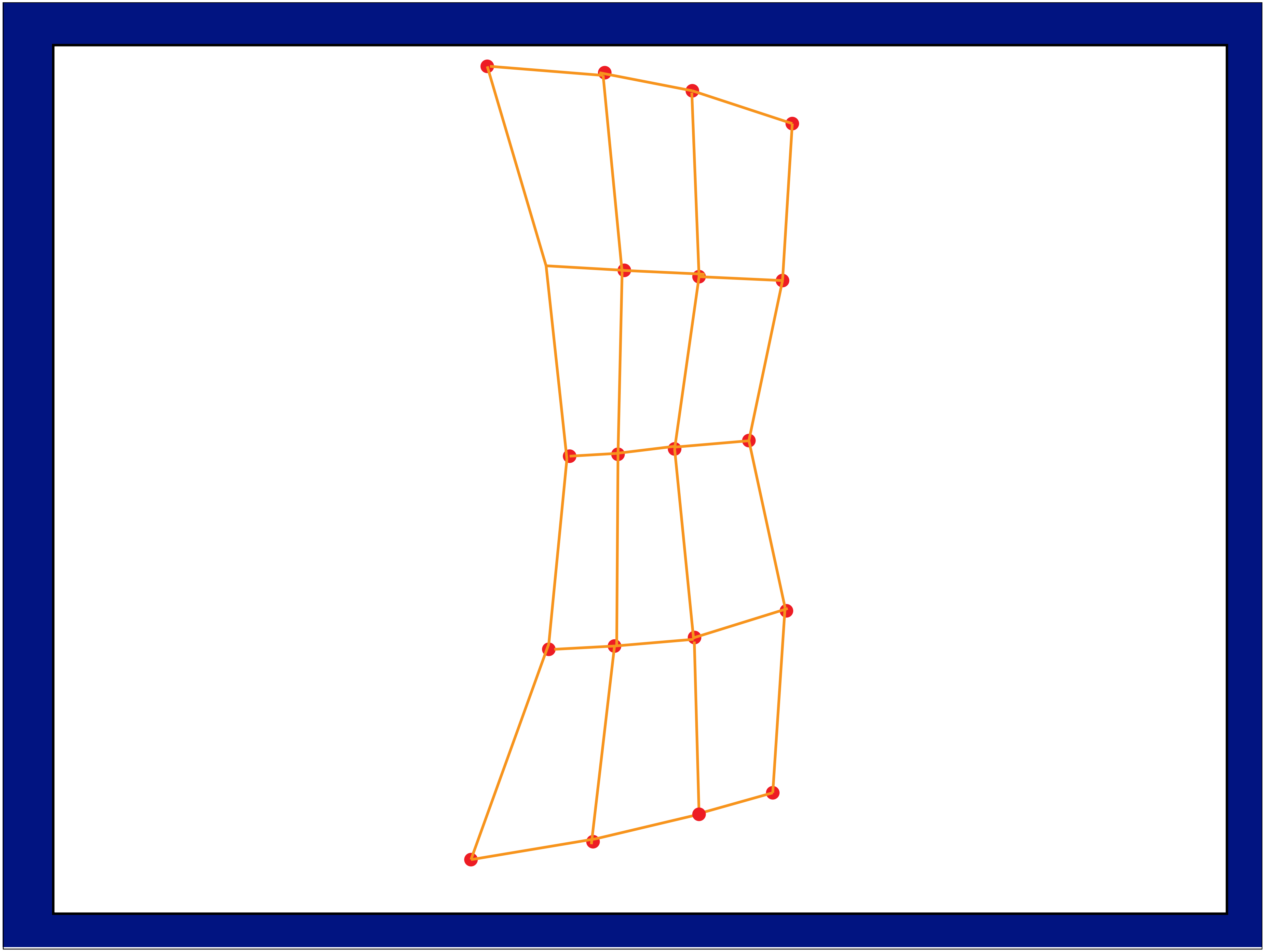
D.A. Forsyth, with slides from John Hart

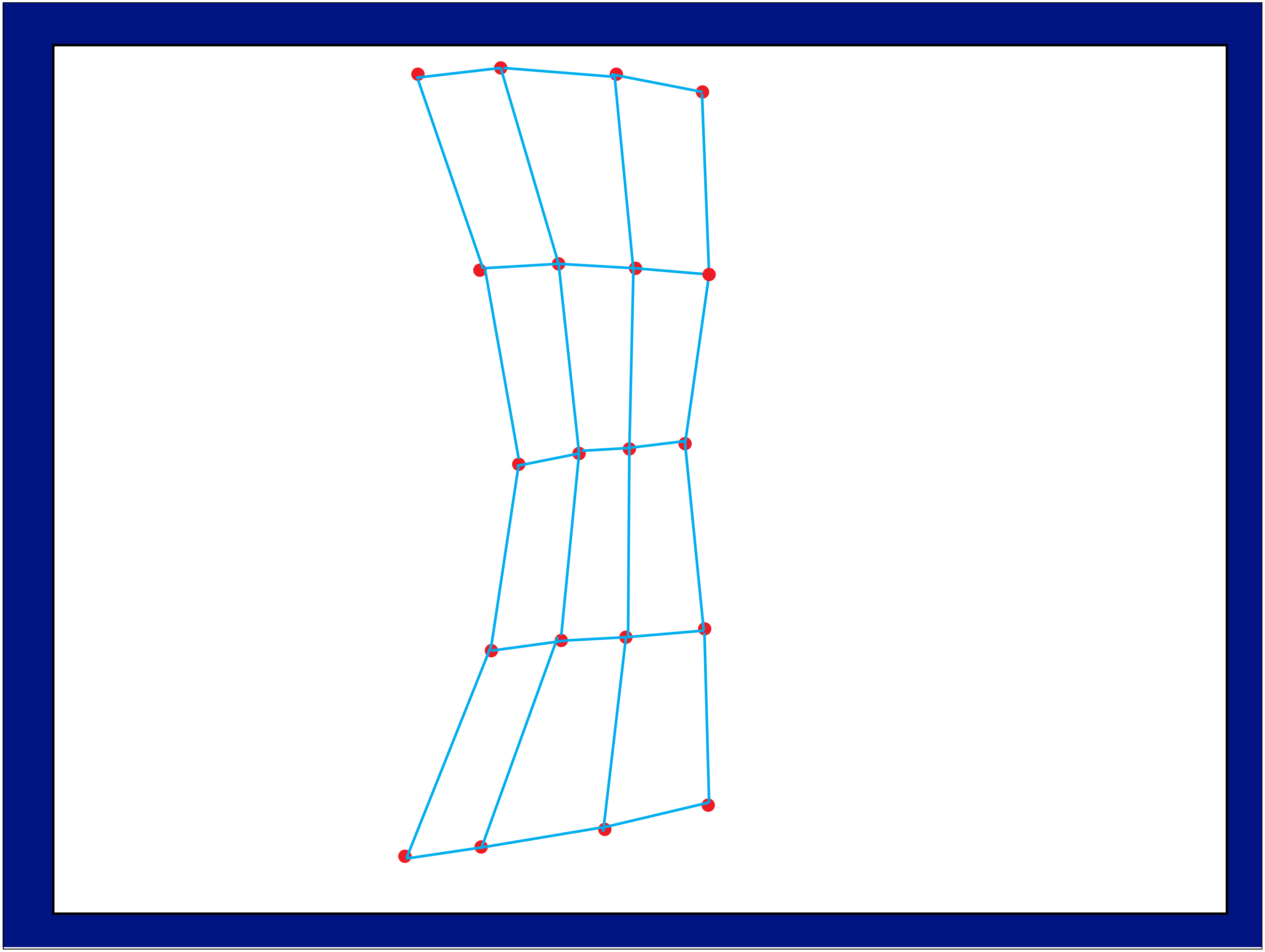


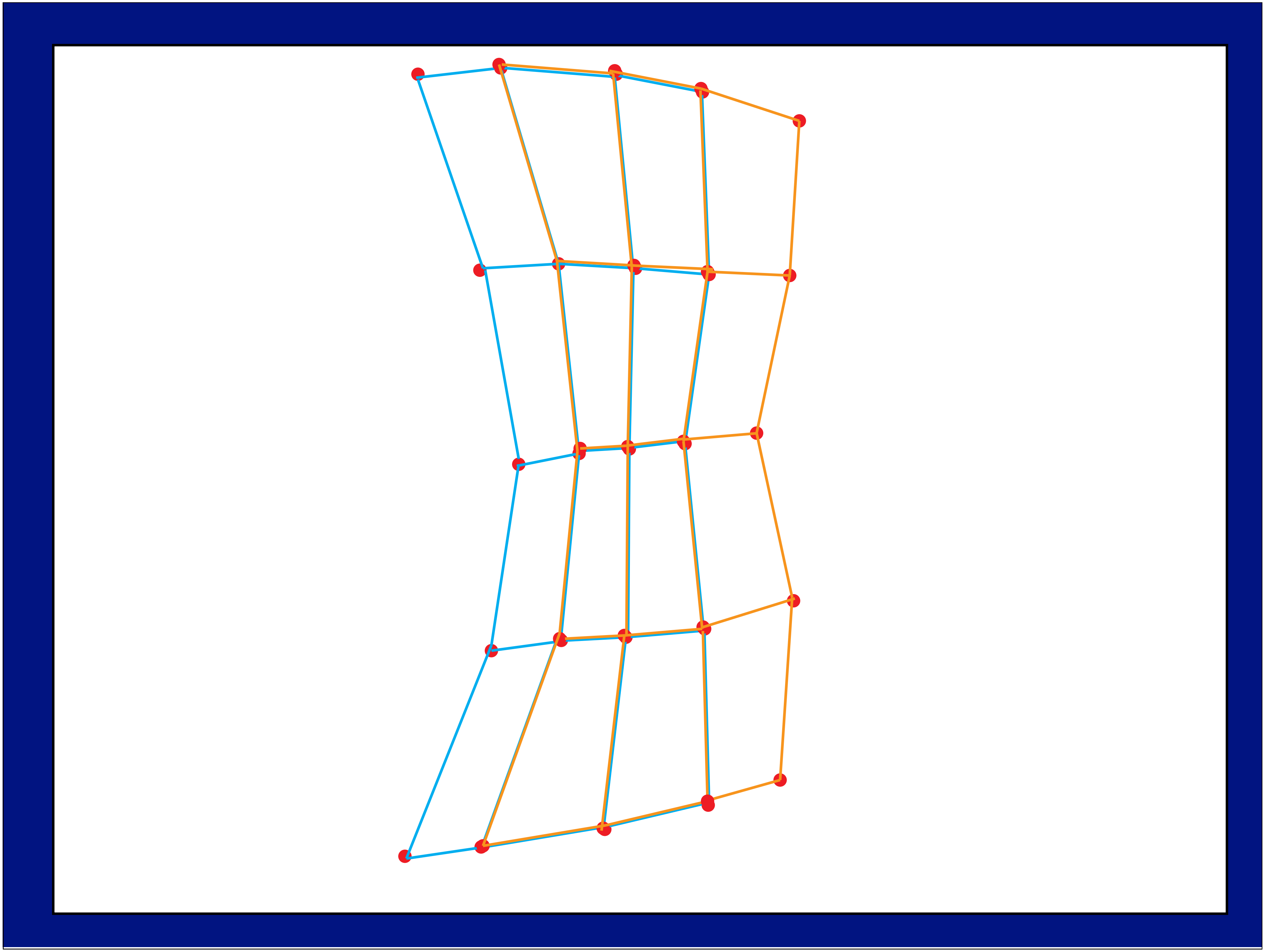




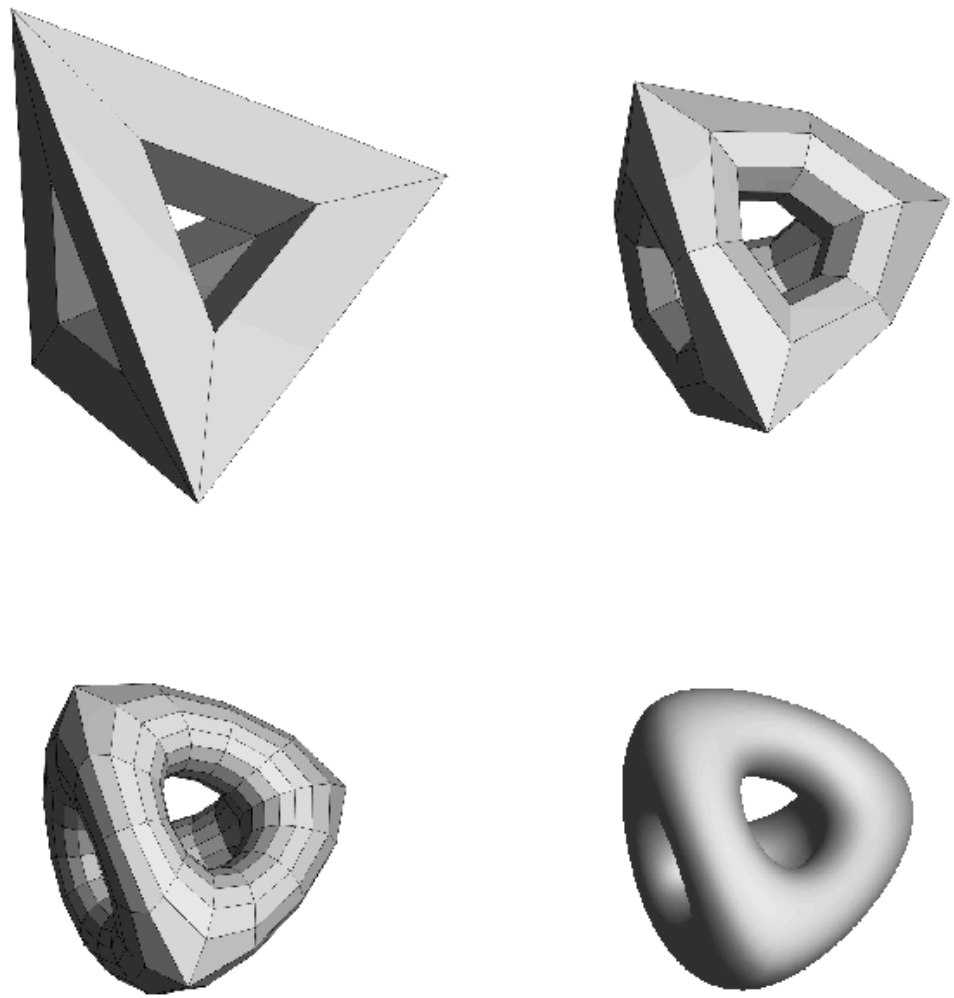








Example

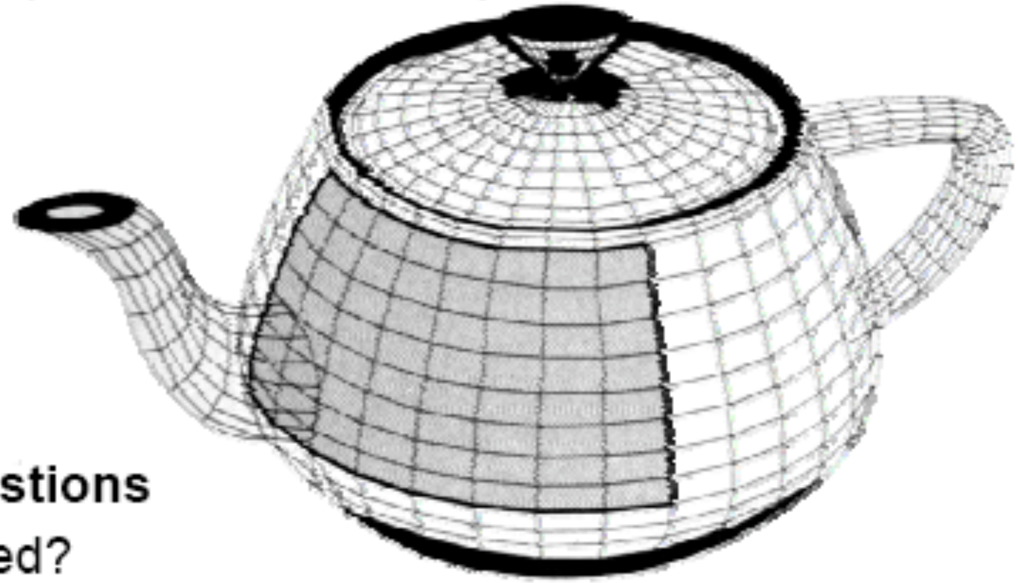


Another Example

Building Objects with Patches

Paste together multiple patches to cover entire object

- the Utah Teapot, for example, is built from 32 patches



This raises some tricky questions

- how many patches needed?
- how to guarantee continuity of patches? while animating!?
- how can we cut holes in the surface?
  - trimming curves — create boundary spline curves on surface