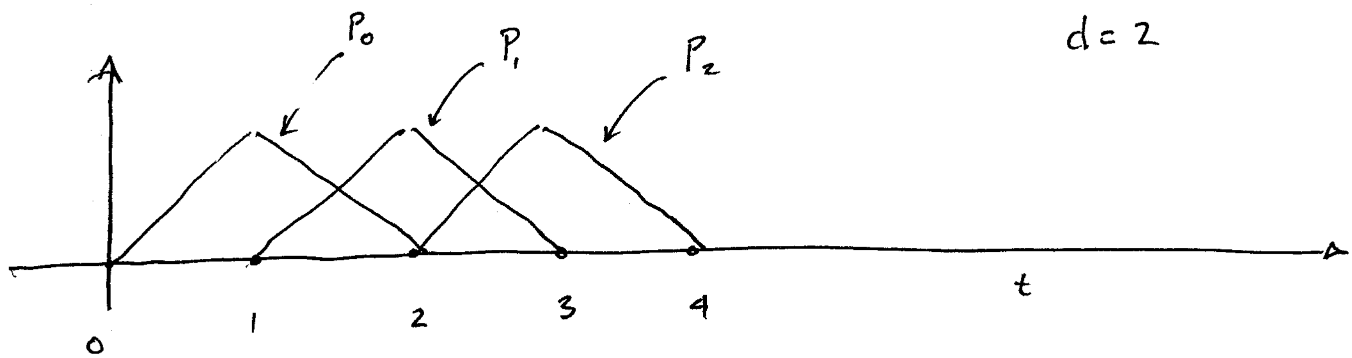
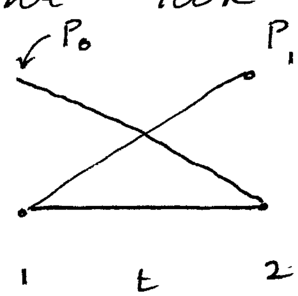
B-Splines and subdivision

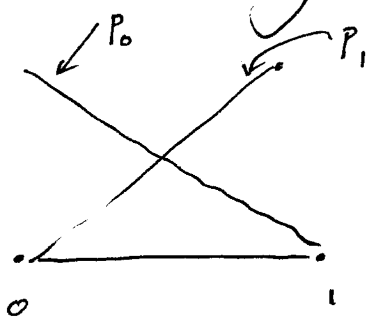
- Assume we have uniform knots



- So if we look at one interval.



- Reparametrize by $u = t - 1$



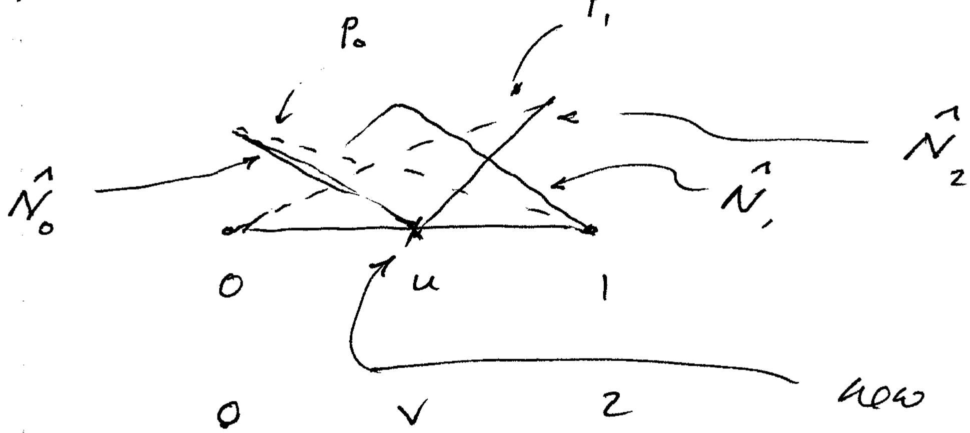
(2)

Now, we will write another parameter,

$v = 2u$, and make a new

set of B splines, w/ uniformly

spaced knots



now

$$N_0 = \frac{1}{2}(2\hat{N}_0 + \hat{N}_1)$$

$$N_1 = \frac{1}{2}(\hat{N}_1 + 2\hat{N}_2)$$

(by inspection)

so

$$p_0 N_0 + p_1 N_1$$

$$= p_0 \hat{N}_0 + \frac{1}{2}(p_0 + p_1)\hat{N}_1 + p_1 \hat{N}_2$$

We have <u>subdivision</u>

- a $d=2$ spline with c.p.s

$$P_0 , P_1 , P_2 \quad \cdots \quad P_n$$

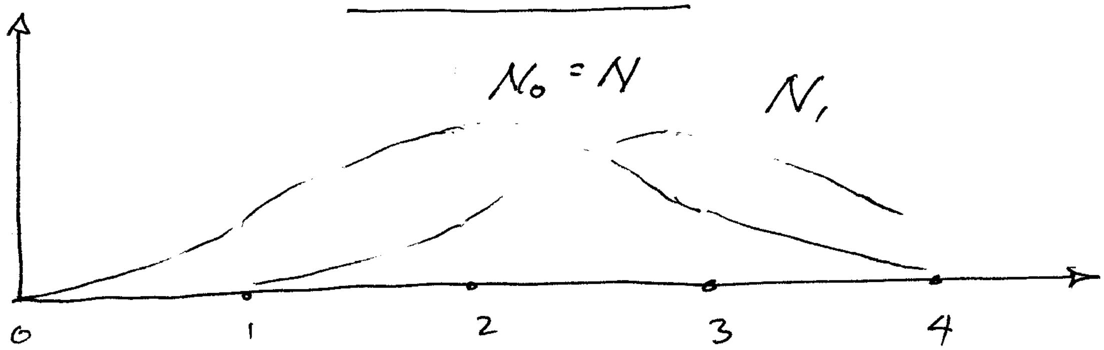
is <u>exactly</u> reproduced by a $d=2$ spline with c.p.'s

$$P_0 , \frac{P_0+P_1}{2} , P_1 \quad \cdots \quad P_{n-1} , \frac{P_{n-1}+P_n}{2} , P_n$$

- Not v. interesting in this form, but works for <u>higher degree</u>:

- $d=4$ is most interesting case

Cubic case



$$N(t) = \begin{cases} \frac{1}{6} t^3 & t < 1 \\ -\frac{1}{2} t^3 + 2t^2 - 2t + \frac{2}{3} & 1 \leq t < 2 \\ \frac{1}{2} t^3 - 4t^2 + 10t - \frac{22}{3} & 2 \leq t < 3 \\ -\frac{1}{6} t^3 + 2t^2 - 8t + \frac{32}{2} & 3 \leq t < 4 \end{cases}$$

Now

$$N_0 p_0 + N_1 p_1 + N_2 p_2 + N_3 p_3$$

$$= p_0 N(t+3) + p_1 N(t+2) + p_2 N(t+1)$$
$$+ p_3 N(t)$$

(Notice how the shift works!)

Now it is straightforward to verify that

$$N(t) = \frac{1}{8} N(2t) + \frac{1}{2} N(2t-1) + \frac{3}{4} N(2t-2) + \frac{1}{2} N(2t-3) + \frac{1}{8} N(2t-4)$$

(Wade through all the cases!).

Now, we have a curve

$$p_3 N(t) + p_2 N(t+1) + p_1 N(t+2) + p_0 N(t+3)$$

We want to write as

$q(t)$ such that $q(2t) = p(t)$

for $t = [0 \cdots 1/2]$

(this is first half)

$$p_0 N(t+3) = p_0 \Big[\tfrac{1}{8} N(2t+6) + \tfrac{1}{2} N(2t+5) + \tfrac{3}{4} N(2t+4) + \tfrac{1}{2} N(2t+3) + \tfrac{1}{8} N(2t+2) \Big]$$

$$p_1 N(t+2) = p_1 \Big[\tfrac{1}{8} N(2t+4) + \tfrac{1}{2} N(2t+3) + \tfrac{3}{4} N(2t+2) + \tfrac{1}{2} N(2t+1) + \tfrac{1}{8} N(2t) \Big]$$

<u>Now</u>:

$$q(2t) = q_0 N(2t+3) + q_1 N(2t+2) + q_2 N(2t+1) + q_3 N(2t)$$

Match $N(2t+3)$ terms:

$$\left.\begin{aligned} q_0 &= \tfrac{1}{2}(p_0 + p_1) \\ q_1 &= \tfrac{1}{8} p_0 + \tfrac{3}{4} p_1 + \text{others} \ldots \end{aligned}\right] \text{ and so on}$$

<u>Now</u> • notice that there is nothing to match the $N(2t+6)$ term which appears in p but not in q.

• <u>but</u> $t \in [0, 1/2]$

$\therefore \quad 2t+6 \in [6,7]$

and N is zero in this range!

we get

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/8 & 3/4 & 1/8 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/8 & 3/4 & 1/8 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

<u>Now</u> q is the <u>first half</u> of the p curve

second half is

$$r(2t-1) = r_0 N(2t+2) + r_1 N(2t+1) + r_2 N(2t) + r_3 N(2t-1)$$

$t \in [\frac{1}{2}, 1]$

Again, match terms

→ Notice that $N(2t-3)$ doesn't match anything

OK, because $t \in [\frac{1}{2}, 1]$ $\therefore$ $2t-3 \in [-2,-1]$ and so N is zero

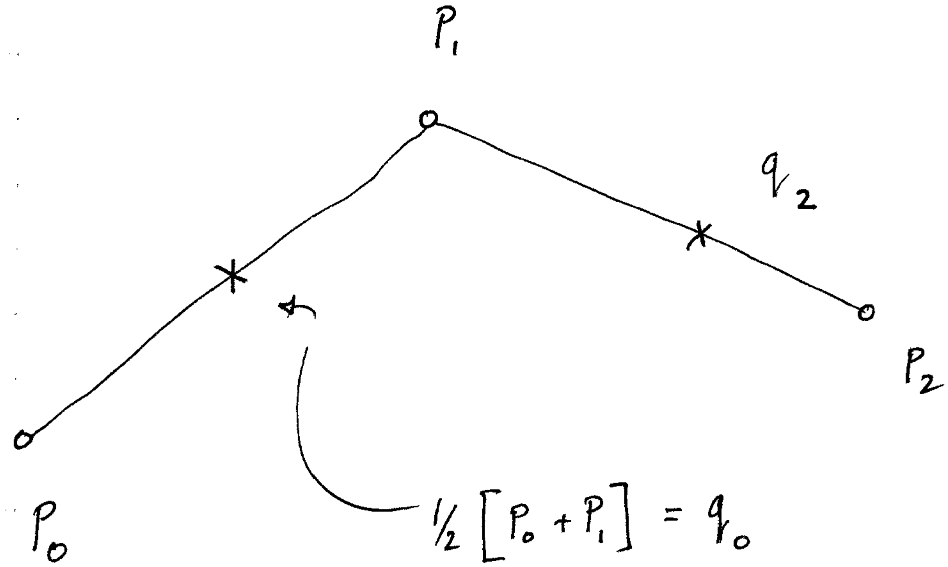
get

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1/8 & 3/4 & 1/8 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/8 & 3/4 & 1/8 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Two ways to read this

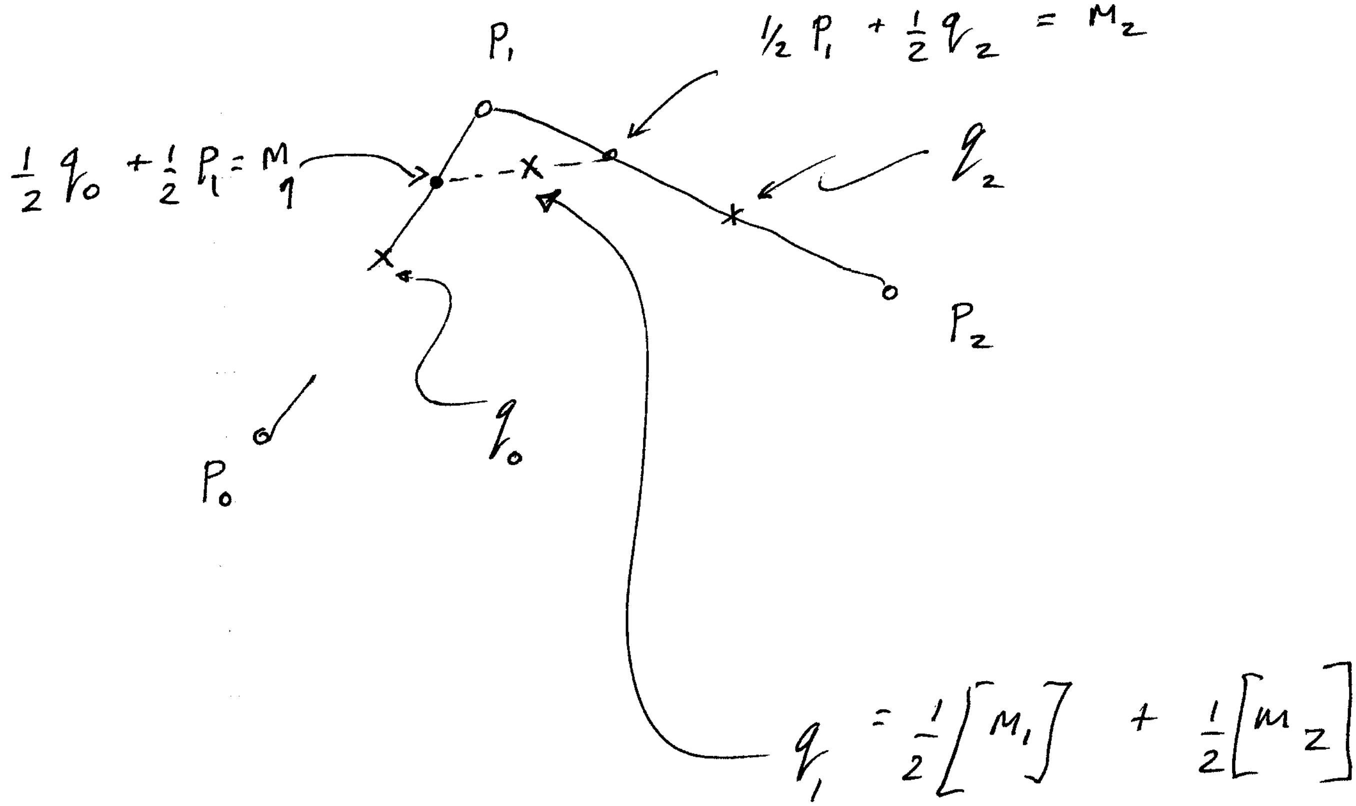
<u>Simple</u>: $p \begin{cases} \to q \\ \to r \end{cases}$ splits

<u>More interesting</u>: we have added pts to control structure, to get cs for same curve



$$q_1 = \frac{1}{8} P_0 + \frac{3}{4} P_1 + \frac{1}{8} P_2$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\left[\frac{1}{2}P_0 + \frac{1}{2}P_1\right] + \frac{1}{2}P_1\right)\right] + \frac{1}{2}\left[\left(\frac{1}{2}P_1 + \frac{1}{2}\left[\frac{1}{2}P_1 + \frac{1}{2}P_2\right]\right)\right]$$



<u>Procedure</u>:

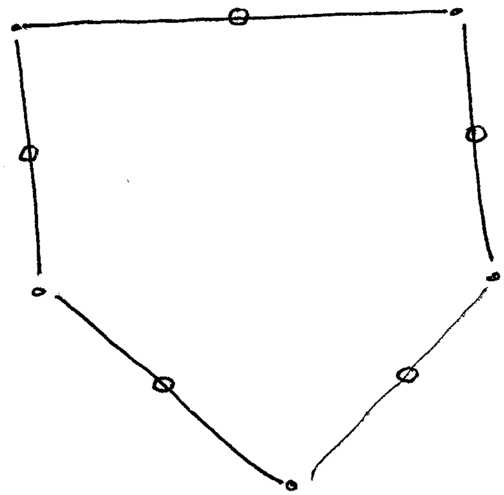
- take closed polygon

I insert midpoints

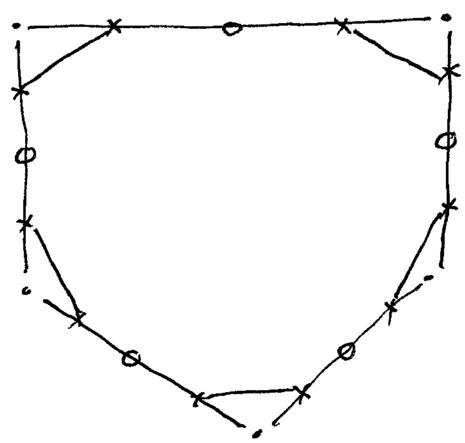
II ~~inser~~ mark midpoints M_i ~~of edges~~ and join with struts

III midpoints of these struts taken w/ points in I, give polygon.

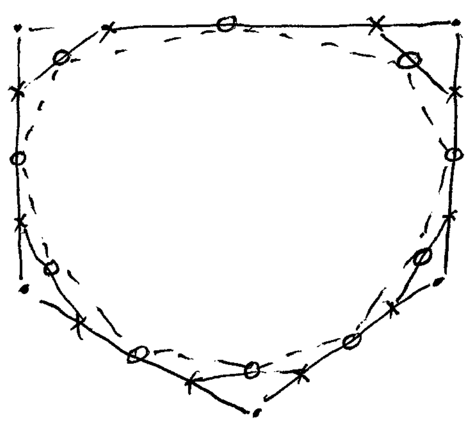
I



II



III



dashed line is <u>new</u> polygon