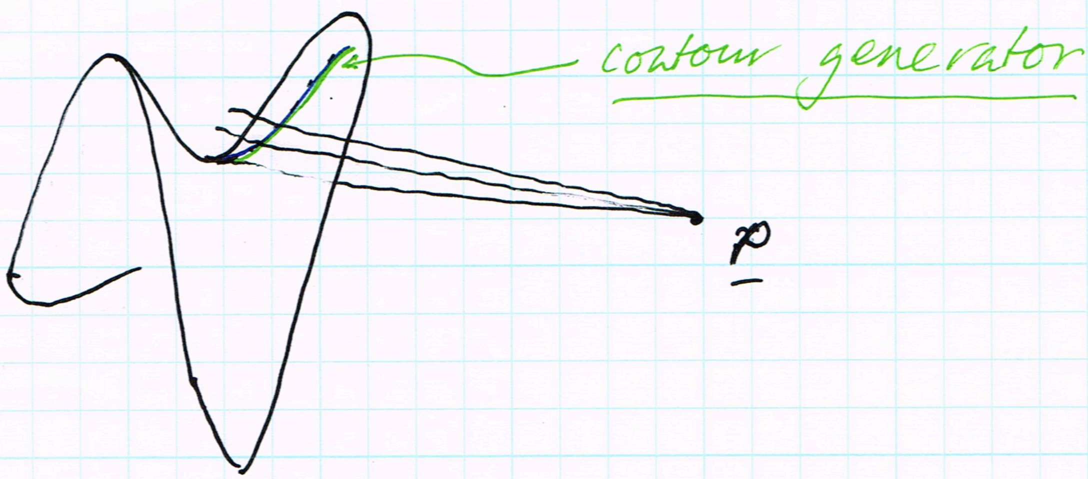
Contour generators and outlines

A contour generator on S is a curve where rays from $\underline{p}$ ← some chosen point are tangent to the surface

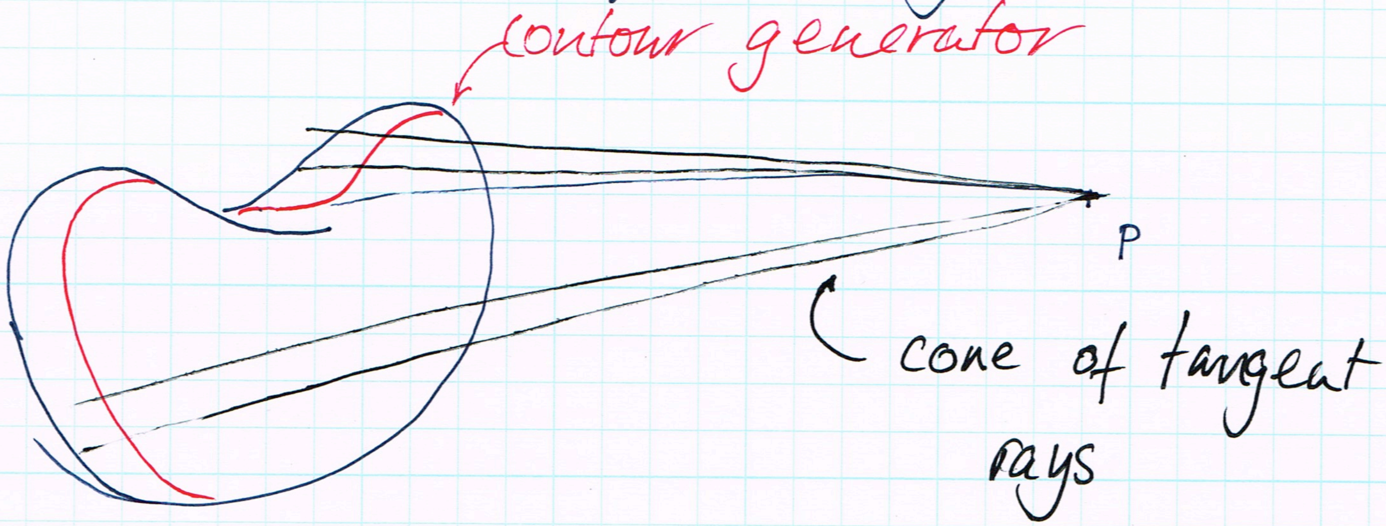


$\underline{p}$ could be infinitely far away, in which case all rays to $\underline{p}$ are $\parallel$

$\therefore$ so rays in dir $\underline{v}$ are tangent to surf

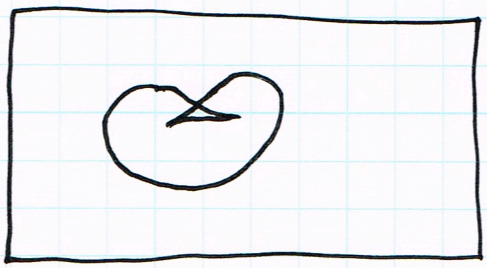
An outline is the projection into the image of a contour generator

- Often helpful to think of this as a cone of tangent rays



- Image is obtained by slicing this cone with a plane
- thinking about the cone avoids confusion caused by rotation about the focal point.

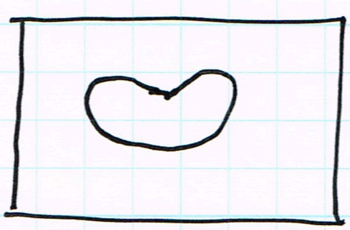
- We usually think about all components (not just visible) of these curves



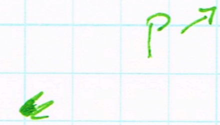
← outline of bean in previous image

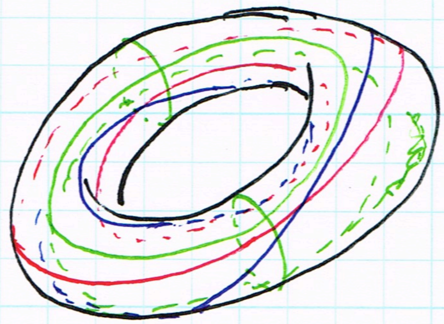
- outlines exhibit complex behavior w/ change of viewpoint, as do contour generators (next p.)

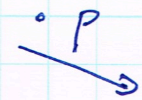
- a <u>silhouette</u> consists of the visible exterior points of outline

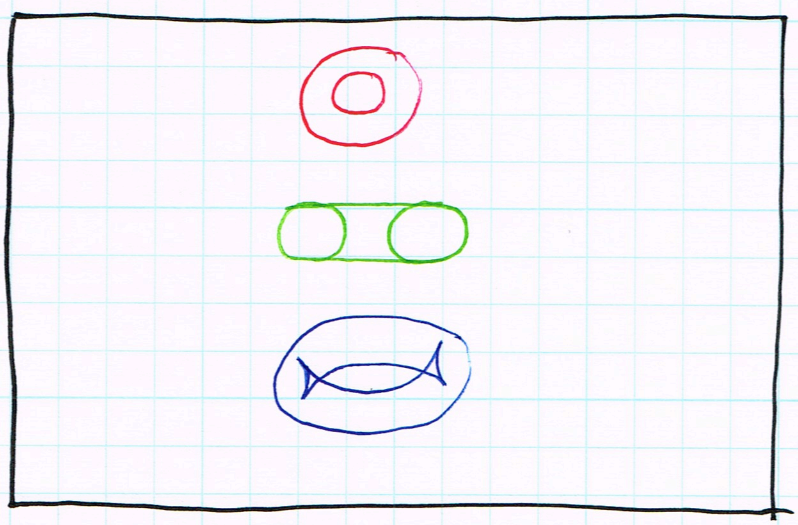


← silhouette of bean.



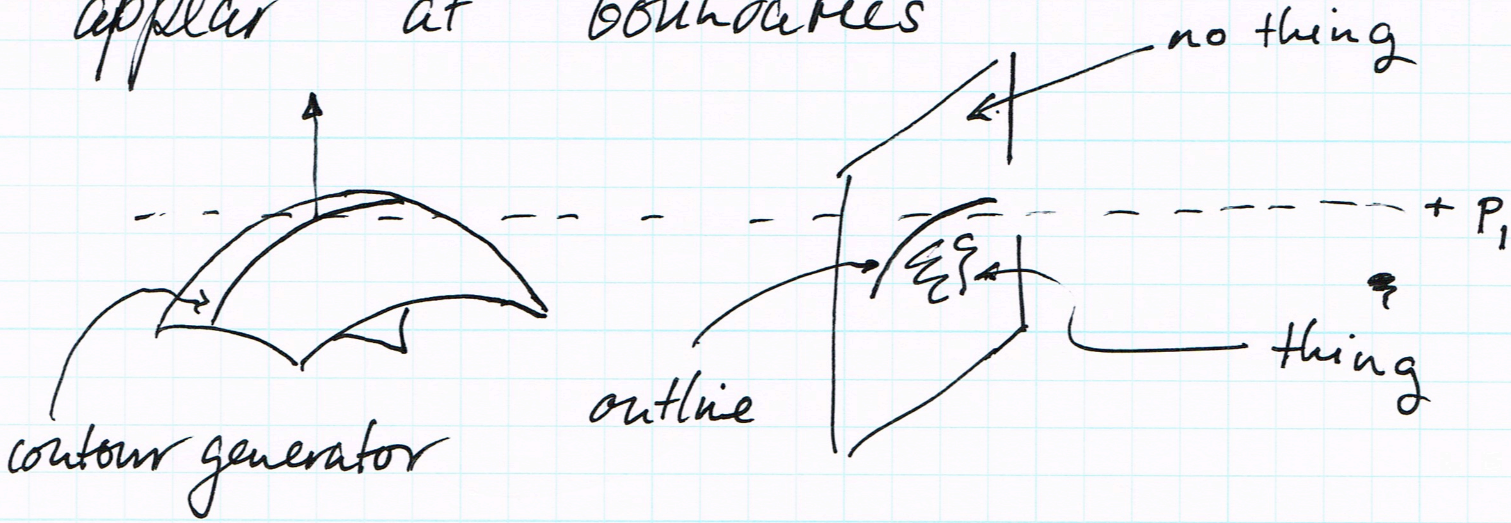




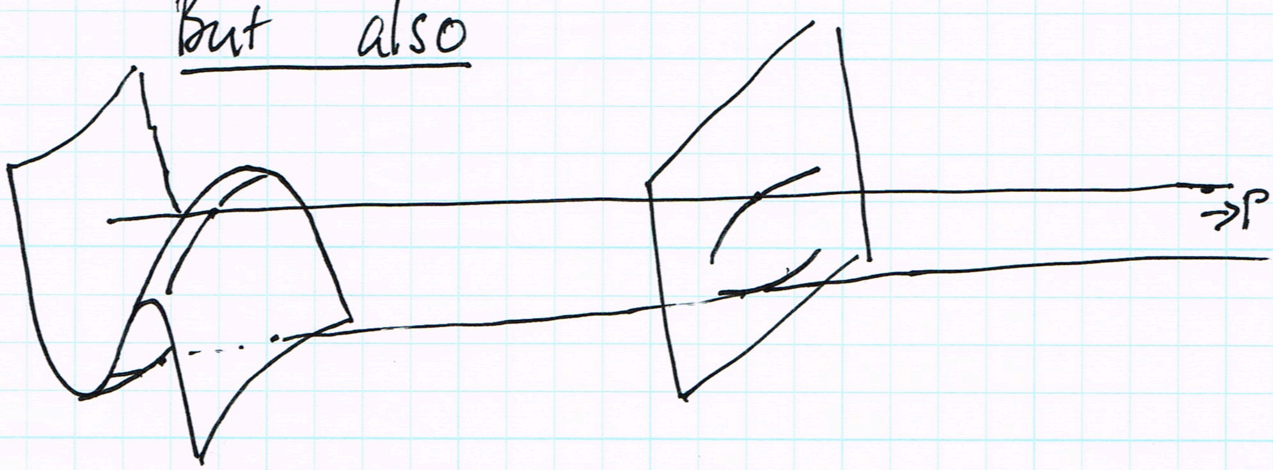


<u>NOTE</u>: outlines + C.G.s are notoriously confusing – famous names have made major errors in print; ~~they~~ C.G.'s are <u>hardly</u> <u>ever</u> plane, a point of much confusion.

These curves are interesting because they appear at boundaries



But also



We tend to keep track of all components because counting visibility is hard

Some properties of C.G.'s + outlines

- The contour generator for a sphere is plane. The outline is a plane section of a right circular cone.

- Proof: symmetry does the work. The geometry is rotationally symmetric about the line from p to center of sphere; C.G. must have this symmetry so be section of sphere by plane normal to axis. Rays form a cone.

- Corollary: perspective views of spheres could be ellipses (in principle, you could calibrate a camera like this)

Q: Why do spheres always look round?

write $\phi = 0$ for an <u>implicit surface</u>

then at x_0, y_0, z_0, Normal is

$$\nabla\phi\Big|_{x_0, y_0, z_0}$$

and tangent plane is

$$\nabla\phi\Big|_{x_0 y_0 z_0} \cdot \underline{x} - \nabla\phi\Big|_{x_0 y_0 z_0} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0$$

(i.e. plane through x_0, y_0, z_0 w/ normal $\nabla\phi$.)

then C.G. is curve (in $\underline{x} = (x, y, z)$)

$$\phi(\underline{x}) = 0$$

$$\nabla\phi^T p - \nabla\phi^T \underline{x} = 0$$

if ϕ is polynomial, all sorts of neat stuff happens. (See my paper, and good luck to you!)

One application:

- The C.G. of a generic quadric is plane

- generic quadric:

$$\frac{x^T A x}{2} + b^T x + c = 0 \quad = \phi$$

for $\det A \neq 0$, $A - A^T = 0$

so C.G. is:

$$\phi = 0$$

$$x^T A p + b^T p - \left[x^T A x + b^T x \right] = 0$$

!

BUT

$$x^T A x + b^T x = 2\left[0 - c \right] - b^T x$$

and result follows

- an <u>Algebraic surface</u> of degree d.

is a set of all points x st

$$\phi(x) = 0$$ where ϕ is polynomial of degree d.

- a generic ~~ray~~ line intersects this surface in d points

(subst. $\underline{x} = \underline{u} + t\,\underline{v}$)

- The C.G., ~~and hence the~~ ~~is~~ can be given by two polynomial eqns

$$\phi = 0 \quad \leftarrow \text{degree } d$$

$$\psi = 0 \quad \leftarrow \text{degree } (d-1)$$

which means the C.G. intersects a generic plane in $d(d-1)$ pts (use Bézout)

The global theory of C.G.'s, outlines is poorly understood in all but a few special cases (algebraic; SOR; some others). Good evidence this is because those cases (which are nasty + complicated) are the easy ones.

Local theory is fairly easy BUT probably not much use (? these curves have global structure, in ALL cases ?)

1) View direction and Tangent to Contour generator are conjugate

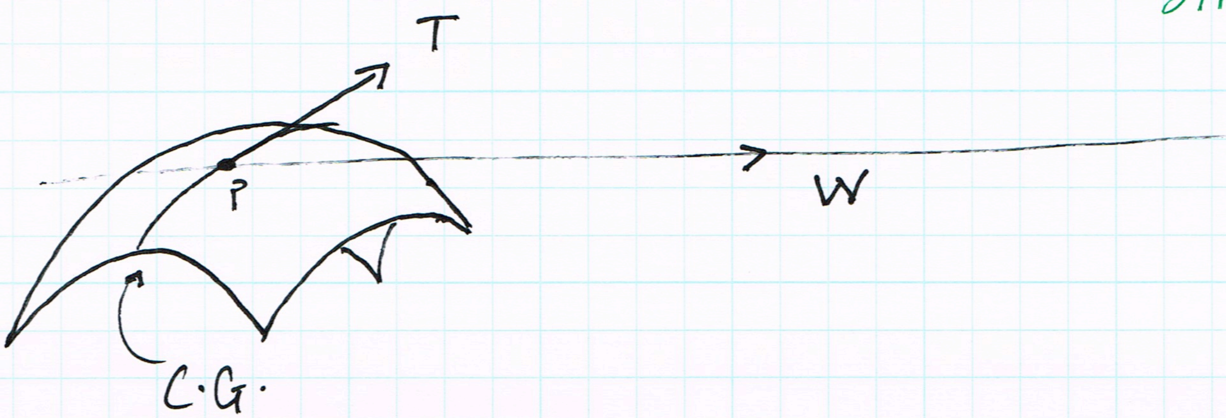
Write W for view dirn (which is a tangent vector at

T for tangent to C.G. point of interest

Then

$$\underbrace{\mathrm{II}(T, W) = 0}$$ at any p.t. on C.G.

→ this condition ≡ T, W are conjugate dir'n's



at p, $N \cdot W = 0$ (defn of C.G.)

so $\nabla_T (N \cdot W) = 0$

directional derivative of $N \cdot W$ in T dir'n

so $(\nabla_T N) \cdot W + \underline{N \cdot (\nabla_T W)} = 0$

$\mathrm{II}(T, W) + 0 = 0$

see next p.

consider $\nabla_T W$; two cases;

(a) View is orthographic $\therefore$ W is const

$\therefore \nabla_T W = 0$

(b) View is perspective, so

$$W = p - f$$

P.O.I. — focal point

$\nabla_T W$: slide p very slightly along T to $p + \varepsilon T$, then form

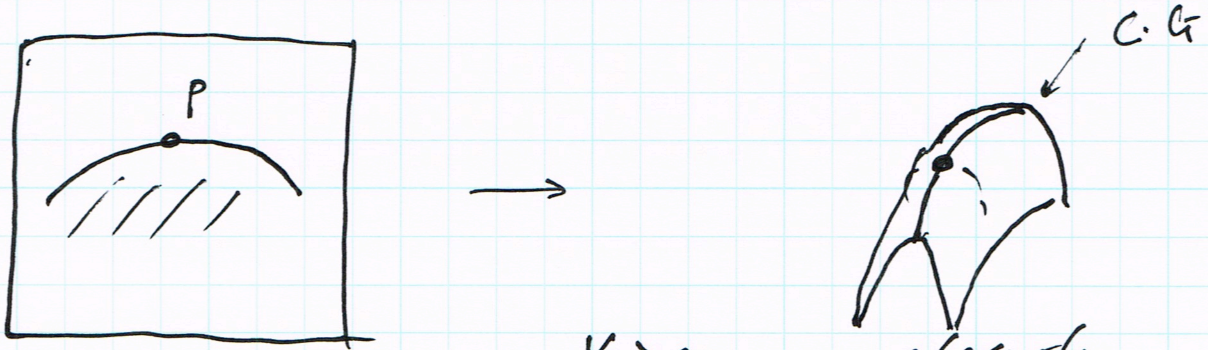
$$\lim_{\varepsilon \to 0} \frac{(p + \varepsilon T - f) - (p - f)}{\varepsilon}$$

this is clearly a tangent vector, so we are done.

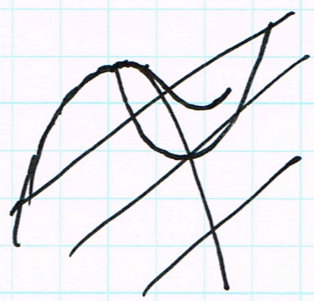
$$\boxed{II(T, W) = 0}$$

we will get Koenderink's thm out of this; worth remembering.

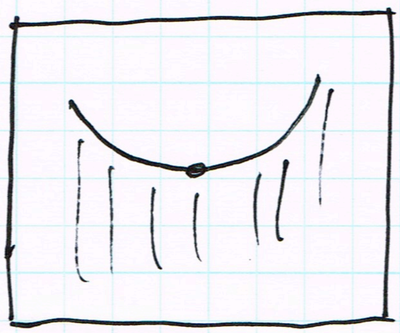
The shape of an outline clearly says something about Gaussian curvature

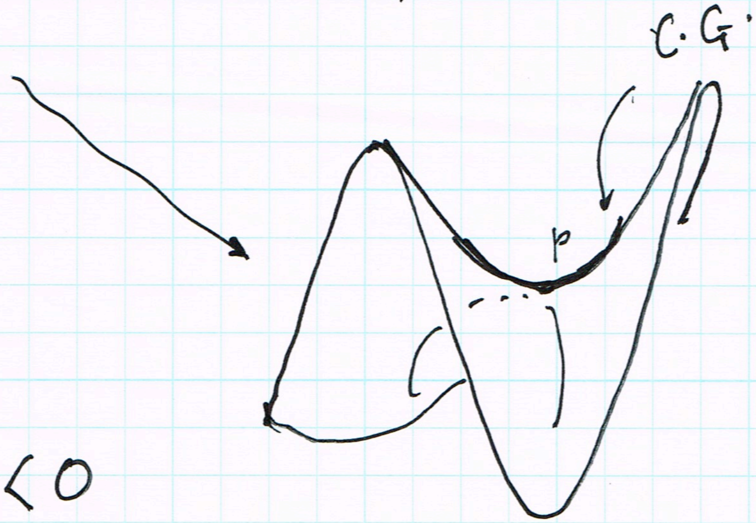


K>0 , clearly



SORRY



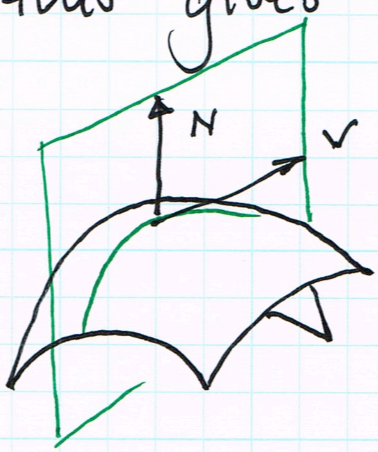


K<0

To go further, we need a new notion

at a pt p. on a surface, and tangent vector V, slice surface with a plane spanned by N, V

- this gives a plane curve with tangent V at p.



The curvature of this plane curve at p. is called the <u>directional</u> curvature of the surface in dir V

<u>Ex</u> Show it is $$\frac{\mathrm{II}(V,V)}{\mathrm{I}(V,V)}$$

You really should do this!

Koenderinks thm (orthographic version)

- We view a smooth surface in orthographic view. Choose a pt p on the <u>outline</u> in the image plane. Write K_o for the curvature of ~~this curve~~ the <u>outline</u> at p. Write V for the view dir'n, and P for the pt on the surface csp to p.

then

$$K_P = K_o \cdot K_V$$

directional curvature at P in dir V.

Gaussian curvature of the surface at P

as in text, curvature of <u>outline</u> at p.

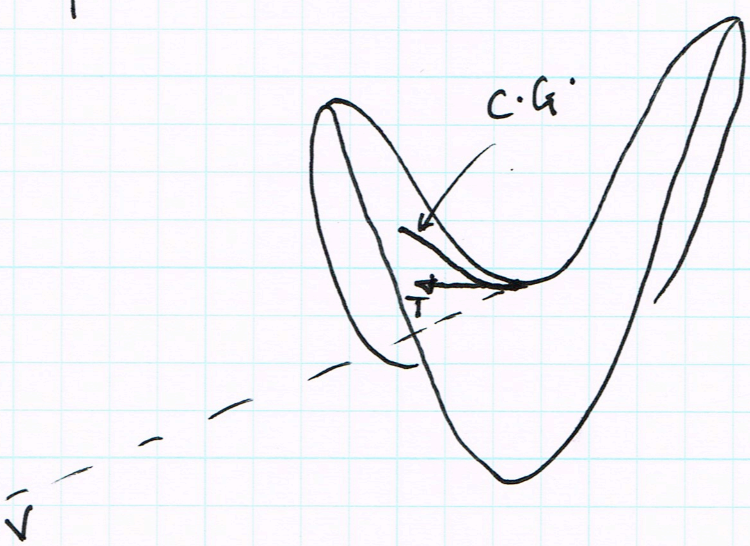
Proof:

- I ignore issues of sign because pix on P 13 sort that out.

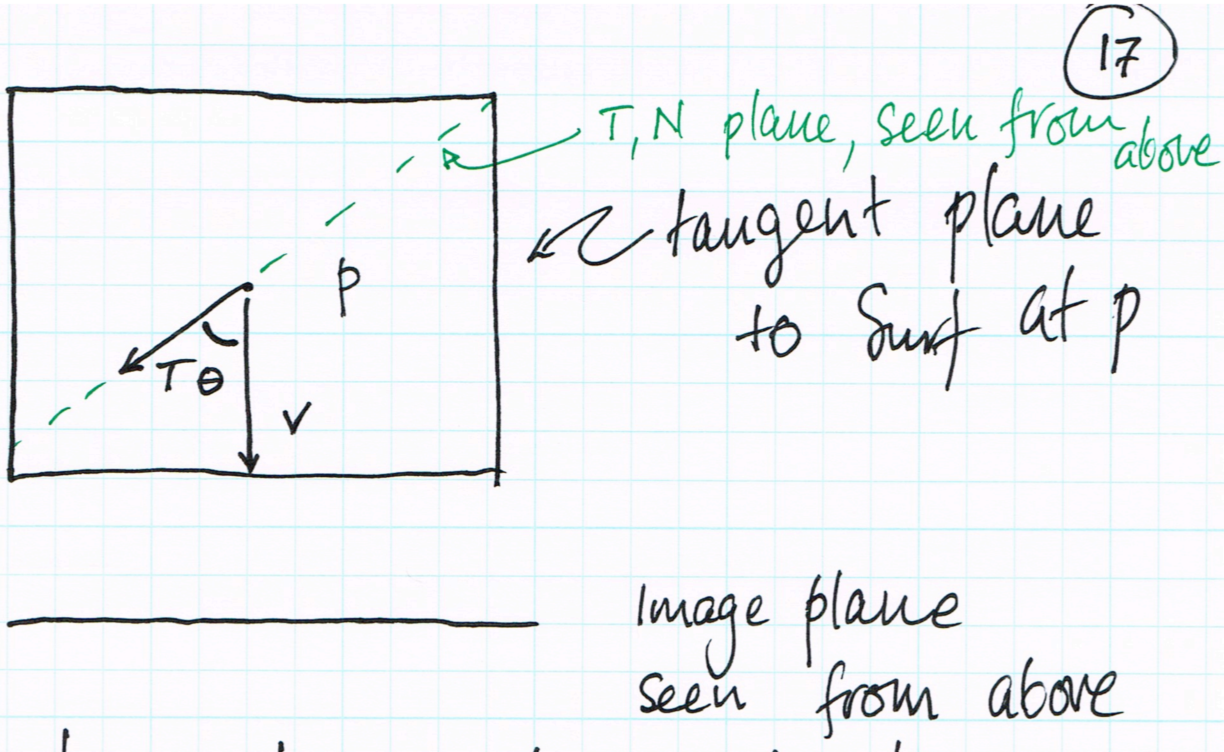
$$K_v = \frac{\mathrm{II}(v,v)}{\mathrm{I}(v,v)}$$

- Now notice that the tangent vector to the C.G.

T is not necessarily $\parallel$ to the image plane $\longrightarrow$ the C.G. is foreshortened to form the outline



easiest to look at this in tangent plane



Q: directional curvature at p in dirn T

is $\dfrac{\mathrm{II}(T,T)}{\mathrm{I}(T,T)} = \kappa_T \longrightarrow$ but this isn't κ_0,

which is foreshortened. Assume p is origin

- the curve is $uT + \frac{\kappa_T}{2}u^2 N$

  in T,N plane.

- in image plane, it is

$$\left(u \overset{\sin}{\cancel{\cos}}\theta\right) X + \frac{\kappa_T}{2}u^2 Y$$

write $u\,\cancel{\cos\theta}\sin\theta = \omega$, then

$$\omega X + \frac{K_I}{2\,\cancel{\cos^2\theta}\sin^2\theta}\,\omega^2 Y$$

so $K_0 = \cancel{\cos\theta}\ \dfrac{K_I}{\sin^2\theta}$

Now defn means
$$K = \frac{II(u,u)\,II(v,v) - II(u,v)^2}{I(u,u)\,I(v,v) - I(u,v)^2}$$

for <u>any</u> u, v st. $\det(u,v) \neq 0$

(<u>Check</u> this)

Now assume that u, v are <u>conjugate</u>.

then
$$K = \frac{II(u,u)\,II(v,v)}{I(u,u)\,I(v,v)\left[1 - \cos^2\phi\right]}$$

angle between u, v

$$= \frac{II(u,u)\,II(v,v)}{I(u,u)\,I(v,v)\sin^2\theta} = K_0 K_v$$

$\square$

Koenderink's thm, perspective case.

- Now assume that focal pt. is r along v
- then

$$K_p = \frac{K_o \cdot K_v}{r}$$

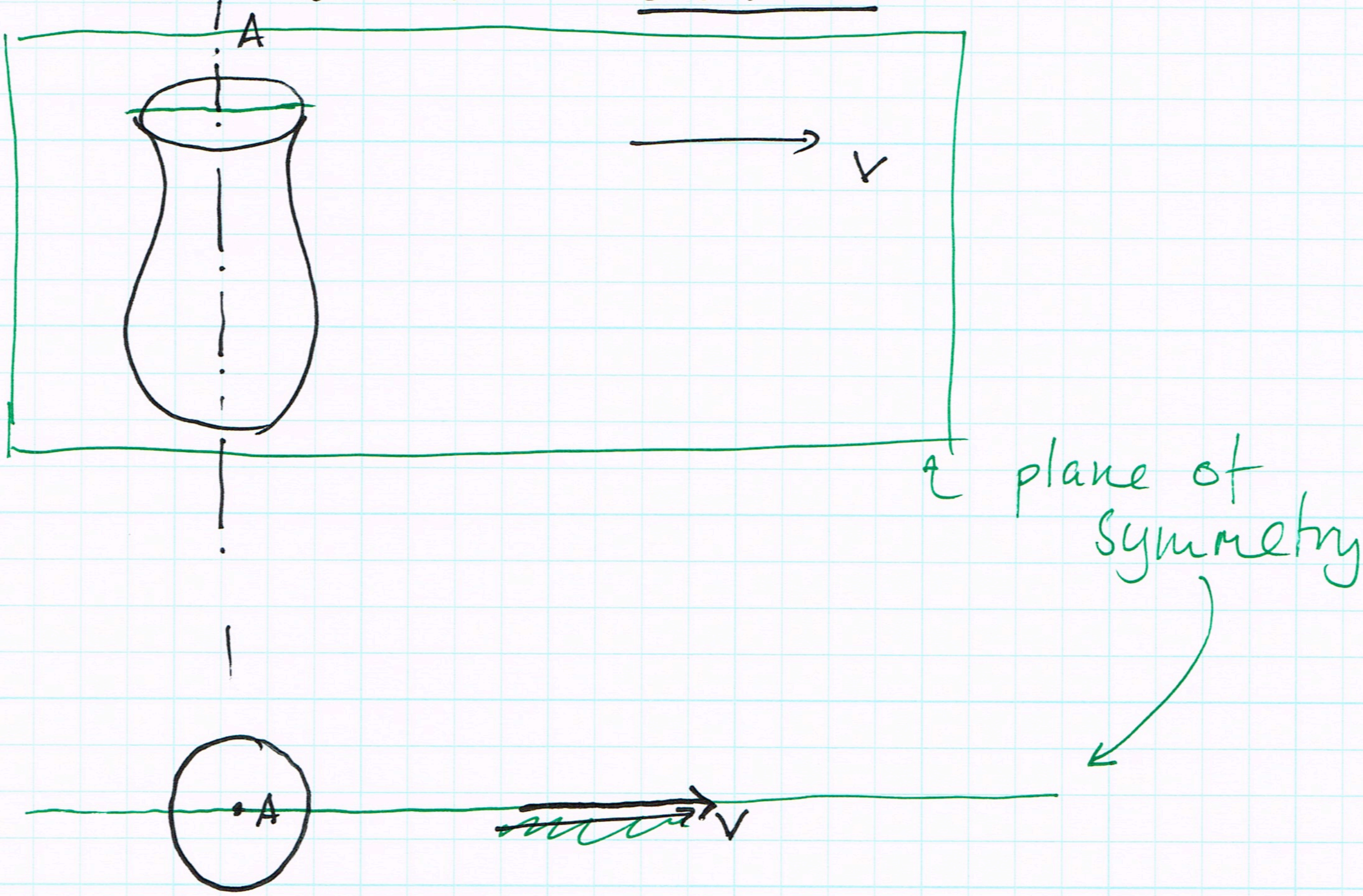
(easy version of proof above; note K_o in this case is $\frac{r\,K_T}{\sin^2\theta}$)

To go much further, we need a (very) little singularity theory — then we can investigate Gauss maps and curvature further.

But first, SOR.

Consider an SOR (surface of revolution) w/ axis A. ~~in~~

- in <u>orthographic</u> view, view dir'n V, C.G. has a <u>reflectional</u> symmetry about plane given by A and V
  - and so does <u>outline</u>



Now drawing above makes it obvious that $\underbrace{\text{C.G. is plane}}$ ← most unusual phenomena

~~Plane~~ Outline w/ reflectional symmetry

⇓

You might be able to find outline of SOR in an image w/o knowing its geometry.

- Very attractive idea - you can find likely instances of S.O.R.'s with very little prior knowledge
  - Difficulty: few SOR's, many symmetric curve frag.'s

SOR - perspective view

- C.G. still has a symmetry, about plane thru A and f
- Outline has a form of symmetry (image plane is not necessarily $\perp$ to plane of symmetry)

SHGC

- take a closed plane curve, C
- Choose a point p on the plane; construct an axis A through p, $\perp$ to the plane
- make a surface by sweeping C along A, scaling by some fn of A

parametric form:

$$C = (f(t),\ g(t),\ 0)$$

$$A = (0,\ 0,\ 1)$$

$$\text{Surf} = (\sigma(s) f(t),\ \sigma(s) g(t),\ s)$$

Orthographic view

- No symmetry, but C.G. has fairly simple form if A $\perp$ to view dir