Contour generators and outlines

A contour generator on $S$ is a curve where rays from $p$ - some chosen point are tangent to the surface.

$p$ could be infinitely far away, in which case all rays to $p$ are $\parallel$ so rays in dir $V$ are tangent to surf.
An outline is the projection into the image of a contour generator.

Often helpful to think of this as a cone of tangent rays.

- Image is obtained by slicing this cone with a plane.
- Thinking about the cone avoids confusion caused by rotation about the focal point.
We usually think about all components (not just visible) of these curves.

Outline of bean in previous image

- Outlines exhibit complex behavior w/ change of viewpoint, as do contour generators (next p.)
- A silhouette consists of the visible exterior points of outline

Silhouette of bean
Note: outlines + C.G.s are notoriously confusing - famous names have made major errors in print; C.G.s are hardly ever plane, a point of much confusion.
These curves are interesting because they appear at boundaries.

But also

We tend to keep track of all components because counting visibility is hard.
Some properties of C.G.'s + outlines

- The contour generator for a sphere is plane. The outline is a plane section of a right circular cone.

Proof: Symmetry does the work. The geometry is rotationally symmetric about the line from p to center of sphere; C.G. must have this symmetry so be section of sphere by plane normal to axis. Rays form a cone.

Corollary: Perspective views of spheres could be ellipses (in principle, you could calibrate a camera like this)

Q: Why do spheres always look round?
write \( \phi = 0 \) for an implicit surface

then at \( x_0, y_0, z_0 \), Normal is

\[ \nabla \phi \bigg|_{x_0, y_0, z_0} \]

and tangent plane is

\[ \nabla \phi \bigg| \cdot x - \nabla \phi \bigg| \bigg( \begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \bigg) = 0 \]

[i.e. plane through \( x_0, y_0, z_0 \) w/normal \( \nabla \phi \).]

then \( \text{CG } \) is curve (in \( x = (x, y, z) \))

\[ \phi(x) = 0 \]

\[ \nabla \phi^T p - \nabla \phi^T x = 0 \]

if \( \phi \) is polynomial, all sorts of neat stuff happens. (see my paper, and good luck to you!)
One application:

- The C.G. of a generic quadric is plane

- generic quadric:

  \[ x^T A x + b^T x + c = 0 = \phi \]

  for \( \text{Let} \ A \neq 0, \ A - A^T = 0 \)

  so \( C.G. \) is:

  \[ \phi = 0 \]

  \[ x^T A p + b^T p - [x^T A x + b^T x] = 0 \]

  BUT

  \[ x^T A x + b^T x = 2[0 - c] - b^T x \]

  and result follows
1. an Algebraic surface of degree \( d \) is a set of points \( x \) st

\[ \phi(x) = 0 \]

where \( \phi \) is polynomial of degree \( d \).

2. a generic line intersects this surface in \( d \) points

\( \text{(subst: } x = y + t v \text{)} \)

3. The C.G., and hence the line can be given by two polynomial eqns

\[ \phi = 0 \quad \text{ (degree } d \text{)} \]
\[ \psi = 0 \quad \text{ (degree } (d-1) \text{)} \]

Which means the C.G. intersects a generic plane in \( d(d-1) \) pts (use Bézout)
The global theory of C.G.'s outlines is poorly understood in all but a few special cases (algebraic, SOK, some others). Good evidence this is because those cases (which are nasty + complicated) are the easy ones.

Local theory is fairly easy but probably not much use (are these curves have global structure, in all cases?)

1) View direction and Tangent to Contour generator are conjugate

Write $W$ for view field (which is a tangent vector at $T$ for tangent to C.G. point & intercept
Then

\[ II(T, W) = 0 \]

at any pt. on CG.

\[ \rightarrow \text{this condition } \Rightarrow T, W \text{ are conjugate} \]

at \( p \), \( N \cdot W = 0 \) (defn of)

\[ \nabla_T (N \cdot W) = 0 \]

[Directional derivative of \( N \cdot W \) in \( T \) dirn]

So

\[ (\nabla_T N) \cdot W + N \cdot (\nabla_T W) = 0 \]

\[ \rightarrow \]

\[ II(T, W) + O = 0 \]

\[ \Rightarrow \text{next p.} \]
Consider $\nabla_T W$; two cases:

1. View is orthographic: $W$ is constant
   $\therefore \nabla_T W = 0$

2. View is perspective, so

   $W = p - \gamma f$
   
   $\text{P.O.I. focal point}$

   $\nabla_T W$: slide $p$ very slightly along $T$ to $p + \varepsilon T$, then form

   $\lim_{\varepsilon \to 0} \frac{(p + \varepsilon T - f) - (p - f)}{\varepsilon}$

   This is clearly a tangent vector, so we are done.

\[ \mathcal{II} (T, W) = 0 \]

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- we still get need only;
- thin out of this body remembering.
The shape of an outline clearly says something about Gaussian curvature.

$K > 0$, clearly

$K < 0$
To go further, we need a new notion

at a pt. p. on a surface, and tangent vector \( V \), slice surface with a plane spanned by \( N, V \).

- This gives a plane curve with tangent \( V \) at \( p \).

The curvature of this plane curve at \( p \) is called the \underline{directional curvature} of the surface in \( \text{dir} \ V \).

**Ex.** Show it is \[ \frac{\mathbb{II}(V,V)}{\mathbb{I}(V,V)} \]

you really should do this!
Koenderink's theorem (orthographic version)

We view a smooth surface in orthographic view. Choose a pt \( p \) on the outline in the image plane. Write \( K_0 \) for the curvature of this curve the outline at \( p \). Write \( V \) for the view dir in, and \( P \) for the pt on the surface esp to \( p \).

Then

\[
K_p = K_0 \cdot K_v
\]

directional curvature at \( P \) in dir \( V \).

Gaussian curvature of the surface at \( P \)

as in text, curvature of outline at \( p \).
Proof:

I ignore issues of sign because pij on P13 sort that out.

\[ k_y = \frac{\mathbb{I}(v, v)}{\mathbb{I}(v, v)} \]

Now notice that the tangent vector to the C.G. T is not necessarily \( \perp \) to the image plane \( \rightarrow \) the C.G. is foreshortened to form the outline.

\[ \vec{v} \]

easiest to look at this in tangent plane
Q: directional curvature at \( p \) in dirn \( T \) is \( \frac{II(T,T)}{I(C,T)} = k_T \rightarrow \) but this isn't \( K_0 \), which is foreshortened. Assume \( p \) is origin.

\[ \text{In the curve is } \ uT + \frac{k_T u^2 N}{2} \text{ in } T,N \text{ plane.} \]

\[ \text{In image plane, it is } \ (u \sin \Theta) X + \frac{k_T u^2 Y}{2} \]
Write \( u \cos \theta = w \), then

\[
\frac{\sin \theta}{K_0} = w X + \frac{K_1}{2 \sin \theta} w^2 Y
\]

so \( K_0 = \frac{K_1}{\sin^2 \theta} \)

Now define \( K = \frac{I(u,u) I(v,v) - I(u,v)^2}{I(u,u) I(v,v) - I(u,v)^2} \)

for any \( u, v \) st. \( \det(u,v) \neq 0 \)

(Check this)

Now assume that \( u, v \) are conjugate.

then \( K = \frac{I(u,u) I(v,v)}{I(u,u) I(v,v) [1 - \cos^2 \phi]} \)

angle between \( u, v \) of \( \frac{I(u,u) I(v,v)}{I(u,u) I(v,v) \sin^2 \theta} = K_0 K_v u, v \)

\( \square \)
Koenderink's slim, perspective case.

Now assume that focal p.t. is $r$ along $V$.

Then

$$K_p = \frac{K_o \cdot K_v}{r}$$

(easy version of proof above; note $K_o$ in this case is $\frac{r K_I}{\sin^2 \Theta}$)

To go much further, we need a (very) little singularity theory — then we can investigate Gauss maps and curvature further.

But first, SOR.
Consider an SOR (surface of revolution) with axis A. In orthographic view, view dirn V, C.G. has a reflectional symmetry about plane given by A and V.

- and so does outline
Now drawing above makes it obvious that C.G. is plane.

Most unusual phenomena.

Outline of reflectional symmetry.

You might be able to find outline of SOR in an image w/o knowing its geometry.

- Very attractive idea - you can find likely instances of SOR's with very little prior knowledge.

- Difficulty: few SOR's, many symmetric curve frag's.
S.O.R - perspective view

- C.C. still has a symmetry about plane thru A and f
- Outline has a form of symmetry (image plane is not necessarily to plane of symmetry)

SHGC
- Take a closed plane curve, C
- Choose a point p on the plane; construct an axis A through p, ⊥ to the plane
- Make a surface by sweeping C along A, scaling by some function of A
Parametric form

WLOG, put curve on \( \mathbb{R}^2 \) plane

\[ C = (f(t), g(t), 0), \quad A = (0, 0, 1) \]

Scaling by \( \sigma \) gets

Surface

\[ (\sigma(s)f(t), \sigma(s)g(t), s) \]

Locally, for any small range of \( s \), \( s \in [s_0, s_0 + \delta s] \)

This surface is a \underline{cone}

(Easiest seen by looking from above)

Now join \( s(t, s) \) to \( s(t, s + ds) \) w/ a line - this must pass thru \((0, 0, 1)\)
Now the outline of a cone is easy (in planes thru line connecting vert and fp)

Right circular case

Now this means I can reconstruct the axis if I have correspondence

Point on axis

There are a variety of hacks to obtain csp, none particularly reliable
IF you see the cross-section at top or bottom, you might be able to reconstruct, with some more geometric constraint.

**Attraction of These Theories:**

Category independent links between outline and geometry, sometimes allowing grouping and reconstruction.

**Disadvantages:**

- Often very weak constraints, results surface model is dubious, at best useful.
- Doesn’t interact well with notions of category well.
- Category notions not recovered by modern language of primitives, composition.
Challenge: let us try to resurrect an easy/useful bit with modern machinery

- Most accessible
  - You can identify the contour of an SOR by image symmetry

- Obstacles
  - Nothing is a true SOR
  - Contour points are hard to find

- Positives
  - We're really good at building and applying detectors
  - Our C.G. is much better than 20 years ago
  - We have mean shift
. replace sor with "pinch"

\[ \Sigma = \{ x | \phi > 0 \}_\text{sor} \quad \text{solid model of sor} \]

\[ \Sigma_0 = \{ x | \phi_0 > 0 \} \quad \text{outside pinch} \]

\[ \Sigma_i = \{ x | \phi_i > 0 \} \quad \text{inside} \]

choose \( \phi \) st. for all \( x \), \( \phi_0 > \phi \)_sor.

. now there are many pinches, choose one with a plausible gap

. consider some focal point \( f \) "outside"

then any ray from \( f \) that enters \( \Sigma_0 \) but not \( \Sigma_i \) could be on a silhouette.
Now we get an image.

**Strategy:** build a detector that responds to points in this volume \( \Sigma_e - \Sigma \).

**Strategy:** do this for "few" pieces of \( \Sigma_e \), and link detector hits w/ mean shift.

**Strategy:** search to find \( \Sigma_e \) appropriate fp's to train detectors.

\[
\text{H} \quad \text{OW}
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