Curves

- You should think of a curve (locally) as an embedding of ~~an~~ a connected open interval of the line

- i.e

$$\underline{x} : t \longrightarrow (x(t), y(t), z(t) \cdots) = \underline{x}(t)$$

- notice we can <u>reparametrize</u> without changing the geometry.

eg.
$$(\cos\theta, \sin\theta)$$
$$\left(\frac{2t}{1+t^2}, \frac{(1-t^2)}{1+t^2}\right)$$

give <u>the same curve</u>.

- Very occasionally, one is justified in fussing about the details of the parametrization

eg
$$(\cos\theta, \sin\theta) \qquad \theta \in [-\infty, \infty]$$
$$\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \qquad t \in (-\infty, \infty)$$

are <u>not</u> the same thing (missing point!)

- We avoid situations where the parametrization <u>folds</u>, like

$$(t^2, t^2-1)$$ ← odd parametrization of part of a line

$$(t(t-1)(t+1), t(t-1)(t+1)-1)$$

even odder param of whole line

Arc length

- assume I have a curve $t \to \underline{x}(t)$

$$= (x_1(t), x_2(t), x_3(t) \dots x_N(t)).$$

- then a step dt in the parametrization has length

$$\sqrt{dx_1^2 + \cdot\cdot\, dx_N^2}$$

$$= \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \cdot\cdot\, \dot{x}_N^2}\; . dt\,.$$

- Now we very often want to parametrize by s (arclength) so that

$$\cancel{ds}\ \sqrt{\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \cdot\cdot} = 1$$

- usually, we don't do this directly

- an arclength parametrization has a bunch of nice properties.

$$\int_0^L ds = \text{length of curve}$$

$$= \int_0^{T(L)} \frac{ds}{dt} \cdot dt.$$

Now $\sqrt{\left(\frac{dx_1}{ds}\right)^2 + \cdots}\; ds = ds$

$$= \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \cdots}\; dt$$

so $\dfrac{ds}{dt} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \cdots}$

and length in t param is

$$\int_0^{T(L)} \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \cdots}\; dt$$

This is often a tricky integral!

· Now assume we have a curve, in any parametrization that doesn't fold at t_0 and in any dimension

$$t \longrightarrow \underline{x}(t)$$

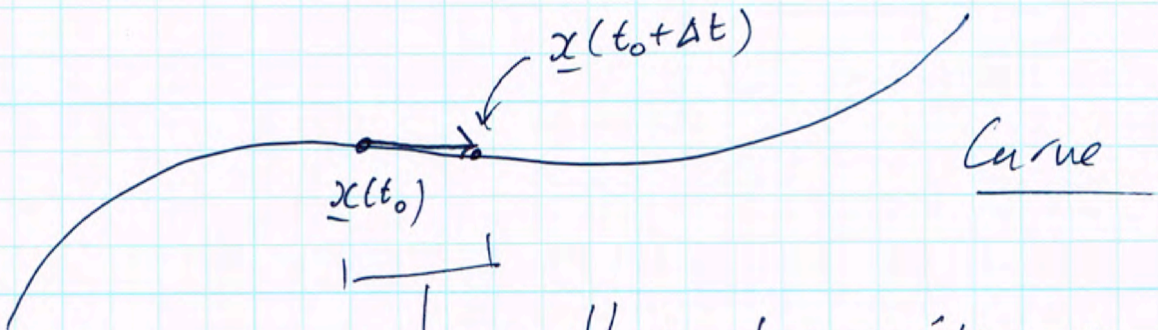
then

$$\left.\frac{d\underline{x}}{dt}\right|_{t=t_0} = \dot{\underline{x}}(t_0)$$

is a vector that is <u>tangent</u> to the curve.

· Think about this using the model of derivatives as limits

$$\dot{\underline{x}}(t_0) = \lim_{\Delta t \to 0} \frac{\underline{x}(t_0+\Delta t) - \underline{x}(t_0)}{\Delta t}$$



these two points are "arbitrarily close"

- Another useful way to think about this is that, at $\underline{x}(t_0)$, the line

$$\underline{x}(t_0) + u\,\dot{\underline{x}}(t_0) \qquad u \in -\infty, \infty$$

(i.e. a line through $\underline{x}(t_0)$ and in dir $\dot{\underline{x}}(t_0)$), is the best line approximating the curve.

- We can say this line has 2-point contact (see pic above).

- Now, if we parametrise by arc length s, the tangent vector is unit

$$\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds} = \left(\frac{dx_1}{ds}\right)^2 + \cdots \left(\frac{dx_N}{ds}\right)^2 = 1$$

- It is <u>usual</u> to call this T

$$T = \frac{d\underline{x}}{ds} = \frac{\frac{d\underline{x}}{dt}}{\left\| \frac{d\underline{x}}{dt} \right\|}$$

very useful. because we can avoid reparametrising!

Phane curves

- Curves dont have much geometry, and plane curves have <u>very</u> little

- parametrize by arc length,

$$\underline{x}(s)$$

- Notice $T \cdot T = 1$

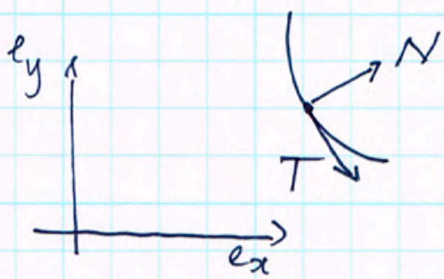
so $T \cdot \frac{dT}{ds} = 0$

- ~~this~~ We construct a <u>UNIQUE</u> ~~not~~ unit normal at s, $N(s)$ by insisting that the transformation $[e_x, e_y] \to [T, N]$ is a proper rotation (positive determinant, no flip)

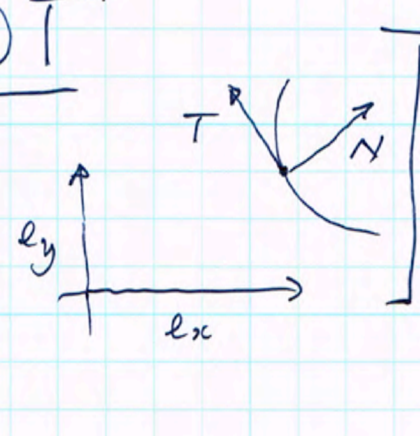
  e_x — unit vector in dir of +ve x axis

  e_y — " " y axis

- eg



<u>BUT NOT</u>



← <u>NO</u> <u>Naughty</u>

- So now we can define

$\kappa(s)$ ← Curvature, has sign.

by

$$\frac{d\underline{T}}{ds} = \kappa(s)\,\underline{N}(s)$$

- Interesting relationship between curvature and Taylor Series

  - Set up a coordinate system about the point of interest ~~s0~~ $(t=0)$ so that ~~to~~
    - curve passes thru origin at $t=0$
    - tangent is e_x
    - normal is e_y.

- I can then simplify parametrisation to get

$$\underline{x}(t) = \left(t, \ \frac{1}{2} a t^2 + O(t^3) \right)$$

- easy to confirm that $T = (1, 0)$ at $t = 0$

$\qquad\qquad N = (0, 1)$ "

- <u>Curvature</u>:

$$\frac{d\underline{x}}{ds} = \frac{(1, \ at + O(t^2))}{\sqrt{1 + (at)^2 + O(t^3)}} = T(t)$$

<u>now</u> $\dfrac{dT}{ds} = \dfrac{dT}{dt} \cdot \dfrac{dt}{ds}$

$$= \left[\frac{(0, a + O(t))}{\sqrt{1 + (at)^2 + O(t^3)}} + \frac{(1, at + O(t^2))}{(1 + (at)^2 + O(t^3))^{3/2}} \cdot \left(-\frac{1}{2}\right) \cdot \begin{pmatrix} 2at \\ + \cdots \end{pmatrix} \right.$$

$$\times \left[\frac{1}{\sqrt{1 + (at)^2 + O(t^3)}} \right]$$

at $t=0$, we have

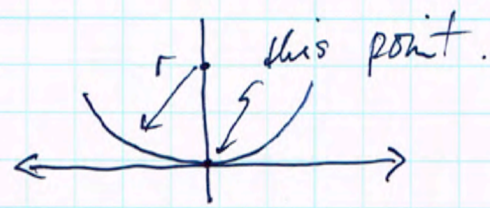
$$\frac{dT}{ds} = (0, a)$$

so $\qquad K = a.$

- the "first order" structure was simplified away by choosing the right coordinate system and parametrisation
- The curvature is what's left.
- <u>Neat interpretation of curvature</u>:

Circle $\quad (r\sin\theta,\ r - r\cos\theta)$

around $\theta = 0$



Taylor series both components, to get

$$\left(r\theta + O(\theta^3),\ r - r\left[1 - \frac{\theta^2}{2} + O(\theta^3)\right] \right)$$

Now ignore terms $O(\theta^3)$, and substitute

$$u = r\theta$$

to get

$$\left(u,\ \frac{1}{r} \cdot \frac{u^2}{2} \right),\quad \text{so}$$

The curvature of a circle radius r is $\frac{1}{r}$ at every point (A)

If a curve has curvature κ at t_0, then a circle, radius $\frac{1}{\kappa}$, fits to second order at that point (B)

- A is easy
- B takes a moment's thought.
  - I can clearly set up a c-sys, param. such that
    - point of interest is origin
    - x-coord is tangent
    - y- " is normal
    - point of interest is hit when param $= 0$
  - Then Taylor series for first coord:
    - has no const term ($= 0$)
    - has non-zero linear term
    -
  - For second:
    - has no const term
    - has no linear term (otherwise x-coord is not tangent)
    - ~~has~~ may have higher terms
- And so we can reparametrize to match a circle (which <u>might</u> have ∞ radius)

In some detail, because its informative

assume we have $(f(u), g(u))$ a curve, interested in $u = u_0$

- reparam with $v = u - u_0$, now interested in $v = 0$
- $(f(v), g(v))$
- translate by moving $(f(0), g(0))$ to the origin
- Now look at Taylor series

$$(\alpha_1 v + \alpha_2 v^2 + \cdots \quad , \beta_1 v + \beta_2 v^2 + \cdots)$$

- ~~write~~ $\cancel{w = \alpha_1 v + \alpha_2 v^2 +} \cdots$

~~Then~~ Now <u>rotate</u>; the right choice of rotation will get us to the form

$$(\gamma_0 v + \gamma_1 v^2 + \cdots \quad , \delta_2 v^2 + \cdots \quad)$$

- now write

$$w = \gamma_0 v + \gamma_1 v^2 + \cdots$$

I can choose $\zeta_2, \cdots$ so that

$$\zeta_2 w^2 + \zeta_3 w^3 + \cdots = \delta_2 v^2 + \delta_3 v^3 \cdots$$

(eg $\zeta_2 \gamma_0^2 = \delta_2$; $\zeta_3(2\gamma_0\gamma_1 + \gamma_0^3) = \delta_3$; etc, term by term matching.

So now my curve is in the form

$$(\omega,\ S_2 \omega^2 + O(\omega^3))$$

(and the $\frac{1}{2}$ is easily dealt with; I could also have used the implicit function thm.)

The nice feature of the tight relationship between derivatives and Taylor series is we can think of everything local in terms of polynomials; usually, this reduces to identifying a small number of terms that matter.

Cultural side note:

- this circle is sometimes called the osculating circle (Lat: osculam dare / to kiss)
- it has 3-point contact (basically, its second order structure) is the same as the curve's

Moving frame:

Notice

$$N \cdot N = 1 \quad , \quad N \cdot T = 0$$

differentiate by s

$$2\, N \cdot \frac{dN}{ds} = 0 \quad , \quad \frac{dN}{ds} \cdot T + N \cdot \frac{dT}{ds} = 0$$

so $\quad \dfrac{dN}{ds} = -\kappa T$

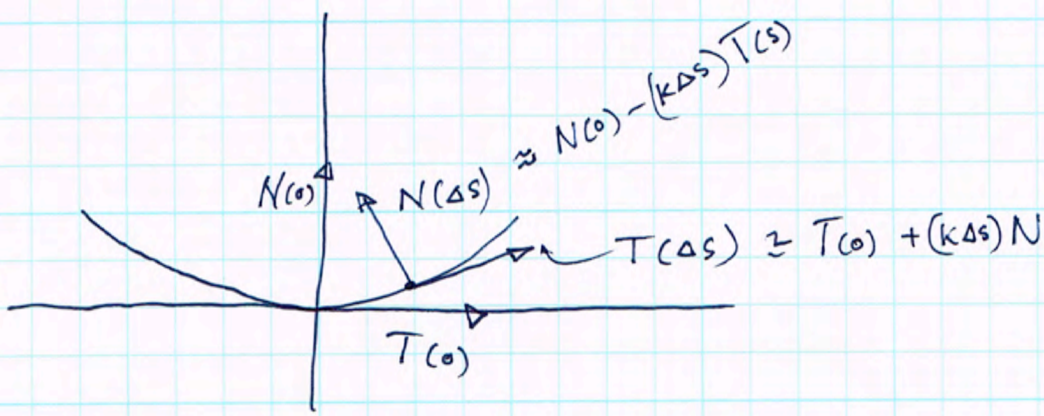
$$\frac{d}{ds}\begin{bmatrix} T \\ N \end{bmatrix} = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix}\begin{bmatrix} T \\ N \end{bmatrix}$$

is another way to catch this

also

$$T(s+\Delta s) \approx T(s) + (\kappa \Delta s) N(s)$$

$$N(s+\Delta s) \approx N(s) \cancel{+} -(\kappa \Delta s)\, T(s)$$



$$\begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix}$$ should look like the derivative of a rotation to you.

Consider the rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(careful about direction of rotation

$$\approx \begin{pmatrix} 1 + O(\varepsilon^2) & \varepsilon + O(\varepsilon^2) \\ -\varepsilon + O(\varepsilon^2) & 1 + O(\varepsilon^2) \end{pmatrix}$$

for ε very small

So derivative at $\theta = 0$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(which you could also do in other ways)

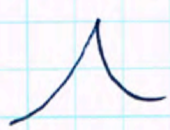
~~But how~~ So a rotation by ε is approximately $Id + \varepsilon\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and you can interpret curvature K as meaning that $[T, N]$ rotates by about $(K\Delta s)$ for a step Δs along curve.

- With a little work, that's all there is to nonsingular plane curves

> **thm**: given $\kappa(s)$, you can reconstruct a curve up to rotation + translation
>
> (fairly obvious, if you know some ODE)

Singular points can be more interesting

eg.  , 

 , etc.

because we need a more detailed vocabulary. More later, perhaps

Space curves:

- more, but not much more, happens here.

$T \cdot T = 1$

so $T \cdot \frac{dT}{ds} = 0$, as before

and we have $N(s) = \kappa \frac{dT}{ds}$

where $\kappa = \frac{1}{\|\frac{dT}{ds}\|}$ (note: always +ve!)

But there's a 2 parameter family of normal directions — a normal plane

so we're not finished

Define $B = \cancel{N \times T}\ T \times N$

unit binormal

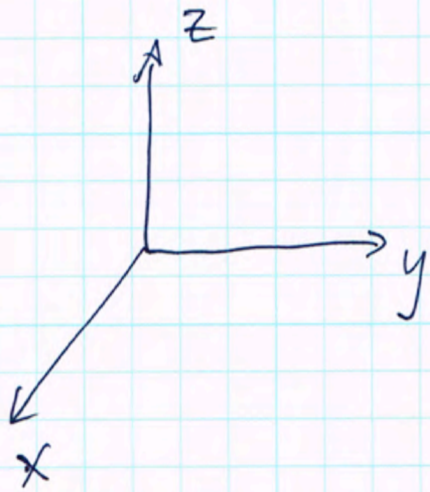
(mnemonic: Button; I look it up!)

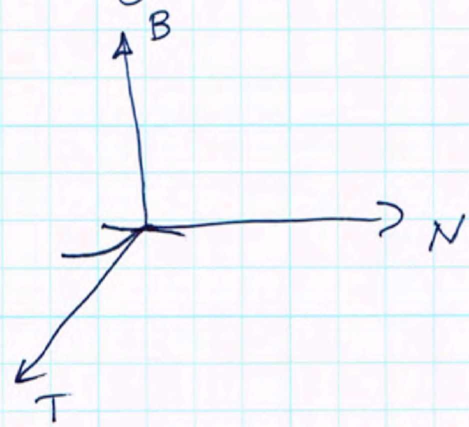
Note following (fairly obvious) points

$B\ T\ N$
- are unit vectors
- orthonormal
- form a coordinate system

$XYZ \xrightarrow{ROT} TNB$ is a rotation w/ $\det = 1$

equivalently TNB is a right-handed c-sys





Now

$$\frac{dT}{ds} = \kappa N \qquad \text{(by defn)}$$

notice $\quad B \cdot B = 1 \quad$ so $\quad B \cdot \frac{dB}{ds} = 0$

$B \cdot T = 0 \quad$ so $\quad B \cdot \frac{dT}{ds} + T \cdot \frac{dB}{ds} = 0$

but $\frac{dT}{ds} = \kappa N$

so $\quad T \cdot \frac{dB}{ds} = 0$

so $\frac{dB}{ds} \cancel{\propto N} = -\tau N$

(because it has no component in B, T directions)

torsion

(note minus sign)

now $\quad \frac{dN}{ds} \cdot T + \frac{dT}{ds} \cdot N = 0$

~~so~~

AND

$$\frac{dN}{ds} \cdot B + N \cdot \frac{dB}{ds} = 0$$

AND $\quad \frac{dN}{ds} \cdot N = 0$

So:

$$\frac{dT}{ds} = \kappa N$$

$$\frac{dN}{ds} = -\kappa T + \tau B$$

$$\frac{dB}{ds} = -\tau N$$

or

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

infinitesimal rotation

coordinate system

Now consider

$$\left(t,\ \frac{\alpha t^2}{2},\ \frac{\beta t^3}{6}\right) \quad \text{at } t=0$$

$$\frac{dt}{ds} = \frac{1}{\sqrt{1+(\alpha t)^2+\left(\frac{\beta t^2}{2}\right)^2}} = 1 + O(t^2)$$

taylor series

$$T:\quad \left(1,\ \alpha t,\ \frac{\beta t^2}{2}\right) \times \left(1 + O(t^2)\right)$$

so at $t=0$, $T = (1,0,0)$

$$\frac{dT}{ds} = \frac{dT}{dt}\cdot\frac{dt}{ds} = (0,\alpha,\beta t)\cdot\left(1+O(t^2)\right) + \left(1,\alpha t,\frac{\beta t^2}{2}\right)\cdot O(t)$$

at $t=0$: $(0,\alpha,0)$ so $\kappa = \alpha$

$N = (0,1,0)$

now $B = T \times N$ so at $t=0$, $B = (0,0,1)$

<u>torsion</u>: we want

$$\frac{dB}{ds} = -\tau N = \frac{dB}{dt}\cdot\frac{dt}{ds}$$

<u>notice</u>: because $N = 0,1,0$, only care about $\frac{dB}{dt}$ for 2nd component.

2nd component of $T \times N$:

$$\left(-\beta t\right)\cdot\left(1 + O(t^2)\right) \qquad (\text{recall } T \times T = 0)$$

so $\tau = \beta$.

Notice we could do all this by the same reasoning as for plane curves; translate, rotate to knock out terms, reparam. Just more bother in 3D case.