

Curves

①

- You should think of a curve (locally) as an embedding of ~~an~~ a connected open interval of the line

- i.e

$$\underline{x}: t \longrightarrow (x(t), y(t), z(t), \dots) = \underline{x}(t)$$

- notice we can reparametrize without changing the geometry.

eg. $(\cos \theta, \sin \theta)$

$$\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right)$$

give the same curve.

- Very occasionally, one is justified in fussing about the details of the parametrization

eg $(\cos \theta, \sin \theta) \quad \theta \in [-\infty, \infty)$

$\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right) \quad t \in (-\infty, \infty)$

are not the same thing (missing point!)

- We avoid situations where the parametrization folds, like

$(t^2, t^2 - 1) \quad \leftarrow$ odd parametrization of part of a line

$(t(t-1)(t+1), t(t-1)(t+1) - 1)$

\leftarrow even odder param of whole line

Arc length

- assume I have a curve $t \rightarrow \underline{x}(t)$
 " $(x_1(t), x_2(t), x_3(t), \dots, x_N(t))$

- then a step dt in the parametrization has length

$$\sqrt{dx_1^2 + \dots + dx_N^2}$$

$$= \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dots + \dot{x}_N^2} \cdot dt$$

- Now we very often want to parametrize by s (arclength) so that

~~that~~

$$\sqrt{\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \dots} = 1$$

- usually, we don't do this directly

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- an arclength parametrization has a bunch of nice properties.

$$\int_0^L ds = \text{length of curve}$$

$$= \int_0^{T(L)} \frac{ds}{dt} dt.$$

Now $\sqrt{\left(\frac{dx_1}{ds}\right)^2 + \dots} ds = ds$

$$= \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dots} dt$$

so $\frac{ds}{dt} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dots}$

and length in t param is

$$\int_0^{T(L)} \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dots} dt$$

↑ this is often a tricky integral!

• Now assume we have a curve, in any parametrization that doesn't fold at t_0 and in any dimension

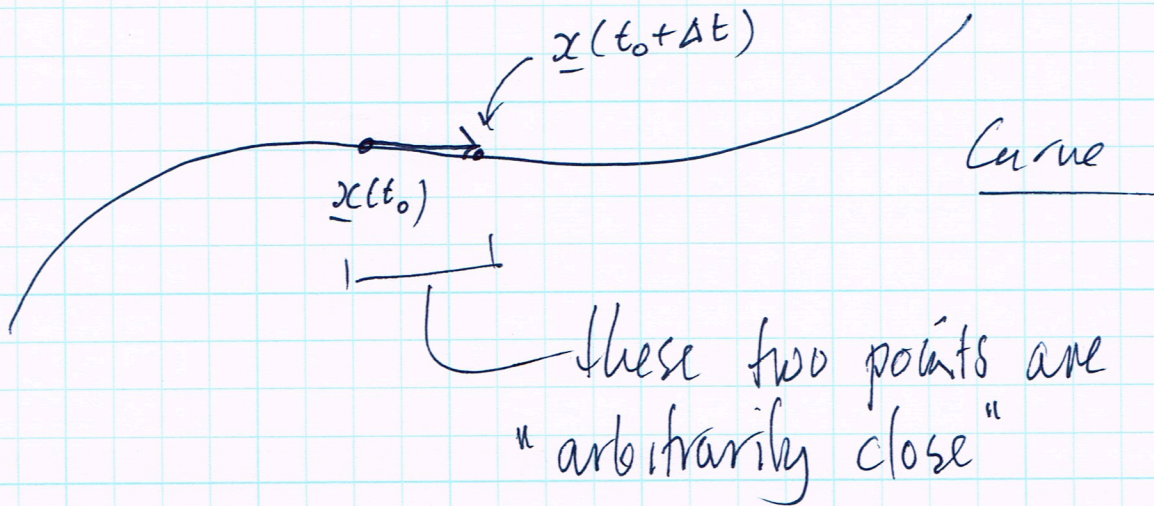
$$t \longrightarrow \underline{x}(t)$$

then $\left. \frac{d\underline{x}}{dt} \right|_{t=t_0} = \underline{\dot{x}}(t_0)$

is a vector that is tangent to the curve.

• Think about this using the model of derivatives as limits

$$\underline{\dot{x}}(t_0) = \lim_{\Delta t \rightarrow 0} \frac{\underline{x}(t_0 + \Delta t) - \underline{x}(t_0)}{\Delta t}$$



• Another useful way to think about this is that, at $\underline{x}(t_0)$, the line

$$\underline{x}(t_0) + u \dot{\underline{x}}(t_0) \quad u \in -\infty, \infty$$

(i.e. a line through $\underline{x}(t_0)$ and in dir $\dot{\underline{x}}(t_0)$) is the best line approximating the curve.

• We can say this line has 2-point contact (see pic above).

• Now, if we parametrise by arc length s , the tangent vector is unit

$$\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds} = \left(\frac{dx_1}{ds}\right)^2 + \dots + \left(\frac{dx_n}{ds}\right)^2 = 1$$

• It is usual to call this T

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$$T = \frac{dx}{ds}$$

$$= \frac{dx}{dt}$$

$$\left\| \frac{dx}{dt} \right\|$$

→ very useful. because we can avoid reparametrising!

Plane curves

• Curves don't have much geometry, and plane curves have very little

• Parametrize by arc length,

$$\underline{x}(s)$$

• Notice $T \cdot \bar{T} = 1$

so $T \cdot \frac{dT}{ds} = 0$

- ~~this~~

We construct a UNIQUE ~~not~~ unit normal at s , $N(s)$ by insisting that the

transformation $[e_x, e_y] \rightarrow [T, N]$

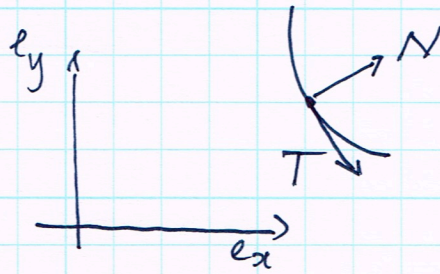
is a proper rotation (positive determinant, no flip)

e_x — unit vector in dir of the x axis

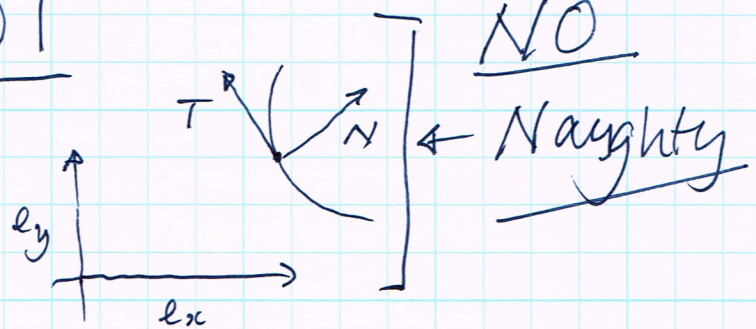
e_y —

y axis

- eg



BUT NOT



• So Now we can define

⑤

$K(s)$ ← curvature, has sign.

by

$$\frac{dT}{ds} = K(s) \underline{N(s)}$$

• Interesting relationship between curvature
and Taylor Series

• Set up a coordinate system
about the point of interest ~~at~~ $(t=0)$

so that ~~for~~

• curve passes thru origin at
 $t=0$

• tangent is e_x

• normal is e_y .

I can then simplify parametrization to get

$$\underline{x}(t) = (t, \frac{1}{2}at^2 + O(t^3))$$

easy to confirm that $T = (1, 0)$ at $t=0$
 $N = (0, 1)$ "

Curvature:

$$\frac{d\underline{x}}{ds} = \frac{(1, at + O(t^2))}{\sqrt{1 + (at)^2 + O(t^3)}} = T(t)$$

now $\frac{dT}{ds} = \frac{dT}{dt} \cdot \frac{dt}{ds}$

$$= \left[\frac{(0, a + O(t))}{\sqrt{1 + (at)^2 + O(t^3)}} + \frac{(1, at + O(t^2)) \cdot (-\frac{1}{2}) \cdot (2at)}{(1 + (at)^2 + O(t^3))^{3/2}} \right]$$

$$\times \left[\frac{1}{\sqrt{1 + (at)^2 + O(t^3)}} \right]$$

at $t=0$, we have

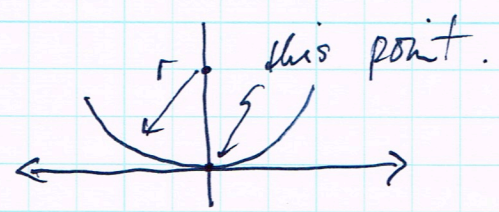
$$\frac{dT}{ds} = (0, a)$$

so $K = a$.

- the "first order" structure was simplified away by choosing the right coordinate system and parametrisation
- The curvature is what's left.
- Neat interpretation of curvature:

Circle $(r \sin \theta, r - r \cos \theta)$

around $\theta = 0$



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Taylor Series both components, to get

$$\left(r\theta + O(\theta^3), r - r\left[1 - \frac{\theta^2}{2} + O(\theta^3)\right] \right)$$

* Now ignore terms $O(\theta^3)$, and substitute

$$u = r\theta$$

to get

$$\left(u, \frac{1}{r} \cdot \frac{u^2}{2} \right), \text{ so}$$

The curvature of a circle radius r is $\frac{1}{r}$ at every point

(A)

If a curve has curvature k at t_0 , then a circle, radius $\frac{1}{k}$, fits to second order at that point

(B)

- A is easy
- B takes a moment's thought.
 - I can clearly set up a c-sys, param. such that
 - point of interest is origin
 - x-coord is tangent
 - y-coord is normal
 - point of interest is hit when param = 0

- Then Taylor series for first coord:
 - has no const term (= 0)
 - has non-zero linear term

- For second:

- has no const term
- has no linear term
- ~~has~~ may have higher terms (otherwise x-coord is at tangent)

- And so we can reparametrize to match a circle (which might have ∞ radius)

In some detail, because its informative
assume we have $(f(u), g(u))$ a curve, interested
in $u = u_0$

• reparam with $v = u - u_0$, now interested in $v = 0$

• $(f(v), g(v))$

• translate by moving $(f(0), g(0))$ to the origin

• Now look at Taylor series

$$(\alpha_1 v + \alpha_2 v^2 + \dots, \beta_1 v + \beta_2 v^2 + \dots)$$

~~write $w = \alpha_1 v + \alpha_2 v^2 + \dots$~~

~~then~~ Now rotate; the right choice of rotation
will get us to the form

$$(\gamma_0 v + \gamma_1 v^2 + \dots, \delta_2 v^2 + \dots)$$

• now write

$$w = \gamma_0 v + \gamma_1 v^2 + \dots$$

I can choose ζ_2, \dots so that

$$\zeta_2 w^2 + \zeta_3 w^3 + \dots = \delta_2 v^2 + \delta_3 v^3 \dots$$

(eg. $\zeta_2 \gamma_0^2 = \delta_2$; $\zeta_3 (2\gamma_0 \gamma_1 + \gamma_0^3) = \delta_3$; etc, term by term
matching.

So now my curve is in the form

$$(w, s_2 w^2 + O(w^3))$$

(and the $\frac{1}{2}$ is easily dealt with; I could also have used the implicit function theorem.)

The nice feature of the tight relationship between derivatives and Taylor series is we can think of everything local in terms of polynomials; usually, this reduces to identifying a small number of terms that matter.

Cultural side note:

- this circle is sometimes called the osculating circle (hat: osculam dare) to kiss
- it has 3-point contact (basically, its second order structure is the same as the curve's)

Moving frame:

Notice

$$N \cdot N = 1, \quad N \cdot T = 0$$

differentiate by s

$$2 N \cdot \frac{dN}{ds} = 0, \quad \frac{dN}{ds} \cdot T + N \cdot \frac{dT}{ds} = 0$$

so $\frac{dN}{ds} = -k T$

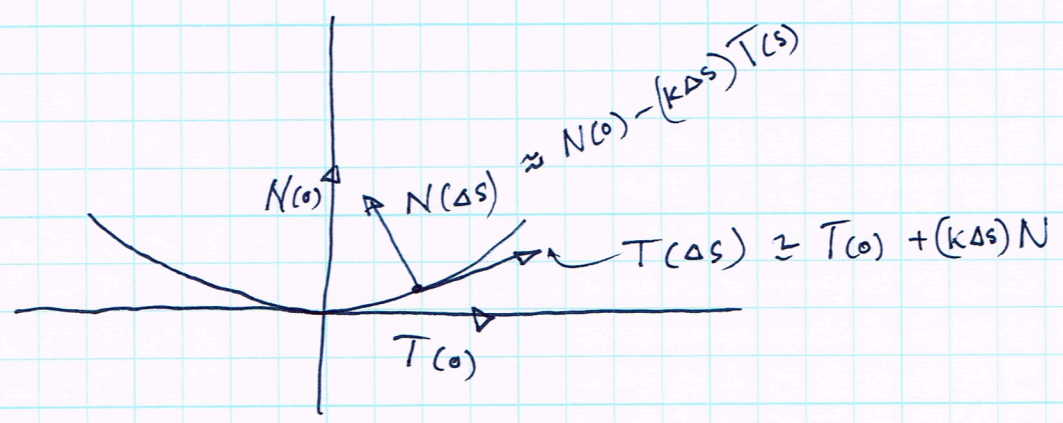
$$\frac{d}{ds} \begin{bmatrix} T \\ N \end{bmatrix} = \begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix}$$

is another way to catch this

also

$$T(s + \Delta s) \approx T(s) + (K\Delta s)N(s)$$

$$N(s + \Delta s) \approx N(s) - (K\Delta s)T(s)$$



$$\begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix}$$

should look like the derivative of a rotation to you.

Consider the rotation

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

(careful about direction of rotation)

$$\approx \begin{pmatrix} 1 + O(\varepsilon^2) & \varepsilon + O(\varepsilon^2) \\ -\varepsilon + O(\varepsilon^2) & 1 + O(\varepsilon^2) \end{pmatrix}$$

for ε very small

so derivative at $\theta = 0$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(which you could also do in other ways!)

~~But~~ now So a rotation by ε

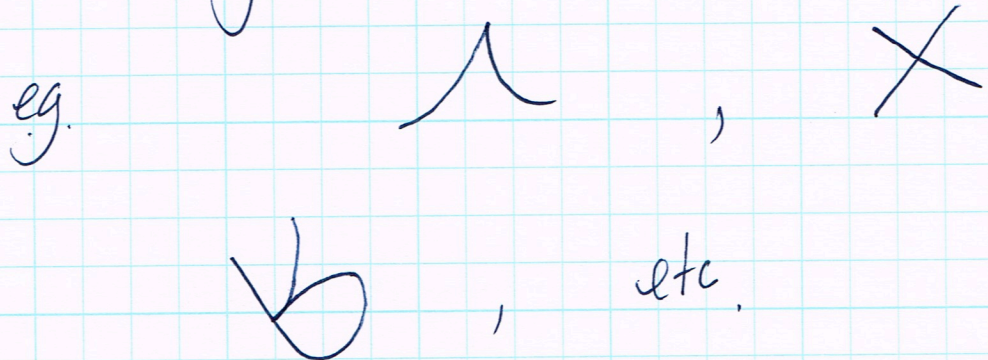
is approximately $\text{Id} + \varepsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and you can interpret curvature K as meaning that $[T, N]$ rotates by about $(K \Delta s)$ for a step Δs along curve.

• With a little work, that's all there is to nonsingular plane curves

then: given $k(s)$, you can reconstruct a curve up to rotation + translation
(fairly obvious, if you know some ODE)

Singular points can be more interesting



because we need a more detailed vocabulary. More later, perhaps

Space curves:

• more, but not much more, happens here.

$$T \cdot T = 1$$

so $T \cdot \frac{dT}{ds} = 0$, as before

and we have $N(s) = \kappa \frac{dT}{ds}$

where $\kappa = \frac{1}{\|\frac{dT}{ds}\|}$ (note: always +ve!)

But there's a 2 parameter family of normal directions — a normal plane

so we're not finished

Define $B = \cancel{N} T \times N$
 ↑ unit binormal

(mnemonic: Button; I look it up!)

Note following (fairly obvious) points

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B, T, N are unit vectors

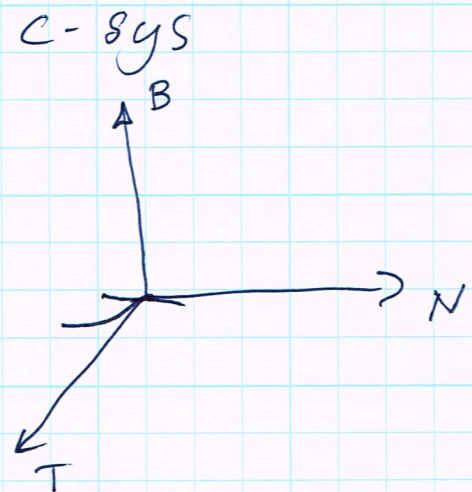
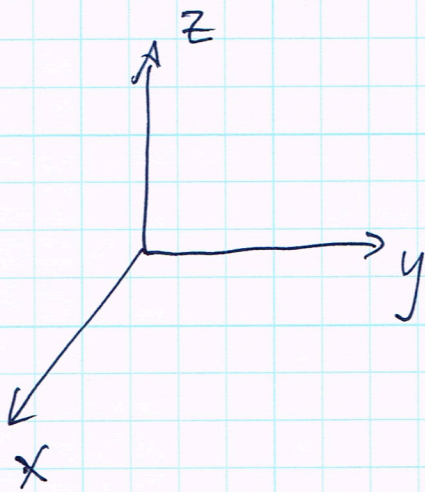
• orthonormal

• form a coordinate system

$XYZ \xrightarrow{\text{ROT}} TNB$ is a rotation w/
 $\det = 1$

equivalently

TNB is a right-handed



Now

$$\frac{dT}{ds} = KN$$

(by defn)

notice

$$B \cdot B = 1 \quad \text{so} \quad B \cdot \frac{dB}{ds} = 0$$

$$B \cdot T = 0 \quad \text{so} \quad B \cdot \frac{dT}{ds} + T \cdot \frac{dB}{ds} = 0$$

$$\text{but } \frac{dT}{ds} = KN$$

$$\text{so } T \cdot \frac{dB}{ds} = 0$$

$$\text{so } \frac{dB}{ds} \propto N = -\tau N$$

torsion

(note minus sign)

(because it has no component in B, T directions)

now

$$\frac{dN}{ds} \cdot T + \frac{dT}{ds} \cdot N = 0$$

so

AND

$$\frac{dN}{ds} \cdot B + N \cdot \frac{dB}{ds} = 0$$

AND

$$\frac{dN}{ds} \cdot N = 0$$

So:

$$\frac{dT}{ds} = kN$$

$$\frac{dN}{ds} = -kT + \tau B$$

$$\frac{dB}{ds} = -\tau N$$

$$\text{or } \frac{d}{ds} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

↑
↑
infinitesimal rotation coordinate system

Now consider

$$\left(t, \frac{\alpha t^2}{2}, \frac{\beta t^3}{6}\right) \quad \text{at } t=0$$

$$\frac{dt}{ds} = \frac{1}{\sqrt{1 + (\alpha t)^2 + \left(\frac{\beta t^2}{2}\right)^2}} = 1 + O(t^2)$$

↑ Taylor series

$$T: \left(1, \alpha t, \frac{\beta t^2}{2}\right) \times (1 + O(t^2))$$

So at $t=0$, $T = (1, 0, 0)$

$$\frac{dT}{ds} = \frac{dT}{dt} \cdot \frac{dt}{ds} = (0, \alpha, \beta t) \cdot (1 + O(t^2)) + \left(1, \alpha t, \frac{\beta t^2}{2}\right) \cdot O(t)$$

at $t=0$: $(0, \alpha, 0)$ so $\kappa = \alpha$

$$N = (0, 1, 0)$$

now $B = T \times N$ so at $t=0$, $B = (0, 0, 1)$

torsion: we want

$$\frac{dB}{ds} = -\tau N = \frac{dB}{dt} \cdot \frac{dt}{ds}$$

notice: because $N = (0, 1, 0)$, only care about $\frac{dB}{dt}$ for 2nd component.

2nd component of $T \times N$:

$(-\beta \epsilon) \cdot (1 + O(\epsilon^2))$ (recall $T \times T = 0$)

so $\tilde{c} = \beta$.

Notice we could do all this by the same reasoning as for plane curves; translate, rotate to knock out terms, reparam. Just more bother in 3D case.