Curves on Surfaces

Length and plane curves

Recall: curve \( U \subset \mathbb{R}^n \rightarrow \mathbb{R}^2 \)

\( \mathbf{x} = x(t) \)

Now we have a curve from \( a \) to \( b \)

- What is shortest such?

Length: \( \int_{t_a}^{t_b} (\dot{x} \cdot \dot{x})^{1/2} \, dt \)

Now assume that \( x(t) \) is the shortest curve.

Then \( \text{length} \left( x + \varepsilon \phi \right) > \text{length}(x) \)

for ANY \( \phi \), small enough \( \varepsilon \)

So \( \frac{d}{d\varepsilon} \text{length} \left( x + \varepsilon \phi \right) \bigg|_{\varepsilon=0} = 0 \) for ANY \( \phi \)

\( \varepsilon = 0 \) IF \( x \) has extremal (min) length
We can work this out

1) Notice that we can rewrite the "nearby" curve as

\[ x + \varepsilon fN \]

where \( N \) is the unit normal (in plane) to curve

2) Now we have

\[
\frac{d}{d\varepsilon} \left[ \int_{t_a}^{t_b} \left( \dot{x} + \varepsilon N + \varepsilon f N \right) \cdot \left( \dot{x} + \varepsilon N + \varepsilon f N \right) \, dt \right] \bigg|_{\varepsilon = 0}
\]

notice \( \dot{x} \cdot N = 0 \), \( \dot{N} \cdot N = 0 \)

So get

\[
\int_{t_a}^{t_b} \frac{f N \cdot \dot{x}}{\sqrt{\dot{x} \cdot \dot{x}}} \, dt = \int_{t_a}^{t_b} f N \cdot T \, dt
\]
Now \( \frac{d}{dt} (N \cdot T) = 0 = N \cdot T + \dot{T} \cdot N \)

so \( \dot{N} \cdot T = -\dot{T} \cdot N = \frac{d}{ds} (T \cdot N) \frac{ds}{dc} \)

\[ \frac{ds}{dc} = \kappa \frac{ds}{dc} \]

so \( \int_{s_0}^{s_f} f(N \cdot T) \, dt = \int_{s_0}^{s_f} f \kappa \, ds \).

Now if \( x \) has extremal length, this is zero for any \( f \)

\[ \therefore \kappa = 0 \quad \text{if} \quad x \text{ has extremal length} \]

i.e. shortest path curve between two points on plane is \( \underline{\text{line}} \) (Shoel Cornue)

\underline{More significant}: we associate curvature with "extra" length

Notice this argument works for 3D with little extra work.
Here's a 3D curve has $k = 0$ everywhere. Show $k = 0$ everywhere, too.

Now consider a curve on a surface.

$$\mathbb{R}^2 \ni u \in V \subseteq \mathbb{R}^2 \rightarrow (u, v) \in V \subseteq \mathbb{R}^2 \rightarrow \text{curve in } \mathbb{R}^3$$

parameter space

$V : x \rightarrow \mathbb{R}^3$ = surface.

Choose two points "quite close", $a$, $b$.

$a \neq b$ here, which can be ignored.

want $x(u(t), v(t))$ that minimizes length.

**NOTICE:** curve's tangent is tangent to surface.

curve's normal is NOT necessarily normal to...
\[ \text{length} = \int_{t_a}^{t_b} \sqrt{\dot{x} \cdot \dot{x}} \, dt. \]

now again, if curve has extremal length then

\[ \text{length (nearby curve)} \leq \text{length (curve)} \]

write \( \mathbf{n}(t) \) for vector field

(a) Tangent to surface

(b) Normal to curve

yields \( \mathbf{n} = \mathbf{N} \text{(surf)} \times \mathbf{T} \text{(curve)} \)

Then nearby curve is

\[ \mathbf{x} + \varepsilon f \mathbf{n} \]

and

\[ \frac{d}{d \varepsilon} \left[ \text{length}(\mathbf{x} + \varepsilon f \mathbf{n}) \right] \bigg|_{\varepsilon=0} = 0 \]

\[ = \int_{t_a}^{t_b} f(\dot{n} \cdot \mathbf{T}) \, dt \]

now \( \mathbf{n} \cdot \mathbf{T} = 0 \) so \( \dot{n} \cdot \mathbf{T} = -n \cdot \dot{T} \)

\[ \dot{T} = k \mathbf{N} \cdot ds \text{ (curv)} \]

so \[ \mathbf{T} = \text{Not the surface normal} \]
\[- \int_{t_a}^{t_b} f \left( N \cdot n \right) \frac{ds}{dt} dt = 0 \]

Now \( kN_{\text{curve}} = x_{ss} \)

so \( \int_{s_a}^{s_b} f \left( x_{ss} \cdot n \right) ds = 0 \)

\( f \) is anything, \( \text{so} \) \( (x_{ss} \cdot n) = 0 \)

\( x_{ss} \cdot n \) is referred to as **geodesic curvature**

= the extent to which curves curvature can be perceived on surface

= \( k_g \)

**Notice**

\( k_g = 0 \iff x_{ss} \in \text{span} \left\{ N, T, \frac{\partial \mathbf{T}}{\partial s} \right\}_{\text{surf curve}} \)

**but** \( x_{ss} = k N_{\text{curve}} \)

which is \( \perp T_{\text{curve}} \)

so \( x_{ss} = \oint k N_{\text{surf}} \)
A curve with everywhere 0 geodesic curvature is a geodesic.

Significant geodesic curvature - you can tell curve is bending away from the tangent ON the surface.
Some examples:

- curve on a sphere.

\[
(x, y, z) = (\cos \theta, \sin \theta, 0)
\]

\[x_{ss} = (-\cos \theta, -\sin \theta, 0)\]

Notice \( x_{ss} \in \text{span}(N) \)

So: this curve is a geodesic, everywhere.

But we can now exploit symmetry, invariance.

- rotating sphere doesn't change lengths.
- rotating sphere maps sphere to sphere.

\[\Rightarrow\] every section of a sphere by plane through normal is a geodesic.
Geodesics are a rich topic

- if there exists a
- there may not be a geodesic from a to b

eg. a plane with a hole in it, a, b on either side of hole.

- in this case, if punctured plane is open, there may not even be a length minimizing curve

- there may not be a unique geodesic from a to b

eg. sphere (geodesics are great circles)

**But**: if a length minimizing curve from a to be exists, it's a geodesic.

write \( d(a, b) = \lim [\text{length} \ (\text{piecewise regular curve}) \ ] \) joining \( a, b \)

(smooth sections joined at turns)
If for any \( a, b \), there is a geodesic from \( a \) to \( b \), surface is geodesically complete.

**Thm. (Hopf-Rinow)**

Connected surface \( S \)

\( S \) is geodesically complete

\( \iff \)

\((S, d)\) is complete metric space

\((i.e. \) sequences that converge in \( d \) have a limit in \( S \) - no holes\)

If either condition is true, for any \( a, b \in S \), there is a geodesic \( \gamma \) such that \( \gamma_{start} = a \), \( \gamma_{end} = b \), \( \text{length}(\gamma) = d(a, b) \).
The differential equation of a geodesic.

We assume we have a geodesic, parametrized by arc length $s$.

$$T(s) = x_u u_s + x_v v_s = x_s$$

$$T'(s) = x_{ss} = x_u u_{ss} + x_v v_{ss} + x_u u_s^2 + 2 x_u u_s v_s v_s + x_v (v_s)^2$$

Now the curve is unit speed, so $T(s) \in \text{Span}\{\mathbf{N}, \mathbf{v}\}$.

But $K_g = \frac{T'(s) \cdot N}{T(s)} = x_{ss} \cdot N$.

Recall $x_{uu} = \Gamma^r_{11} x_u + \Gamma^r_{12} x_v + \mathbf{e} \mathbf{N}$.

etc.

And note that tangential component of $T(s) = K_g(s) \mathbf{N}(s)$.

So $K_g(s) \mathbf{N}(s) = \left(u_{ss} u_s^2 + u_s^2 \Gamma^1_{11} + 2 u_u v_s \Gamma^1_{12} + v_s^2 \Gamma^1_{22}\right) x_u$

$$+ \left(v_{ss} u_s^2 + u_s^2 \Gamma^2_{11} + 2 u_u v_s \Gamma^2_{12} + v_s^2 \Gamma^2_{22}\right) x_v$$
We can get two valuable things from this

1: DE for a geodesic:

2: Expression for kg:

1: on a geodesic, $kg n_s = 0$ because $kg = 0$

So we must have

$$u_{ss} + u_s^2 \Gamma^1_n + 2u_s v_s \Gamma^1_{12} + v_s^2 \Gamma^1_{22} = 0$$

$$v_{ss} + v_s^2 \Gamma^2_n + 2u_s v_s \Gamma^2_{12} + v_s^2 \Gamma^2_{22} = 0$$

This is second order ODE in $u(s), v(s)$

2) through each pt, in each given dir, we can construct a unique geodesic

(existence, uniqueness)

(hence, the plane sections thin for ODE)

of pt $a$ are ALL geodesics on sphere.
2: an expression for \( K_g \)

\[
\mathbf{n} \text{ is unit } \left( \mathbf{n} \equiv \frac{\mathbf{N} \times \mathbf{T}}{\mathbf{N}_{\text{surf}}} \right)
\]

\[
\therefore \quad k_g \mathbf{n} \cdot \mathbf{n} = k_g
\]

\[
= (k_g \mathbf{n}) \cdot \left( \frac{\mathbf{N} \times \mathbf{T}}{\mathbf{N}_{\text{surf}}} \right)
\]

Recall

\[
\mathbf{M} \cdot (\mathbf{N} \times \mathbf{N}) = \det \left[ \mathbf{M} \mathbf{V} \mathbf{N} \right] = \mathbf{N} \times \mathbf{V}
\]

\[
= \det [\mathbf{N} \mathbf{M} \mathbf{V}]
\]

\[
= (\mathbf{N} \times \mathbf{M}) \cdot \mathbf{V}
\]

so we care about

\[
(\mathbf{T} \times k_g \mathbf{n}) \cdot \mathbf{N}_{\text{surf}}
\]

- expand this
- apply:

\[
x_u \times x_v = \left( \mathbf{E} \mathbf{E} - \mathbf{F}^2 \right) \mathbf{N}_{\text{surf}}
\]

and \( \mathbf{N} \cdot \mathbf{N} = 1 \)
- and bash, to get.
\[ K^g(s) = \sqrt{EG - F^2} \cdot \text{det}\left( \begin{array}{cc} u_s & u_{ss} + u_s^2 \Gamma_{11} + 2u_s v_s \Gamma_{12} + v_s^2 \Gamma_{22} \\ v_s & v_{ss} + v_s^2 \Gamma_{11} + 2u_s v_s \Gamma_{12} + v_s^2 \Gamma_{22} \end{array} \right) \]

### Three:

Too good things follow, rather easily:

1) In any parametrization, geodesic eqns are

\[
\begin{align*}
&u'' + (u')^2 \Gamma_{11}' + 2u' v' \Gamma_{12}' + (v')^2 \Gamma_{22}' = 0 \\
&v'' + (v')^2 \Gamma_{11}^2 + 2u' v' \Gamma_{12}^2 + (v')^2 \Gamma_{22}^2 = 0
\end{align*}
\]

( get by repair eqns, properties of det)

2) An expression for \( K^g \) in any param

Write \( \beta(t) = x(u(t)), v(t) \) and \( t_0 \) is point

Then

\[
K^g = \frac{\sqrt{EG - F^2}}{(\beta'(t_0) \cdot \beta'(t_0))^{3/2}} \cdot \text{det}\left( \begin{array}{cc} u'' + u'^2 \Gamma_{11} + 2u' v' \Gamma_{12} + v'^2 \Gamma_{22} \\ v'' + v'^2 \Gamma_{11} + 2u' v' \Gamma_{12} + v'^2 \Gamma_{22} \end{array} \right)
\]

3) Geodesics are intrinsic, as is \( K^g \).
Looking at the eggs for geodesics, we have seen something similar before.

Covariant derivative expression

Cf. this is a covariant deriv, but it takes some work.

Issue we are interested in curve. Tangent \( T \), but this isn't a vector field on surface -
only on curve.

Write \( T = \sum_i x_i \hat{e}_i \)

Curve is: \( x(u(t), v(t)) \)

\( T \) (tangent) is: \( x_u u + x_v v \)

As a differential operator, tangent is

\[
\frac{u \partial}{\partial u} + \frac{v \partial}{\partial v} = \frac{d}{dt}
\]
Now notice

\[ \nabla_T^T = \prod \left[ \left( \frac{u \partial}{\partial u} + \frac{v \partial}{\partial v} \right) \left[ u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right] \right] \]

proj to surf

\[ = \prod \left[ \langle \dddot{u} x_u + u \dddot{x}_{uu} + 2 \overset{\cdot}{u} \overset{\cdot}{v} x_{uv} + v \dddot{x}_{vv} \rangle \right] \]

The easiest way to see these terms is to remember

\[ \frac{u \partial}{\partial u} + \frac{v \partial}{\partial v} = \frac{d}{dt} \]

\[ = \left\langle \dddot{u} + u \dddot{x}_u + 2 \overset{\cdot}{u} \overset{\cdot}{v} \dddot{x}_{uv} + v \dddot{x}_{vv} \right\rangle x_u \]

\[ + \left\langle \dddot{v} + v \dddot{x}_v + 2 \overset{\cdot}{v} \overset{\cdot}{u} \dddot{x}_{vu} + u \dddot{x}_{uv} \right\rangle x_v \]

So for a geodesic

\[ \nabla_T T = 0 \]