Curves on Surfaces

. length and plane curves

recall: curve $U \subset \mathbb{R} \longrightarrow \mathbb{R}^2$

$t \qquad x(t)$

now we have a curve from a to b

– what is shortest such?

length: $\displaystyle\int_{t_a}^{t_b} (\dot{x} \cdot \dot{x})^{1/2}\, dt$

Now assume that $x(t)$ is the shortest curve –

then $\text{length}(\underline{x} + \varepsilon \underline{\phi}) \geqslant \text{length}(\underline{x})$

for ANY ϕ, small enough ε

so $\displaystyle\frac{d}{d\varepsilon} \text{length}(\underline{x} + \varepsilon \underline{\varphi}) \Big|_{\varepsilon=0} = 0$ for ANY φ

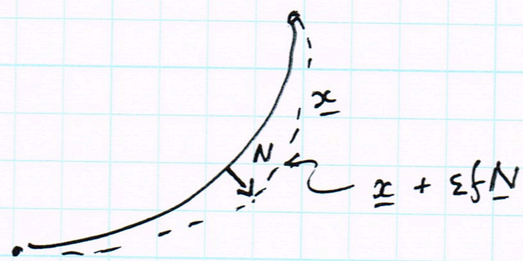
IF x has extremal (min) length

We can work this out

1) Notice that we can rewrite the "nearby" curve as

$$\underline{x} + \varepsilon f \underline{N}$$

where N is the unit normal (in plane) to curve



2) Now have

$$\frac{d}{d\varepsilon}\left[\int_{t_a}^{t_b} \sqrt{(\dot{x} + \varepsilon \dot{f} N + \varepsilon f \dot{N}) \cdot (\dot{x} + \varepsilon \dot{f} N + \varepsilon f \dot{N})} \, dt \right]\Bigg|_{\varepsilon = 0}$$

notice $\dot{x} \cdot N = 0$; $\dot{N} \cdot N = 0$

So get $$\int_{t_a}^{t_b} \frac{f \dot{N} \cdot \dot{x}}{(\dot{x} \cdot \dot{x})^{1/2}} \, dt = \int_{t_a}^{t_b} f \dot{N} \cdot T \, dt$$

now $\frac{d}{dt}(N \cdot T) = 0 = \dot{N} \cdot T + \dot{T} \cdot N$

so $\dot{N} \cdot T = -\dot{T} \cdot N = -\left(\frac{dT}{ds} \cdot N\right) \frac{ds}{dt}$

$= \kappa \frac{ds}{dt}$

so $\int_{t_a}^{t_b} f(\dot{N} \cdot T)\, dt = \int_{s_a}^{s_b} f \kappa \, ds.$

Now if x has extremal length, this is zero for <u>any</u> f

$\therefore \quad \kappa = 0 \quad$ if $\underline{x}$ has extremal length

i.e. Shortest ~~pata~~ curve between two points on plane is <u>line</u> (Shock horror)

<u>More significant</u>: we associate curvature with "extra" length.

Notice this argument works for 3D with little extra work.

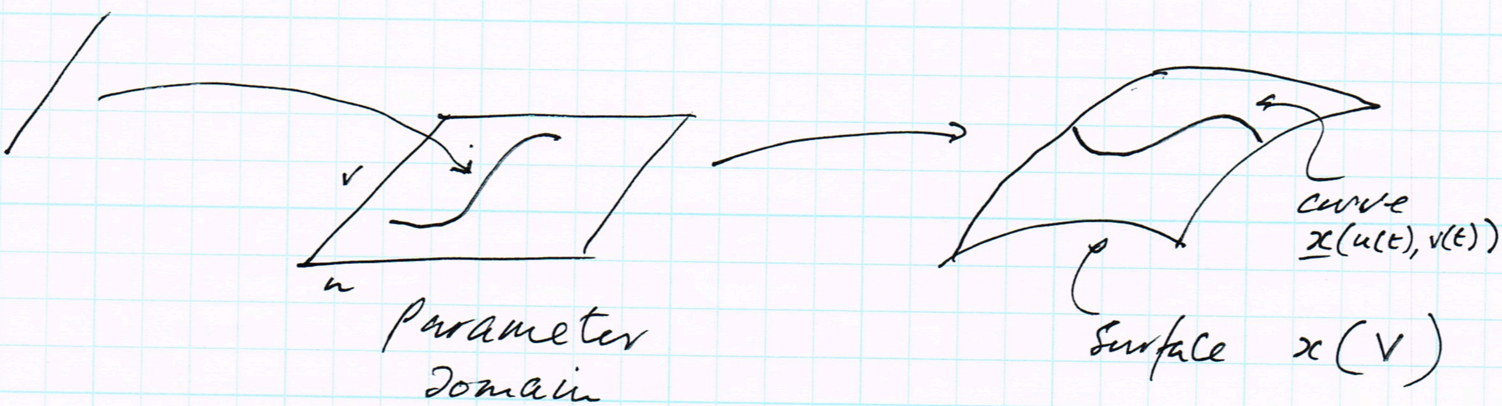
HINT OR DISTRACTION OR EXERCISE

- a 3D curve has $\kappa = 0$ everywhere. Show $\tau = 0$ everywhere, too

Now consider a curve on a surface

$$\mathbb{R}^2 \supset U \ni t : \longrightarrow (u,v) \in V \subset \mathbb{R}^2 \quad \leftarrow \text{curve in parameter space.}$$

$$V : \xrightarrow{x} \mathbb{R}^3 \quad \leftarrow \text{surface.}$$



Choose two points "quite close", a, b

a <u>lot</u> of subtlety here, which can be ignored

want $\underline{x}(u(t), v(t))$ that <u>minimizes</u> length.

<u>NOTICE</u>: curve's tangent is tangent to surface;

curve's normal is <u>NOT</u> necessarily

<u>Length</u> $= \displaystyle\int_{t_a}^{t_b} \sqrt{\dot{\underline{x}} \cdot \dot{\underline{x}}}\; dt.$

now again, if curve has extremal length then

length (nearby curve) $\geq$ length(curve)

write $\underline{n}(t)$ for vector field

(a) Tangent to surface

(b) Normal to curve.

Then nearby curve yields $\underline{n} = \underset{\cdot(\text{surf})}{N} \times T_{(\text{curve})}$ is

$$\underline{x} + \varepsilon f \underline{n}$$

and $\dfrac{d}{d\varepsilon}\left[\text{length}(\underline{x} + \varepsilon f \underline{n}) \right]\Big|_{\varepsilon=0} = 0$

$$= \int_{t_a}^{t_b} f(\dot{\underline{n}} \cdot \underline{T})\, dt$$

now $\underline{n} \cdot \underline{T} = 0$ so $\dot{\underline{n}} \cdot \underline{T} = - n \cdot \dot{T}$

$\dot{T} = \kappa \underset{\text{curve}}{N} \cdot \dfrac{ds}{dt}$ so

Not the surface normal

$$-\int_{t_a}^{t_b} f\,\kappa\left(N_{curve} \cdot n\right) \frac{ds}{dt}\,dt = 0$$

now $\kappa N_{curve} = x_{ss}$

so $$\int_{s_a}^{s_b} f\,(x_{ss} \cdot n)\,ds = 0$$

f is anything, so $(x_{ss} \cdot n) = 0$

$x_{ss} \cdot n$ is referred to as geodesic curvature

= the extent to which curves curvature can be perceived on surface.

$= \kappa_g$

NOTICE

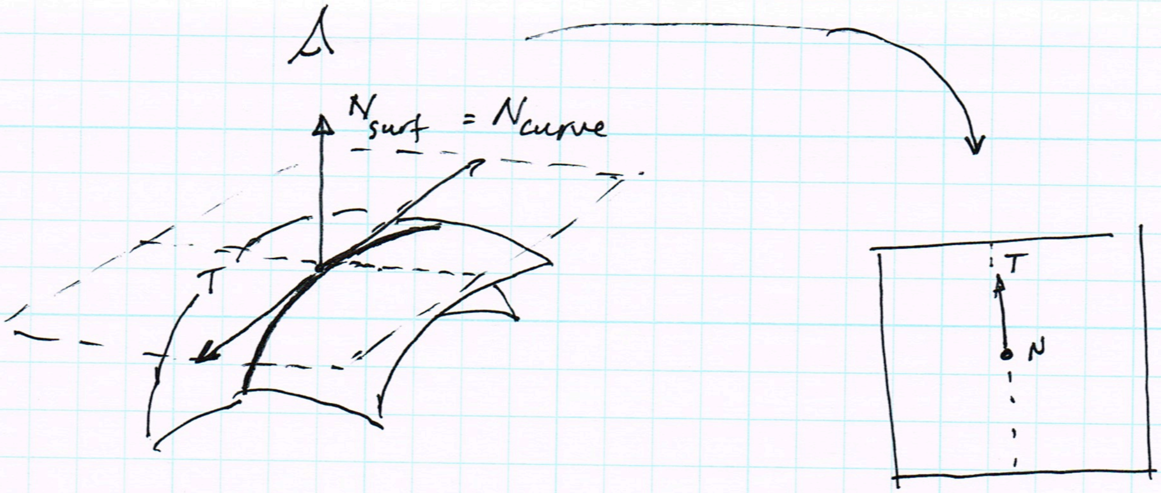
$$\kappa_g = 0 \iff x_{ss} \in \text{span}\{N_{surf}, T_{curve}\}$$

but $x_{ss} = \kappa N_{curve}$

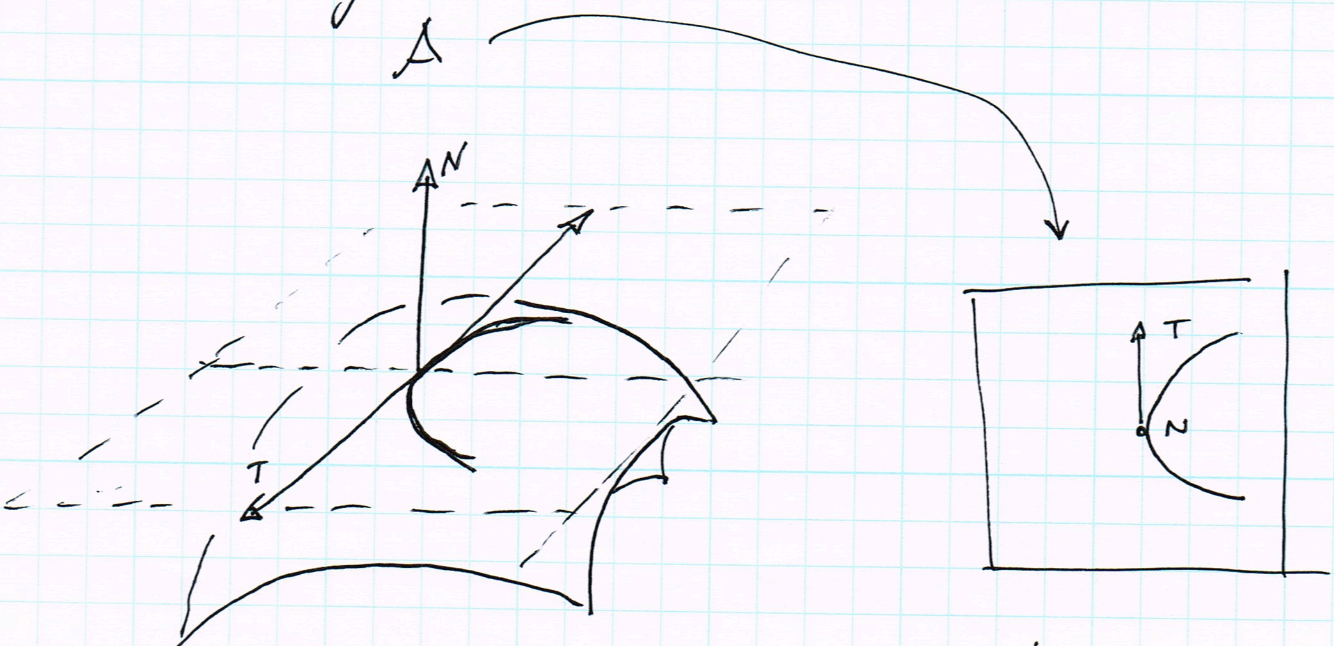
which is $\perp T_{curve}$

so

$$x_{ss} = \kappa N_{surf}$$



0 geodesic curvature



significant geodesic curvature
– you can tell curve is bending away from the tangent <u>ON</u> the surface

A curve with <u>everywhere</u> 0 geodesic curvature is a <u>geodesic</u> :

Some examples:

- curve on a sphere,

$$(\cos \xi, \sin \xi, 0) .$$ equator

$$\underline{x}_\xi = (-\sin \xi, \cos \xi, 0)$$ ⟸ Notice this is **unit** tangent.

$$\underline{x}_{ss} = (-\cos s, -\sin s, 0)$$

Notice $\underline{x}_{ss} \in \text{span}(N)$

So: this curve is a geodesic, **everywhere**

But we can now exploit symmetry, invariance

- rotating sphere doesn't change lengths
- rotating sphere maps sphere to sphere

⟹ every section of a sphere by plane thru. normal is a geodesic

<u>Geodesics</u> are a <u>rich</u> topic

- ~~if there exists a~~
- there may not be a geodesic from a to b

e.g. a plane with a hole in it, a, b on either side of hole.
— in this case, if punctured plane is open, there may not even be a length minimizing curve

- there may not be a unique geodesic from a to b.

e.g. sphere (geodesics are great circles)

<u>But</u>: if a length-minimizing curve from a to b exists, it's a geodesic.

write $d(a,b) = \inf[\text{length}(\text{piecewise regular curve joining } a, b)]$

(smooth sections joined at turns)

if for any a, b, there is a geodesic from a to b, surface is <u>geodesically complete</u>

<u>Thm</u>: (Hopf-Rinow) connected surface S

S is Geodesically complete surface $\iff$ (S, d) is Complete metric space

(i.e. ~~...~~ sequences that converge in d have a limit in S – no holes)

if either condition is true, for any $a, b \in S$ there is a geodesic γ st

~~γ~~ $\gamma_{start} = a$, $\gamma_{end} = b$, $length(\gamma) = d(a, b)$

The differential equation of a geodesic.

~~tina~~ assume we have a geodesic, param **by arclength** s

$$T(s) = x_u u_s + x_v v_s = \underline{x}_s$$

$$T_s(s) = x_{ss} = x_u \cdot u_{ss} + x_v v_{ss} + x_{uu} u_s^2 + 2 x_{uv} u_s v_s + x_{vv} (v_s)^2$$

now the curve is unit speed, so $\underline{T}_s(s) \in \text{Span}\{\underline{N}_{surf}, \underline{n}\}$

$\rightarrow$ but $\kappa_g = \underline{T}_s(s) \cdot \underline{n} = \underline{x}_{ss} \cdot \underline{n}$

recall $x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + eN$

etc.

and note that tangential component of

$$T_s(s) = \kappa_g(s)\, n(s)$$

so $$\kappa_g(s)\, n(s) = \left(u_{ss} + u_s^2 \Gamma_{11}^1 + 2u_s v_s \Gamma_{12}^1 + v_s^2 \Gamma_{22}^1\right) \underline{x}_u + \left(v_{ss} + u_s^2 \Gamma_{11}^2 + 2u_s v_s \Gamma_{12}^2 + v_s^2 \Gamma_{22}^2\right) \underline{x}_v$$

We can get ~~two~~ ~~several~~ two valuable things from this

1: <u>DE for a geodesic</u>:

2: <u>Expression for ~~k~~ κ_g</u>:

<u>1</u>: on a geodesic, $\kappa_g n_s = 0$ because $\kappa_g = 0$

So we <u>must</u> have

$$u_{ss} + u_s^2 \Gamma_{11}^1 + 2 u_s v_s \Gamma_{12}^1 + v_s^2 \Gamma_{22}^1 = 0$$

$$v_{ss} + u_s^2 \Gamma_{11}^2 + 2 u_s v_s \Gamma_{12}^2 + v_s^2 \Gamma_{22}^2 = 0$$

this is second order ODE in $u(s), v(s)$

$\Rightarrow$ through each pt, in each given dir, we can <u>construct</u> a unique geodesic

(existence, uniqueness thm for ODE)

(hence, the plane sections of p 7a are ALL geodesics on sphere.

~~Ke~~ 2: an expression for κ_g

$\underline{n}$ is unit $\left(\underline{n} = N_{surf} \times T \right)$

$\therefore$

$$\kappa_g \underline{n} \cdot \underline{n} = \kappa_g$$

$$= (\kappa_g \underline{n}) \cdot (N_{surf} \times T)$$

recall $M \cdot (V \times N) = \det[M V N] = \cancel{N \times V \cdot}$

$$= \det[N M V]$$

$$= (N \times M) \cdot V$$

so we care about

$$(T \times \kappa_g \underline{n}) \cdot \underline{N}_{surf}$$

- expand this
- apply: $x_u \times x_v = \left(\sqrt{EG - F^2}\right) N_{surf}$ and $N \cdot N = 1$
- and bash, to get.

$$k_g(s) = \sqrt{EG-F^2}\,\det\begin{pmatrix} u_s & u_{ss} + u_s^2\Gamma_{11}^1 + 2u_sv_s\Gamma_{12}^1 + v_s^2\Gamma_{22}^1 \\ v_s & v_{ss} + v_s^2\Gamma_{11}^2 + 2u_sv_s\Gamma_{12}^2 + v_s^2\Gamma_{22}^2 \end{pmatrix}$$

three:
~~two~~ good things follow, rather easily:

1) in *any* parametrization, geodesic eqns are

$$u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 = 0$$
$$v'' + (v')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2 = 0$$

(get by repar eqns, properties of det)

2) an expression for k_g in *any* param

- write $\beta(t) = \underline{x}(u(t), v(t))$ and t_0 is point of interest

then

$$k_g = \frac{\sqrt{EG-F^2}}{(\beta'(t_0)\cdot\beta'(t_0))^{3/2}}\cdot\det\begin{pmatrix} u' & u'' + u'^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + v'^2\Gamma_{22}^1 \\ v' & v'' + u'^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + v'^2\Gamma_{22}^1 \end{pmatrix}$$

3) Geodesics are INTRINSIC, as is K_g.

looking at the eqns for geodesics, we have seen something similar before.

→ cf. Covariant derivative expressions

this is a covariant deriv, but it takes some work.

Issue: we are interested in curve Tangent T, but this isn't a vector field on surface — only on curve.

~~write~~ $\cancel{T = \sum_i \dot{x}_i}$

Curve is: $\underline{x}(u(t), v(t))$

T (tangent) is: $x_u \dot{u} + x_v \dot{v}$

as a differential operator, tangent is

$$\dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} = \frac{d}{dt}$$

Now notice

$$\nabla_T \vec{T} = \Pi\left[\left(\dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v}\right)\left[\dot{u}\,\underline{x}_u + \dot{v}\,\underline{x}_v\right]\right]$$

proj to surf

$$= \Pi\left[\ddot{u}\,\underline{x}_u + \ddot{v}\,\underline{x}_v + \dot{u}^2\,\underline{x}_{uu} + 2\dot{u}\dot{v}\,\underline{x}_{uv} + \dot{v}^2\,\underline{x}_{vv}\right]$$

easiest way to see these terms is to remember

$$\dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} = \frac{d}{dt}$$

$$= \left[\ddot{u} + \dot{u}^2\,\Gamma^1_{11} + 2\dot{u}\dot{v}\,\Gamma^1_{12} + \dot{v}^2\,\Gamma^1_{22}\right]\underline{x}_u$$

$$+ \left[\ddot{v} + \dot{u}^2\,\Gamma^2_{11} + 2\dot{u}\dot{v}\,\Gamma^2_{12} + \dot{v}^2\,\Gamma^2_{22}\right]\underline{x}_v$$

So for a geodesic

$$\boxed{\nabla_T T = 0}$$