The mean shift update now becomes

\[
\mathbf{m}_k = -\frac{\sum_i k' \cdot \mathbf{v}_i}{\sum_i k'}
\]

This is a tangent vector at \( y_k \)

and then we form

\[
y_k = \exp (\mathbf{m}_k y_k).
\]

Notice

- In \( \mathbb{R}^n \), this is just what we're used to
- For anything else, we need to be good at \( \exp \).
Some examples:

- lie groups.

(a lie group is a group that is a differentiable manifold, where the group's action on itself $G \times G \to G$ is differentiable; they have lots of interesting properties; all our usual matrix groups are lie groups)

- assume a matrix representation

- define two MATRIX operators

$$e^M = \sum_{i=0}^{\infty} \frac{1}{i!} M^i$$
\[
\log(M) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (M - \text{Id})^i
\]

You can verify that \(\log(e^M) = M\) \(\quad e^{\log^M} = M\) etc.

These series may come with requirements on \(M\) for convergence!

One can now show (O’Neill, ch 11) that

\[
\exp(M) = X e^{[X^{-1}, M]}
\]

Matrix: a point on group
if \( \exp_x (V) = Y \) then

\[
V = x \log [ x^{-1} Y ]
\]

None of this is particularly attractive, but in special cases, we can do things

e.g.: \( SO(3) \) (3D rotations)

notice: if \( M(\phi) \) is \( \in SO(3) \)

then \( M^T M = I_d \)

so \( M^T \nabla_\phi M + \nabla_\phi (M^T M) M = 0 \)

now consider tangents at origin, \( M = I_d \)

\[
\nabla_\phi M + \nabla_\phi M^T = 0
\]
We always work with the natural inner product on the matrix space. So if \( U, V \) are tangent vectors represented as matrices we use

\[
\langle U, V \rangle = \sum_{ij} U_{ij} \theta^{ij} v_{ij} \\
= \text{Trace} \left( U^T V \right)
\]
So tangent space at origin consists of $N \in \mathbb{R}^N$ such that $N + N^T = 0$.

Write

$$[\omega]_x = \begin{bmatrix}
0 & -w_z & w_y \\
w_z & 0 & -w_x \\
-w_y & w_x & 0
\end{bmatrix}$$

(here $\omega = [\omega_x, \omega_y, \omega_z]$ is axis of rotation, $\|\omega\|$ is magnitude)

Then

$$e^x = I_d + \frac{\sin \|\omega\|}{\|\omega\|} [\omega]_x + \left(1 - \cos \|\omega\|\right) \frac{[\omega]_x^2}{\|\omega\|^2}$$

(called the Rodrigues formula — expand, and match terms w/ Taylor series)
It's also straightforward to extract axis angle from a rotation. Consider $M \in \text{SO}(3)$

$Mv = v$ if $v$ is axis

Other two eigenvalues reveal angle.

If eigenvalues of 2D rotation $\lambda_1, \lambda_2 = e^{\pm i\theta}$

This seems to work rather well for clustering pose. (Subbarao + Meer paper)
Working on Grassmannians

(Earlier notes were a little sloppy; better here)

\[ G_{\mathbb{R}^n}^p = \{ \text{set of } p \text{-dimensional flats through the origin in } \mathbb{R}^n \} \]

(which is the same as \( p-1 \) d flats in \( \mathbb{R}^{p-1} \))

Multiple representations available. We will parametrize with

\[ [Y] = n \times p \text{ matrix} \]

\[ \text{st } Y^TY = I_d \]
Notice that multiple Y's can represent the same point on the Grassmannian.

- If $M_{p \times p}$ is a $p \times p$ rotation, then $YM$ represents the same point.

Interpretation:

. $p$ orthogonal basis for vectors for the space we wish to rep'n
. $M_{p \times p}$, right action is a rotation of that basis w/in that space.
Tangent vectors:

- Easy argument (there's a harder argument in Edelman).

\[ (Y + 3\Delta)^T (Y + 3\Delta) = 1d \]

want \( \Delta \) st. to first order.

so \( Y^T \Delta = 0 \) \( \in \mathbb{R}^{n \times p} \)

Tangent space is \( p(n-p) \) dimensional, as it should be.
\langle \Delta, \Gamma \rangle = \text{tr}(\Delta^T \Gamma) \quad \text{as before.}

Then, there is a parametric form for \( \exp \):

\[
\exp_y(H) = [\gamma \gamma' \, u] \begin{pmatrix} \cos \Sigma^* \\ \sin \Sigma^* \end{pmatrix} V^T
\]

where \( u \Sigma V^T \) is SVD of \( H \).

Proof: Edelman, p15, easy!

AND

\[
M = \exp_y(H)
\]

\[
H = A \sin^{-1}(s) \otimes B^T
\]
Where

\[ \text{Asd}^T = M - YY^T M \]

\[ BCD^T = Y^T M \]

\[ C^T C + S^T S = 1d \]

\[ S, C \text{ diagonal.} \]

(Subbarao + Meer - Haven't checked this!)