A little singularity theory

①

consider a function $f : x \in U \subset \mathbb{R} \longrightarrow \mathbb{R}$

so 1D $\rightarrow$ 1D

and assume that $|f'| > 0$ for all x

now consider $g(w)$.

$$\frac{d}{dw} f(g(w)) = f' \cdot g'$$

so choose g s.t. $g' = \frac{1}{f'}$

then $f(g(w)) = w + w_0$ for the domain

OR

consider $h(w)$ s.t. $h' = \frac{1}{f'}$

and form $h(f(x)) = x + x_0$

I have constructed an equivalence.

$$\begin{tikzcd}
{} \arrow[r, "\alpha"] & {} \\
{} \arrow[u, "f"] \arrow[r, "\beta"'] & {} \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

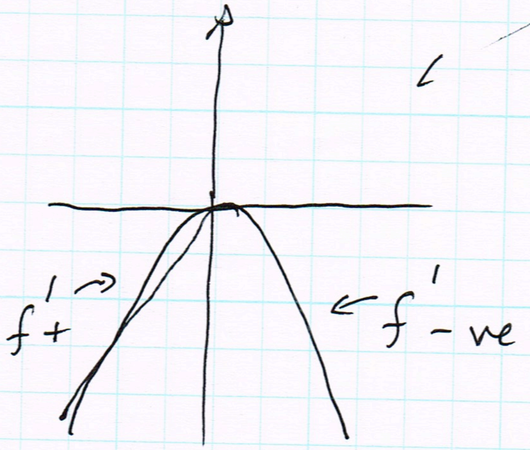
here α, β are smooth, differentiable, 1-1 and could use either my h or g to get $\alpha \circ f \circ \beta^{-1} = \mathrm{Id}$.

– in this sense (Change of coords in domain, range) all fn's with non-zero derivative are <u>the same</u> (the identity)

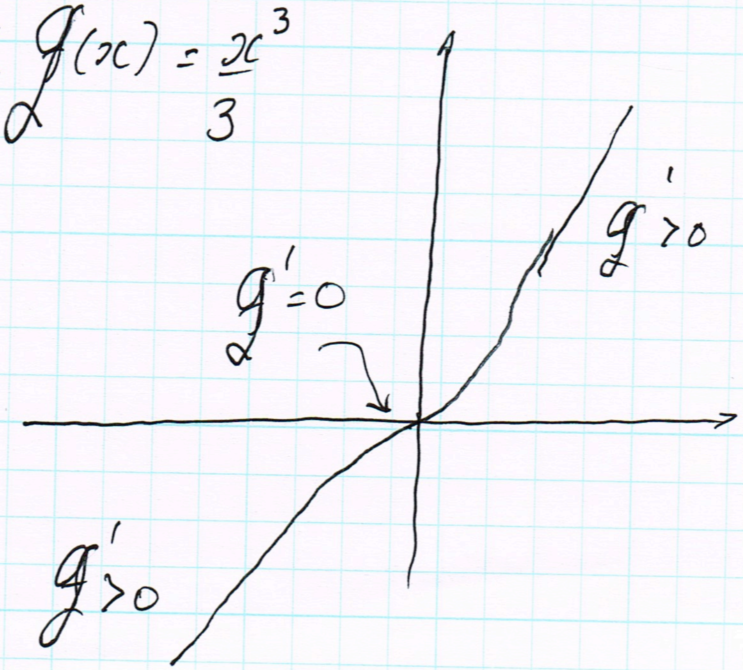
<u>Not</u> true if there is <u>some</u> pt x_0 st $f' = 0$ at x_0. $\boxed{\text{critical point}}$

- without loss of generality, choose $x_0 = 0$

compare $f(x) = -\frac{x^2}{2}$



with $g(x) = \frac{x^3}{3}$



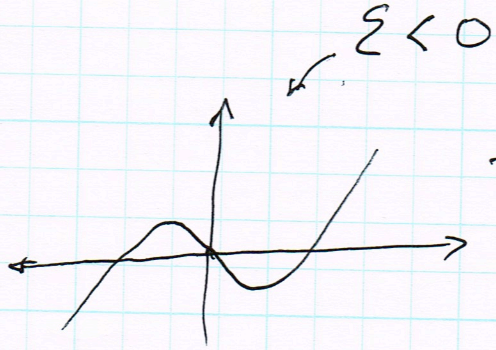
<u>Notice</u> $f + \varepsilon\phi$ gives me the same picture as f, for small enough ε, <u>any</u> φ

<u>Ex</u>: Why?

<u>BUT</u>

$g(x) + \varepsilon x$

$\varepsilon > 0$ — <u>No</u> critical ~~looks like $g(x)$~~ points.



<u>Two</u> critical points

$g(x)$ is referred to as being <u>unstable</u> at 0 — or: "has an unstable critical point at zero"

<u>Notice</u>: $\frac{x^3}{3} + ax$, $a > 0$ has <u>no</u> c.p. near~~at~~ zero and is stable

$\frac{x^3}{3}$ unstable c.p. at zero

$\frac{x^3}{3} - ax$, $a > 0$ has 2 stable c.p.s near zero

In many cases we can consider only stable phenomena.

Thm: (MORSE).

at a stable critical point, ~~a~~ any smooth fn of N vars can be written locally as:

$$\underbrace{\qquad\qquad}_{\text{Some vars missing.}} + \underbrace{x_i^2 + x_{i+1}^2 + \cdots}_{\text{Some +ve}} \underbrace{-x_{i+j}^2 - \cdots - x_N^2}$$

(with appropriate changes of coord in domain and range.)

Won't prove; never used directly

Now consider maps from 2D → 2D ⑥

~~cases~~: We work <u>locally</u>. Maps take

- $(x_1, x_2) \longrightarrow (y_1, y_2)$

WLOG, we work around $(0,0)$ in source + target.

<u>Cases</u>:

- <u>f</u> such that $[Df]$ (Jacobian derivative) has rank 2. Straightforward fiddling with construction above means

$$\underline{f} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ (i.e. the identity)

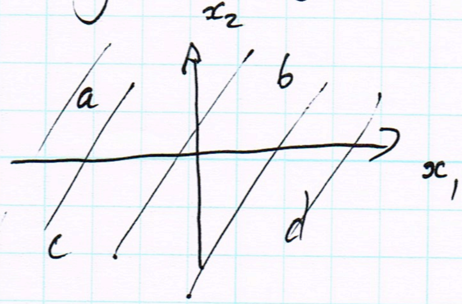
locally around $(0,0)$.

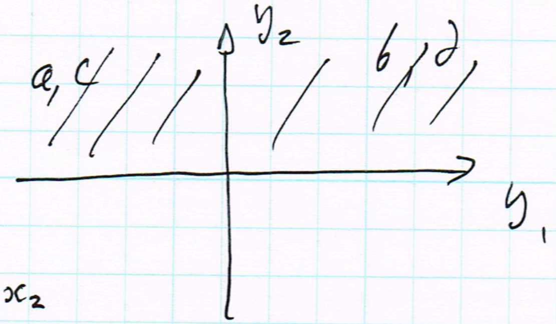
<u>THM</u>: Stable (i.e add small g to f and its still identity)

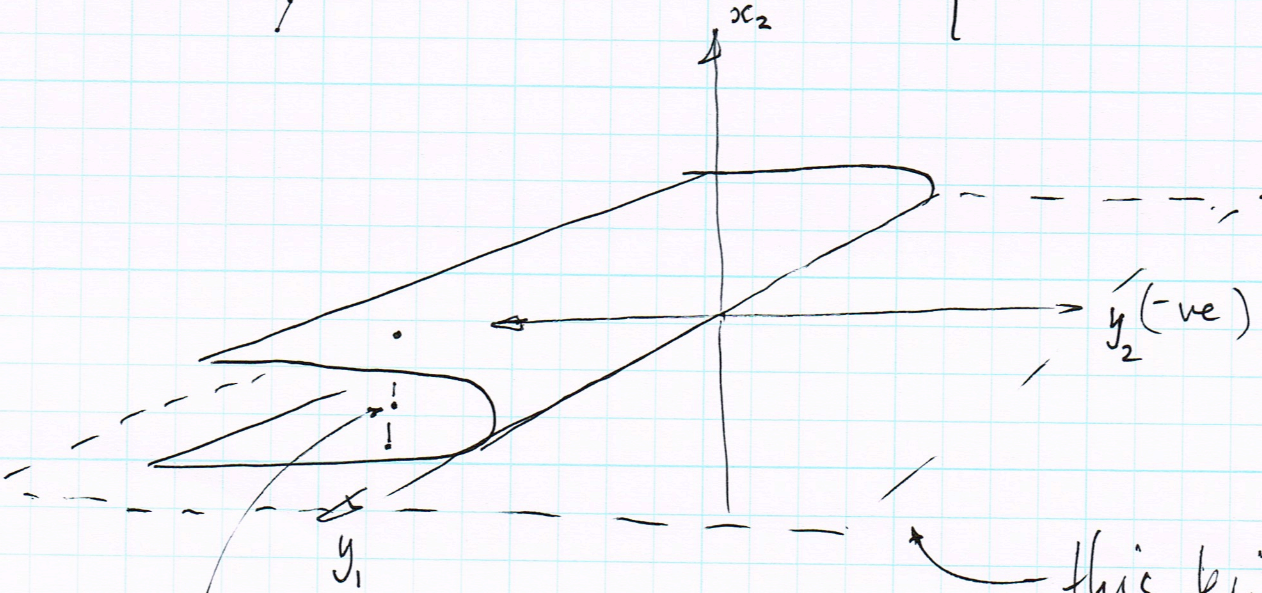
$[D_x f]$ has rank 1

a) $f(x) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2^2 \end{pmatrix}$ (around 0,0)

many ways to think of this







target space

this kind of drawing is more usual

there are 2 (x_1, x_2)'s csp to this target point.

CASE: $$\begin{pmatrix} x_1^3 + x_1 x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 . \end{pmatrix}$$ (around 0,0).

Jacobian:

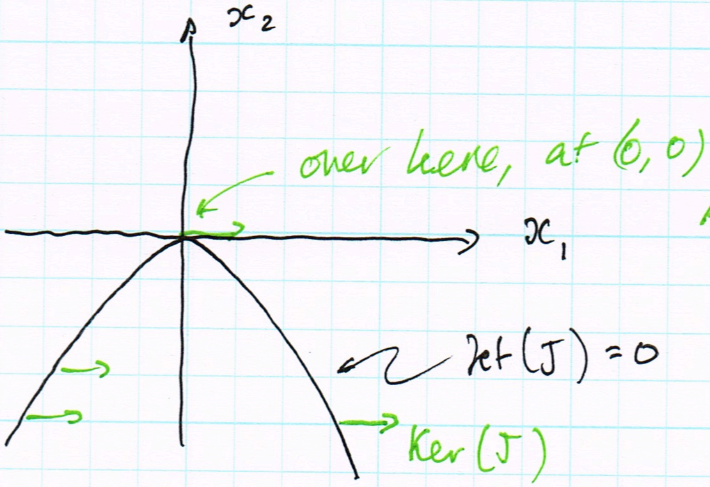
$$\begin{pmatrix} 3x_1^2 + x_2 & x_1 \\ 0 & 1 \end{pmatrix}$$

So

$$\det(J) = 3x_1^2 + x_2$$

Notice when $\det(J) = 0$, $\mathrm{Ker}(J) = \mathrm{Span}(1, 0)$

So



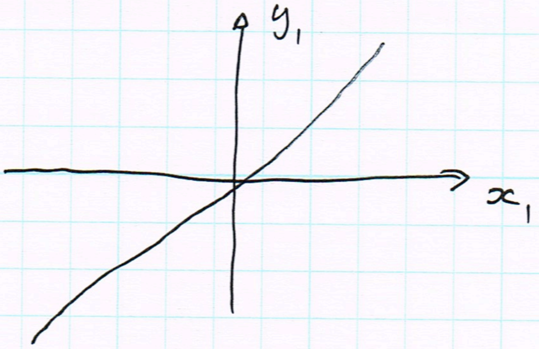
Ker(J) is tangent to Det(J)

Try to plot:

for $x_2 > 0$

$y_1(x_1)$ is 1-1

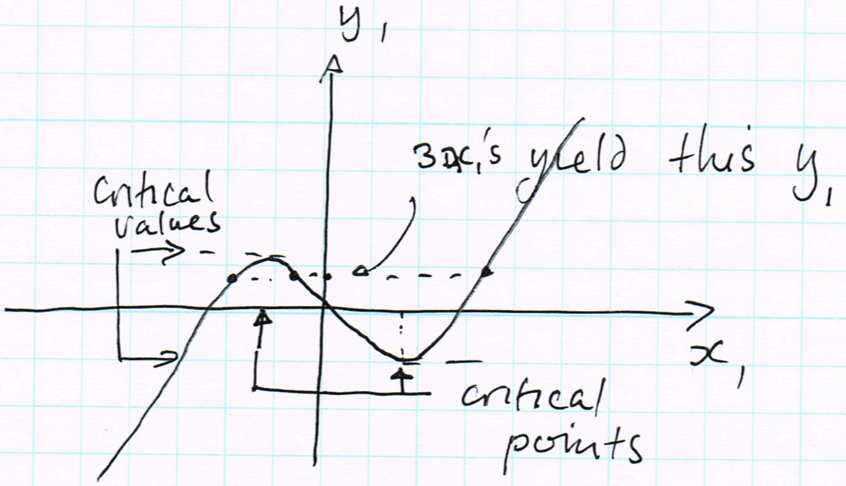
and fn is 1-1



for $x_2 < 0$

$y_1(x_1)$ is $3 \rightarrow 1$

for some x_1



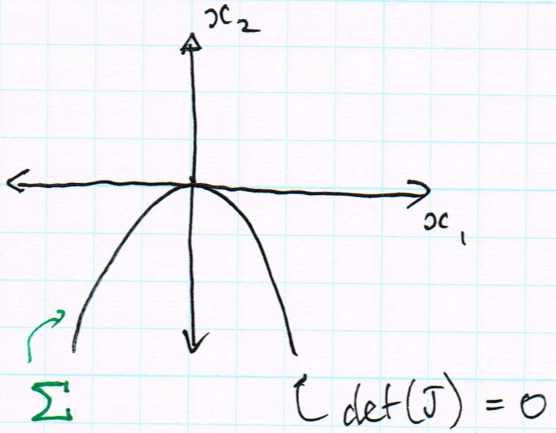
write. $x_2 = -u^2$

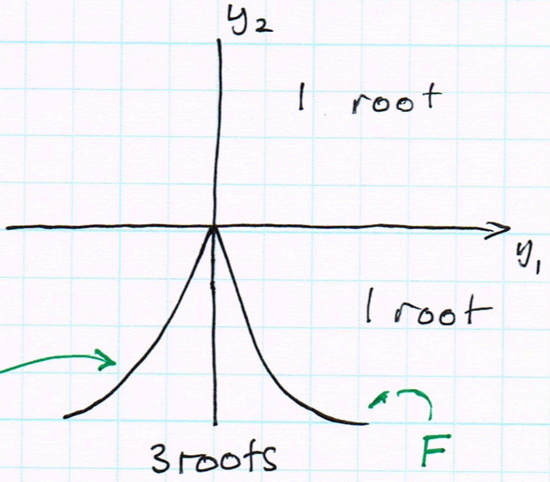
then $y_1(x_1) = x_1^3 - u^2 x_1$

so: critical points at $x_1 = \pm\sqrt{\frac{u^2}{3}}$

critical values: $\pm u^3 \cdot \frac{2}{3\sqrt{3}}$

So



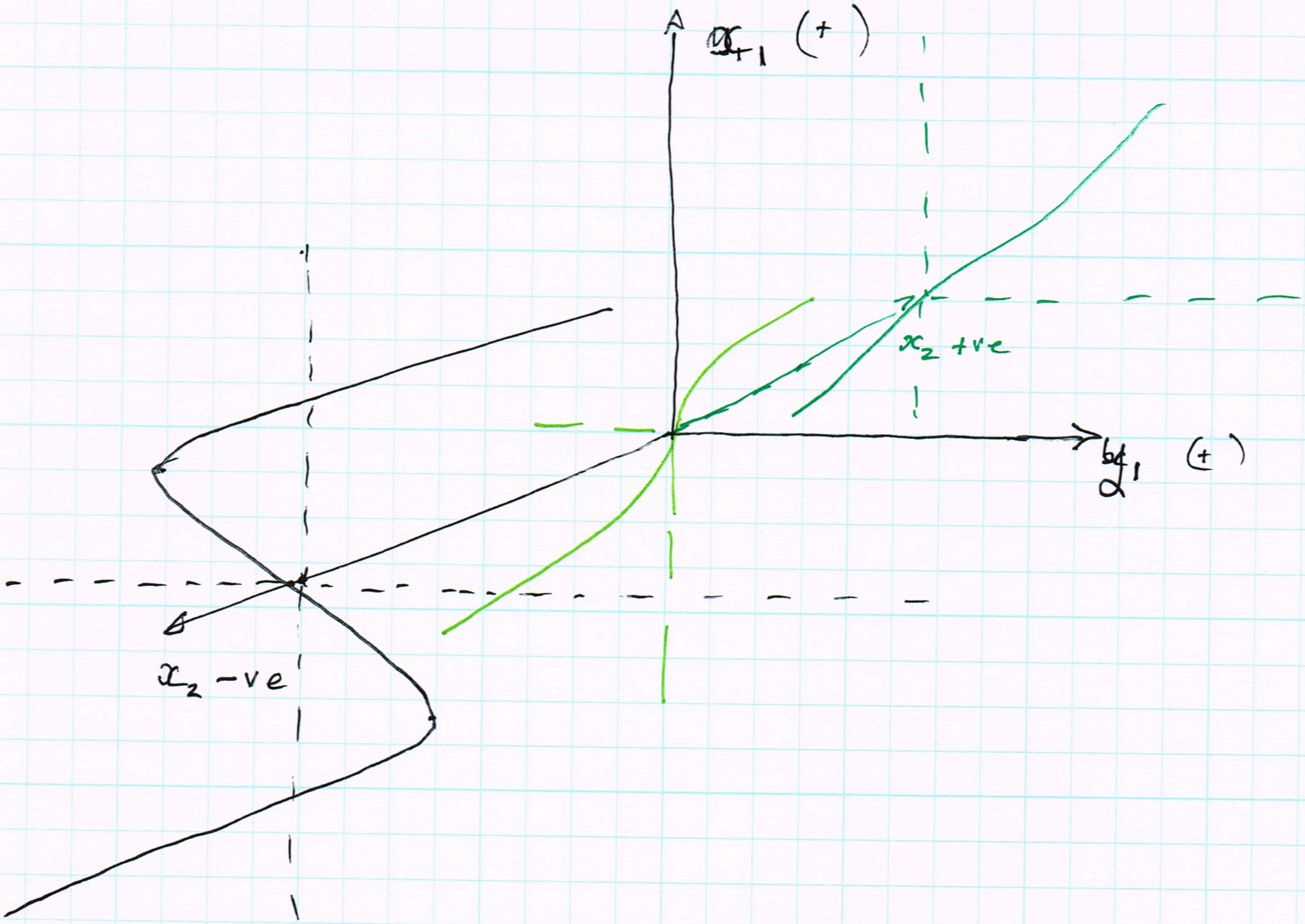


3 roots ≡ 3 (x_1, x_2) pairs csp to this (y_1, y_2) pair.

$\left(u^3 \cdot \frac{2}{3\sqrt{3}}, -u^2\right)$

Semi cubical parabola.

Notice $F = f(\Sigma)$



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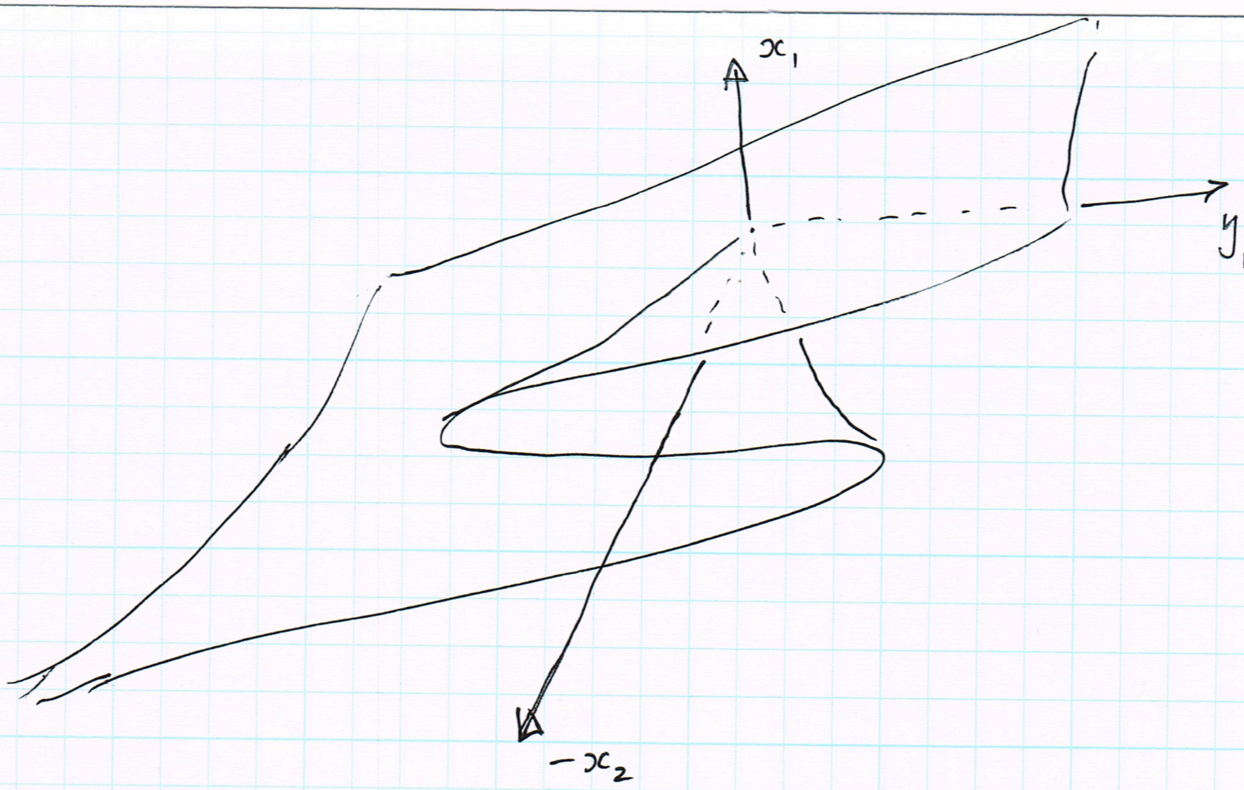
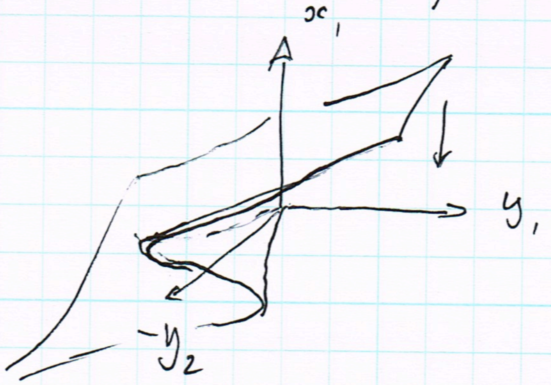
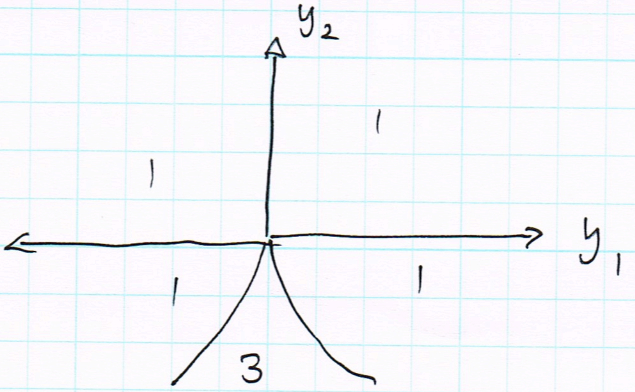


Now this is a nice compact drawing, because $x_2 = y_2$, so we can subst. and see the image of map by squashing the x_1 dir



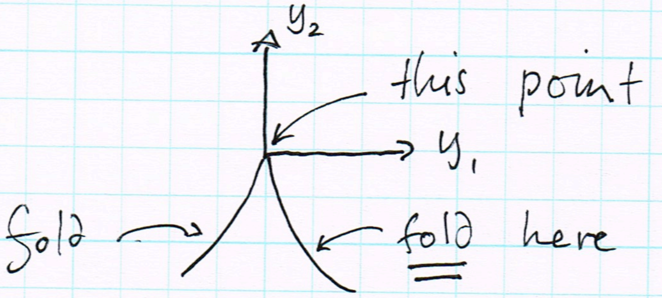


Called a cusp:

Thm: (Whitney, Zeep)

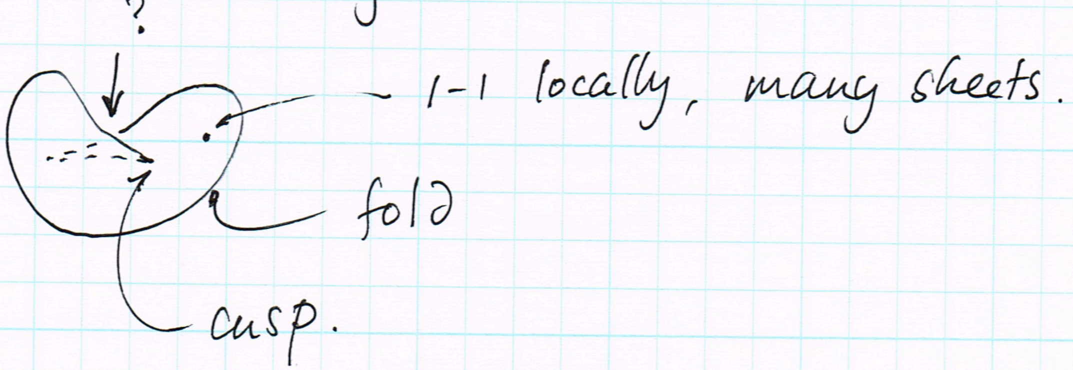
the only stable maps from 2D to 2D are identity, fold, cusp (up to change of coords).

Notice: a cusp is the way folds appear



We see these in outlines:

view an ~~outl~~ image as a 2D → 2D map,
surface param → image coords



? is clearly a singular point of some form but not of a 2D - 2D map (not a local effect). To understand ? you need to study singularities of maps

Surface ⟶ lines in 3D.

All this has been worked out, is known, and has been completely classified! (see URL).

Great math, no impact on recognition

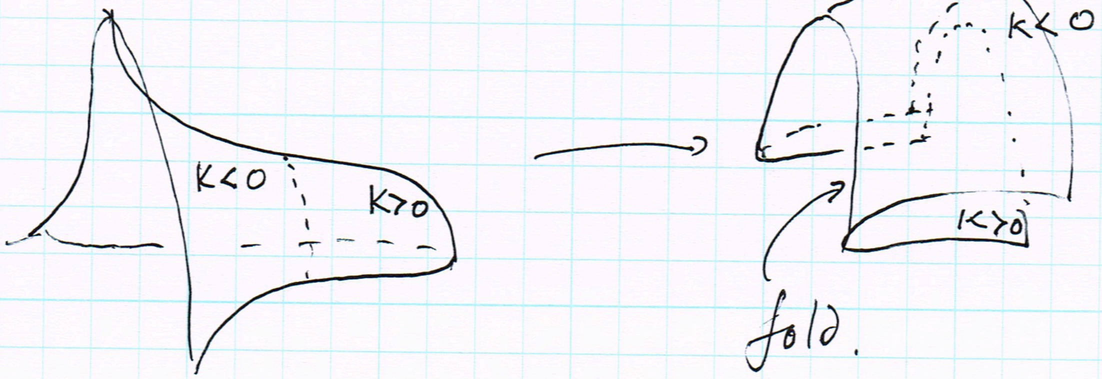
We can get some neat stuff from 2D → 2D.
Consider Gauss map.

$$\begin{tikzcd} \text{Surf par} \arrow[r, "\text{Gauss}"] & \text{unit sphere} \\ (u,v) & N \end{tikzcd}$$

if $K>0$, $K<0$, gauss map is 1—1 (saw this)

$K=0$ — at *least* fold.

e.g.



Some *simple* lighting consequences.

Specularities and Diff. geom:

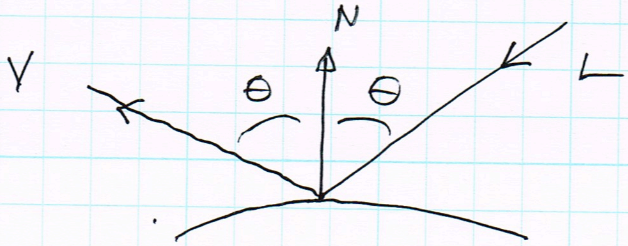
- Specular (mirror-like) reflection can occur when N, V, L

view dir ↑ ↑ lighting dir

have the properties

1) N, V, L are coplanar

2) $\angle N,V = \angle N,L$

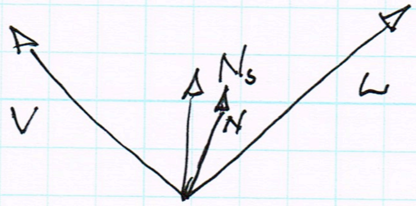


← usual picture

This model is rather idealized!

Some consequences:

- fix view dirn (orthography), light dir (Sun).
- assume that surface isn't perfectly specular, so that



N_s = true specular normal

N = normal.

∴ "bright" if $N \cdot N_s \geq 1 - \varepsilon$.

- this gives a small specular blob, consistent w/ experience
- Shape of blob reveals D.G. of surface.
  - choose c-sys, so that

$$N_s = 0, 0, 1$$

$$\text{Surf} = \left(u, v, -\tfrac{1}{2}(\kappa_1 u^2 + \kappa_2 v^2)\right)$$

Then

$$N = \frac{(\kappa_1 u,\ \kappa_2 v,\ 1)}{\sqrt{1 + \kappa_1^2 u^2 + \kappa_2^2 v^2}}$$

So boundary of specularrity is

$$\kappa_1^2 u^2 + \kappa_2^2 v^2 = \omega$$

Notice: ~~if curvature~~

- ellipse, oriented along principal dirn's
- major, minor axis $\propto$ radius of princ. curvature
- long in low curvature dirn's

Now consider how N_s behaves

- we have some surface — at how many points is N_s "right" for a specularity?

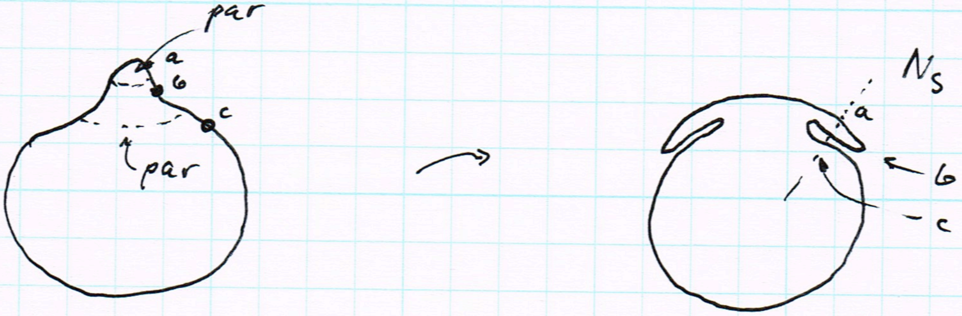
Recall earlier discussion of Gauss-Bonnet, ⑲
where we established that any surface
that is compact, w/o boundary, ~~covers the~~
has at least one point w/ any normal

- → there is always at least 1 specularity on such a surface

- → for generic configurations, there is always an odd number of specularities

(generic ≡ almost every ≡ stable under small move)

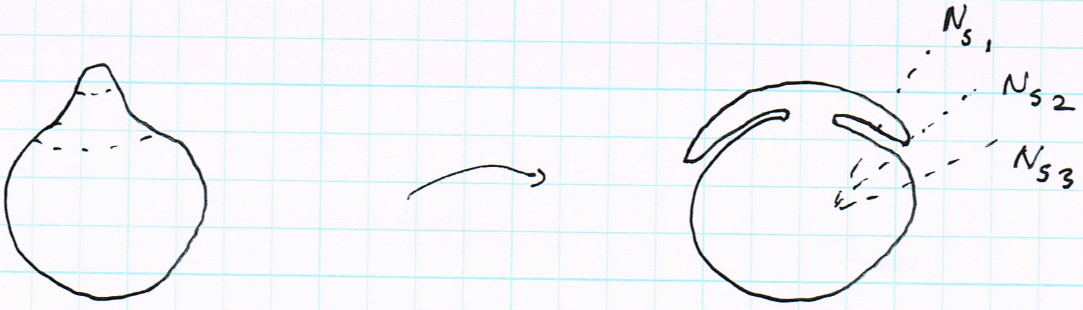
This is because there is always an odd number of sheets on the Gauss map, except at folds, cusps, which appear only on curves) example



(Section thru Gauss map)

Notice if I wiggle it a bit, a, b, c, move a bit, but nothing exciting happens

BUT a large move has consequences



at N_{s_1}, 3 specs

N_{s_2}, 2, one on par line

N_{s_3}, 1

- Specularities are born and die, in pairs, at parabolic lines when object moves.

ISOPHOTES

- an **ISOPHOTE** is a curve of constant brightness on a surface
- in one, **EXTREMELY DUBIOUS** model of brightness, surf has radiosity

$$\underset{\substack{\uparrow\\ \text{surface normal}}}{N} \cdot \underset{\substack{\uparrow\\ \text{Source direction}}}{S}$$

- like poverty, etc. you will see this often; like poverty, etc. it's a bad thing, but seems to be hard to suppress.
  - Does not account for interreflections
  - **Shadows** ? $\max(N \cdot B, 0)$?

Gives some neat geometry.

In this model, an isophote is a curve of constant $N \cdot B$

$\equiv$ ~~a gm~~ Choose Gauss map so B is the north pole — then isophotes are constant height plane sections.