at each point, there are two orthornormal directions in which the directional curvature is extremal.

$$\boxed{\begin{array}{l}\text{principal directions} \\ \qquad\quad \text{curvatures}\end{array}}$$

for a surface

$$\left(s,\ t,\ \frac{1}{2}\left(\kappa_1 s^2 + \kappa_2 t^2\right) + O(3)\right)$$

<u>Compute tangents:</u>

- think of surface as map from a piece of $\mathbb{R}^2 \rightarrow \mathbb{R}^3$
- $(s,t) \longrightarrow \underline{x}(s,t)$.
- Then $\dfrac{\partial \underline{x}}{\partial s}$ must be tangent, by the same argument as for curves

  $\dfrac{\partial \underline{x}}{\partial t}$ "
- and N is unit vector $\perp$ to tangents

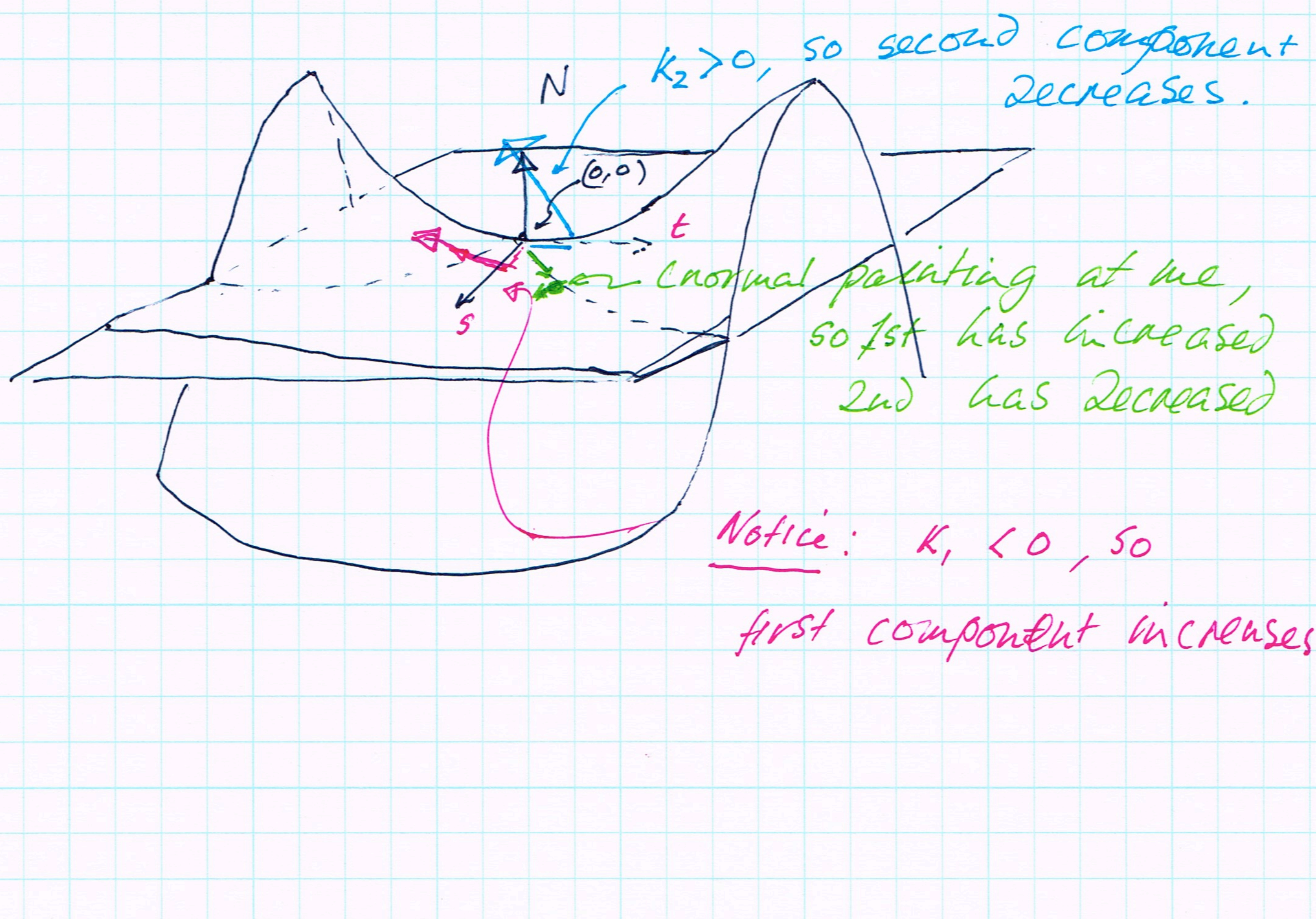
so

T_1: $(1, 0, K_1 s)(1 + O(2))$

T_2: $(0, 1, K_2 t)(1 + O(2))$

N: $(-K_1 s, -K_2 t, 1)(1 + O(2))$

A small step away from $(0,0)$ to $(\Delta u, \Delta v)$ in the tangent plane causes the normal to <u>swing</u> to $(-K_1 \Delta u, -K_2 \Delta v, 1)$



- Now consider a "box" $(0,0) \to (\varepsilon, 0) \to (\varepsilon, \varepsilon)$
  $\to (0, \varepsilon)$
  $\to (0,0)$

- To first order, 3rd normal component doesn't change
  - on gauss map, we get "box"

$(0,0,1) \to (-k_1\varepsilon, 0, 1) \to (-k_1\varepsilon, -k_2\varepsilon, 1) \to (0, -k_2\varepsilon, 1) \to (0,0,1)$

- i.e ratio of areas is

$$\boxed{\text{Gaussian curvature} = k_1 k_2}$$

- Notice that rotating the coordinate system in the tangent plane will get us non-zero st terms in the quadratic form — this expression applies only in the right coordinate system.

<u>But</u> • consider a new coordinate system in tangent plane

$$\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \ \ R \ \ \end{bmatrix}}_{\text{rotation}} \begin{bmatrix} s \\ t \end{bmatrix}$$

then

$$\begin{bmatrix} s & t \end{bmatrix} \underbrace{\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}}_{Q} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} u & v \end{bmatrix} \underbrace{\begin{bmatrix} \ \ R \ \ \end{bmatrix} \begin{bmatrix} K_1 & \\ & K_2 \end{bmatrix} \begin{bmatrix} \ \ R^T \ \ \end{bmatrix}}_{RQR^T} \begin{bmatrix} u \\ v \end{bmatrix}$$

<u>we say</u>: the action of the rotation on the quadratic form takes $Q \longrightarrow RQR^T$

<u>Notice</u>

$$\det(Q) = \det(RQR^T) = K_1 K_2$$

– it's <u>invariant</u> to the rotation!

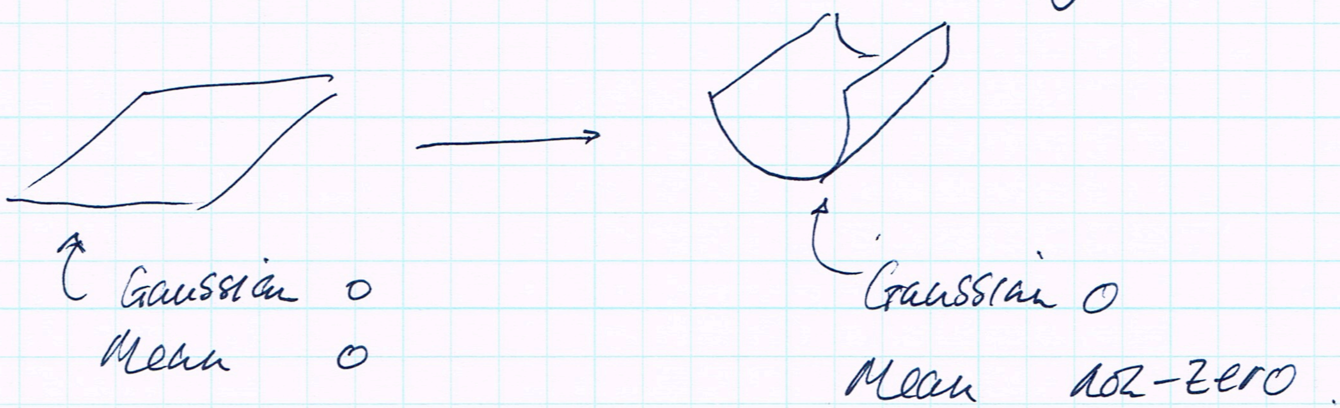
Notice that there is a <u>second</u> invariant

$$\mathrm{Trace}(Q) = \mathrm{Trace}(R\,Q\,R^T) = \kappa_1 + \kappa_2$$

$$\boxed{\text{Mean curvature} = \kappa_1 + \kappa_2}$$

Gaussian curvature has to do with <u>area</u>
(demo w/ piece of paper)

Mean curvature with <u>bending</u>



Clearly, there are surfaces w/ non-zero G.C. and zero M.C.

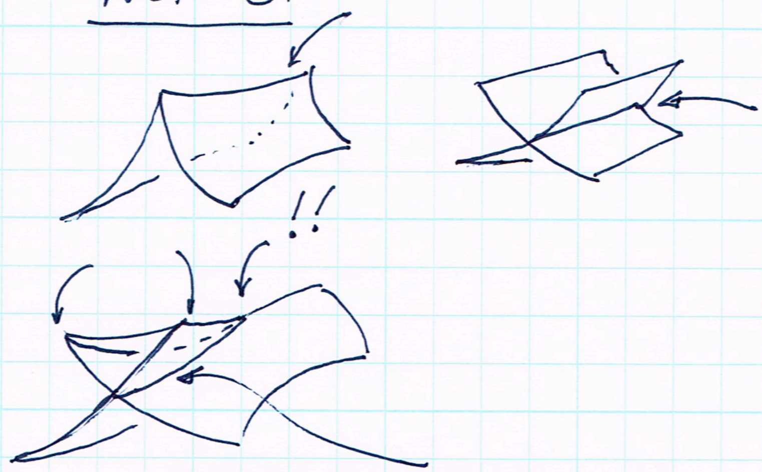
$\Rightarrow$ here G.C. is <u>always</u> -ve.

At this point, we need more powerful machinery – we don't want to constantly reparametrize.

- we are interested in surfaces <u>away</u> from singular points

<u>OK</u>

<u>NOT OK</u>



<u>formally</u>

$$\underline{x} : U \in \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

- Such that $\underline{D\underline{x}}$ has full rank (=2)

[Jacobian of $\underline{x}$
Derivative of $\underline{x}$]

- Notice that anything smooth can (locally) be reparametrized to be like this

- In $\mathbb{R}^3$, we have a metric we're used to working with — we can tell the length of a vector, or the angle between 2 vectors, easily.

- Define

$$x_s = \frac{\partial x}{\partial s} \cdot \cancel{\frac{1}{\|\frac{\partial x}{\partial s}\|}} \quad \leftarrow \quad \text{tangent in } s \text{ direction}$$

$$x_t = \frac{\partial x}{\partial t} \quad \leftarrow \quad \text{tangent in } t \text{ direction}$$

→ these two <u>aren't</u> necessarily unit, <u>and</u> they're not necessarily orthogonal either.

→ They span a <u>tangent plane</u> at each point p, often T_p

→ There's one, usually different, Tangent plane at each point.

→ They're <u>not</u> parallel, because Dx has full rank.

- Now at any point p, we can specify any tangent vector by using $\underline{x}_s$, $\underline{x}_t$ as basis elements

$$V = a\,\underline{x}_s + b\,\underline{x}_t$$

tangent vector

and

$$V \cdot V = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_s \cdot \underline{x}_t \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and if we have

$$u = c\,\underline{x}_s + d\,\underline{x}_t$$

$$V \cdot u = \begin{pmatrix} a \\ b \end{pmatrix} \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_t \cdot \underline{x}_s \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

- this object allows us to measure lengths and angles in the given parametrization — ~~change param~~, ~~and then change~~,
- Quadratic form — $\boxed{\text{The first fundamental form}}$

- often write $I(u,v)$
  for $\begin{pmatrix} a \\ b \end{pmatrix} \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_s \cdot \underline{x}_t \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$

- We can now consider

$$N = \text{the } \underline{\text{unit}} \text{ normal.}$$

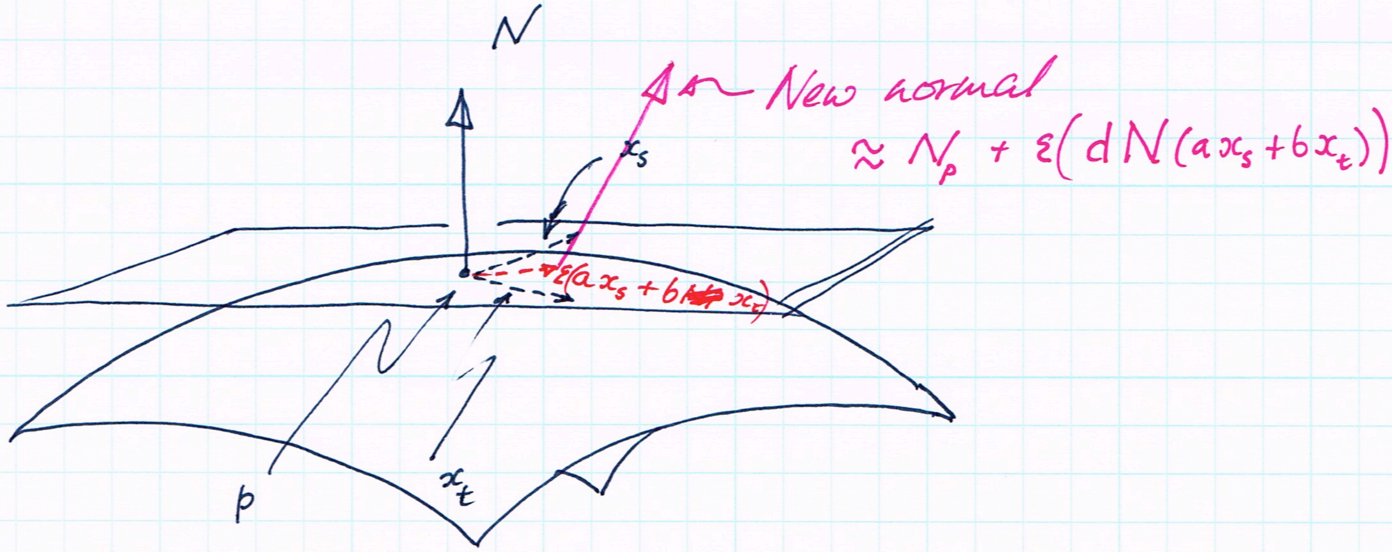
$$= \frac{\underline{x}_s \times \underline{x}_t}{\| \underline{x}_s \times \underline{x}_t \|}$$

defined for our patch, because Dx has full rank.

- because N is <u>unit normal</u>, we have

$$N \cdot N = 1 \quad , \quad N \cdot x_s = 0 \, , \quad N \cdot x_t = 0$$

- now consider moving away from $\underline{p}$, by a small ~~amount, a~~ step in the tangent plane at p.
  - represent as $\varepsilon(a\,\underline{x}_s + b\,\underline{x}_t)$



- if we think of N as a <u>map</u> from 2D $_{(s,t)}$ to 2D [this is the Gauss map]

  points on the sphere

  - it must have a derivative

    dN

  - which is a linear map from plane to plane

    tangent plane to surf

    tangent plane to sphere.

This may worry you – why doesn't the normal swing "in 3D"?

$$N \cdot N = 1 \qquad \text{so } N_s \cdot N = 0, \quad N_t \cdot N = 0$$

and the derivative in some new dirn (in T_p) u has $\frac{\partial}{\partial u} = a\frac{\partial}{\partial s} + b\frac{\partial}{\partial t}$ so $N_u \cdot N = 0$

- the normal only moves on T_p for infinitesimal steps.

- particularly interesting is

$$\boxed{-\underbrace{I}_{\text{first fundamental form}}(\underbrace{dN(u)}_{\text{another tangent vector}}, \underbrace{V}_{\text{tangent vector}}) = \underbrace{II(u, V)}_{\text{SECOND FUNDAMENTAL FORM}}}$$

this is easy to work out in coords:

let $V = v_0 \underline{x}_s + v_1 \underline{x}_t$

$u = u_0 \underline{x}_s + \underline{u}_1 \underline{x}_t$

$$dN(u) = u_0 \cancel{x_s} N_s + u_1 \cancel{x_t} N_t$$

$$I(dN(u), V) = (u_0 \quad u_1) \begin{pmatrix} \underline{N}_s \cdot \underline{x}_s & \underline{N}_s \cdot \underline{x}_t \\ \underline{N}_t \cdot \underline{x}_s & \underline{N}_t \cdot \underline{x}_t \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

this might worry you, because it looks like it isn't symmetric

→ untidy

→ you have to remember order of arguments

<u>BUT</u>

$$N \cdot x_s = 0 \qquad\qquad N \cdot x_t = 0$$

$$\therefore \quad N_t \cdot x_s = -N \cdot x_{st} = N_s \cdot x_t$$

$$N_s \cdot x_s = -N \cdot x_{ss} \quad ; \quad N_t \cdot x_t = -N \cdot x_{tt}$$

So

II is symmetric

Key result

$$\boxed{\begin{aligned} K &= \text{Gaussian curvature} = \det(-\mathrm{I}^{-1}\,\mathrm{II}) \\ H &= \text{Mean curvature} = \operatorname{trace}(-\mathrm{I}^{-1}\,\mathrm{II}) \end{aligned}}$$

which we can establish a bunch of different ways

Advantage: • don't need to compute taylor series at some location

Elegant way to see that K and H are as given.

① for our surfaces

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(t) \qquad \text{at } (0,0)$$

$$II = -\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} + O(t) \qquad \text{at } (0,0)$$

So at $(0,0)$ $\det(I^{-1} II) = k_1 k_2$

$\operatorname{trace}(I^{-1} II) = k_1 + k_2$

② rotating and translating a surface <u>cannot</u> change I or II

- entries are dot products of vectors that also rotate; translation doesn't change the vectors

③ What about change of parametrization? we could think about

$x(s(u,v), t(u,v))$

New parameters — but parametrizing THE SAME geometry.

$$\begin{bmatrix} \underline{x}_u^T \\ \underline{x}_v^T \end{bmatrix} = \begin{bmatrix} s_u & t_u \\ s_v & t_v \end{bmatrix} \begin{bmatrix} \underline{x}_s^T \\ \underline{x}_t^T \end{bmatrix}$$

I haven't been careful about rows + cols till now, because there wasn't a need; but vectors are col. vectors, and this is 2×3

Derivative of repar map, transp

write $J = \begin{bmatrix} s_u & s_v \\ t_u & t_v \end{bmatrix}$

So $$\mathrm{I}^{(u,v)} = \begin{bmatrix} \underline{x}_u^T \\ \underline{x}_v^T \end{bmatrix} \begin{bmatrix} \underline{x}_u & \underline{x}_v \end{bmatrix} = J^T \, \mathrm{I}^{(st)} \, J$$

$$\mathrm{II}^{(u,v)} = \begin{bmatrix} N_u^T \\ N_v^T \end{bmatrix} \begin{bmatrix} \underline{x}_u & \underline{x}_v \end{bmatrix} = J^T \, \mathrm{II}^{(st)} \, J$$

So $$\left[\mathrm{I}^{(u,v)}\right]^{-1} \left[\mathrm{II}^{(u,v)}\right] = J^{-1} \left[\mathrm{I}^{(st)}\right]^{-1} \mathrm{II}^{(st)} \cdot J$$

now $\det(AB) = \det(A)\det(B)$ and $\operatorname{trace}(ABC) = \operatorname{trace}(BCA)$

So $$\det\left(-\left[\mathrm{I}^{(u,v)}\right]^{-1} \mathrm{II}^{(u,v)}\right) = \det\left(-\left[\mathrm{I}^{(st)}\right]^{-1} \mathrm{II}^{(st)}\right)$$

and $$\operatorname{trace}\left(-\mathrm{I}^{-1}\mathrm{II}\right)^{(st)} = \operatorname{trace}\left(-\mathrm{I}^{-1}\mathrm{II}\right)^{(u,v)}$$

<u>NOTE</u>: I have used a standard, superpowerful, geometric argument.

• Prove something in an easy coordinate system, then show that change of coords doesn't matter.

<u>NOTE</u>: We think of K, H as local geometric properties of surfaces <u>BECAUSE</u> they're invariant to rigid motion and reparametrization

- Sometimes, we are interested in <u>other</u> groups (rigid motion + scale; affine tx; projective tx).
- All the above has analogous, much fiddlier, constructions for these cases
  - can be looked up, or worked out; not usually worth it.

Some notation:

it is traditional to write

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x_s \cdot x_s & x_s \cdot x_t \\ x_s \cdot x_t & x_t \cdot x_t \end{pmatrix} \qquad \mathrm{I}$$

and

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} N \cdot x_{ss} & N \cdot x_{st} \\ N \cdot x_{st} & N \cdot x_{tt} \end{pmatrix} \qquad \mathrm{II}$$

now we know that

$$\begin{pmatrix} \underline{N}_s^T \\ \underline{N}_t^T \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} = a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \underline{x}_s^T \\ \underline{x}_t^T \end{pmatrix} \quad \text{for some } a_{ij}$$

(because $N_s \in T_p$, etc) but we don't know a_{ij}

→ easy to get

$$\begin{bmatrix} \underline{N}_s^T \\ \underline{N}_t^T \end{bmatrix} \begin{bmatrix} \underline{x}_s & \underline{x}_t \end{bmatrix} = -\mathrm{II} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \cdot \mathrm{I}$$

so $$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = -\dot{\mathrm{I\!I}}\, \mathrm{I}^{-1}$$

and so $K = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, $H = \mathrm{tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$

Now $\{x_u, x_v, N\}$ is a basis for <u>any</u> vector at p (not just tangent.

Recall the situation w/ curves — we had a frame, and much geometry was revealed by what happened if we took a small step

$$\underline{x}_{ss} = \Gamma^1_{11}\underline{x}_s + \Gamma^2_{11}\underline{x}_t + L_1 \underline{N}$$

$$\underline{x}_{st} = \Gamma^1_{12}\underline{x}_s + \Gamma^2_{12}x_t + L_2 \underline{N} \qquad (= x_{ts})$$

$$\underline{x}_{tt} = \Gamma^1_{22}\underline{x}_s + \Gamma^2_{22}\underline{x}_t + L_3 \underline{N}$$

$$\underline{N}_s = a_{11}\underline{x}_s + a_{12}\underline{x}_t$$

$$\underline{N}_t = a_{12}\underline{x}_s + a_{22}\underline{x}_t$$

above is basically just notation ($\{x_u, x_v, N\}$ is a complete basis, so there must be an expansion)

We can fill in some detail.

$$x_{ss} \cdot N = L_1 (N \cdot N) = L_1 = e$$

Similarly $\quad \cancel{x_{st}} \; L_2 = f \quad , \quad L_3 = g$

which basis vector

$$\boxed{\Gamma^{i}_{jk} \quad \leftarrow \quad \text{Christoffel symbols}}$$

which derivative

<u>Notice</u> $\quad E_s = 2(x_{ss} \cdot x_s) \qquad E_t = 2(x_{st} \cdot x_s)$

$$F_s = (x_{ss} \cdot x_t) + (x_s \cdot x_{st})$$

$$F_t = (x_{st} \cdot x_t) + (x_s \cdot x_{st})$$

$$G_s = 2(x_{st} \cdot x_t) \qquad G_t = 2(x_{tt} \cdot x_t)$$

linear algebra yields

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_s \\ F_s - \frac{1}{2} E_t \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_t \\ \frac{1}{2} G_s \end{pmatrix}$$

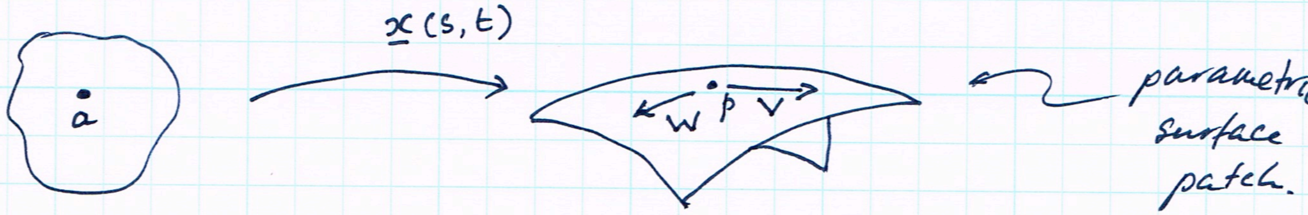
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} F_t - \frac{1}{2} G_s \\ \frac{1}{2} G_t \end{pmatrix}$$

(you can look these up, or derive them; the form isn't that significant)

<u>SIGNIFICANT</u>: Christoffel symbols can be recovered from I and its derivatives. — no use of the embedding.

I as a metric:

- recall what I does.



U = parametric domain
(s,t)

- Given tangent vectors V, W on surface at p, I can measure lengths, angles ~~at~~ by

  $$\text{length}(V) = (V \cdot V)^{1/2}, \quad \text{etc}, \text{ using}$$

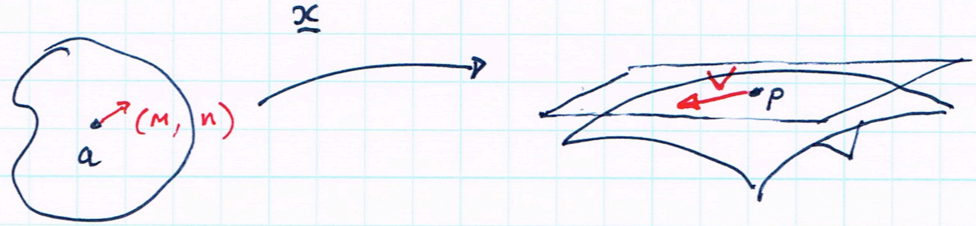
  dot product in 3-space

- ~~I~~ there is a natural rep'n of ~~the~~ V, W at p in terms of basis $\underline{x}_s$, $\underline{x}_t$

  $$V = m\,\underline{x}_s + n\,\underline{x}_t \qquad \text{etc.}$$

- Now I could represent V (on T_p) as (m,n) ~~@~~ (at a, $\underline{x}: a \to p$, in par domain).

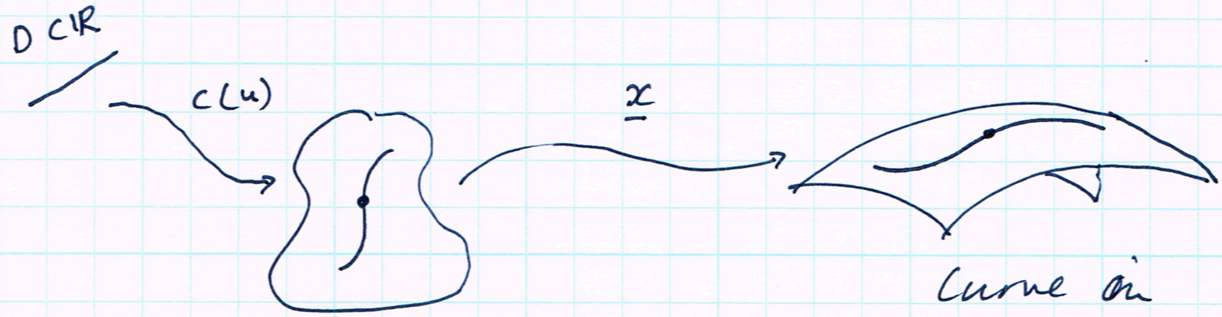
I allows me to measure lengths, angles <u>there</u>



$I\left(\begin{pmatrix} m \\ n \end{pmatrix}, \begin{pmatrix} m \\ n \end{pmatrix}\right) = V \cdot V$

etc for angles

So I can measure the length of a curve on the surface



$u \in \mathbb{R}^2$

curve in par domain

$s(u), t(u)$

Curve in Surface

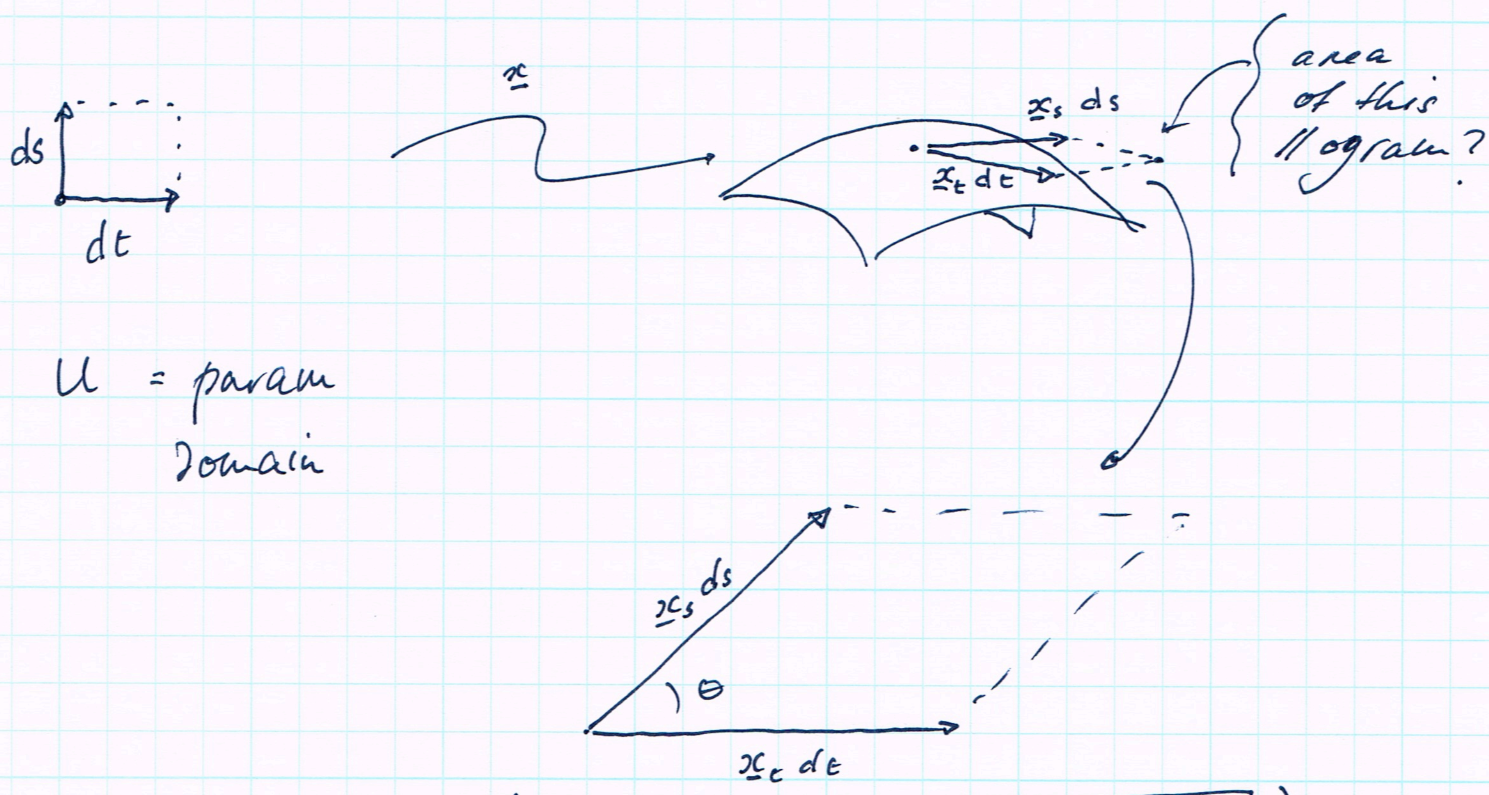
$\underline{x}(s(u), t(u))$.

So length $= \displaystyle\int_D \sqrt{I\left(\begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}, \begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}\right)}\, du$

tangent to curve in surface pars

Area:

- I get the area of a patch on a surface by adding up the area of elementary patches
- Divide relevant piece of parametrisation domain into infinitesimal quads ds, dt and add up areas



Area:

$$(ds\,dt)\left(\|x_s\|\,\|x_t\|\sqrt{1-\left[\frac{(\underline{x}_s\cdot\underline{x}_t)}{\|x_s\|\,\|x_t\|}\right]^2}\right)$$

So

$$\underline{\text{Area}} = ds\,dt \left[\sqrt{\|x_s\|^2 \|x_t\|^2 - (x_s \cdot x_t)^2} \right]$$

$$= ds\,dt \left[\sqrt{EF - G^2} \right]$$

this is det I

so area cut out by $U \in$ domain

$$\int_U \det(I)\, ds\, dt$$

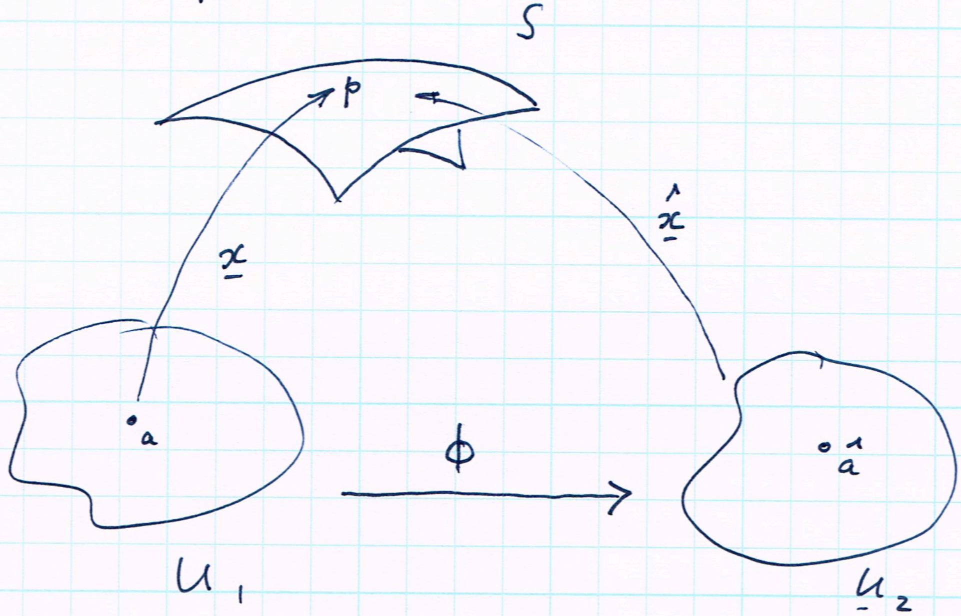
(not usually easy!)

Notice that in these calculations, we used I, __not__ $\underline{x}$.

- i.e. if I have a I over a domain, I can complete length, area w/o knowing $\underline{x}$

Reparametrizing a Surface

· the same surface can have many different parametrizations



· consider two parametrizations, $\underline{x}$, $\hat{\underline{x}}$ of a surface S; there must be some ϕ, 1-1 linking them. In this picture

$$\hat{\underline{x}} \circ \phi = \underline{x}$$

Now since the <u>parametrization</u> can't affect lengths, angles on surf, we must have

for $\underline{x}, \hat{\underline{x}}$ case — Derivative of ϕ at a.

$$I_a^{(x)}(u, v) = I_{\phi(a)}^{(\hat{x})}\left(d\phi(u), d\phi(v)\right)$$

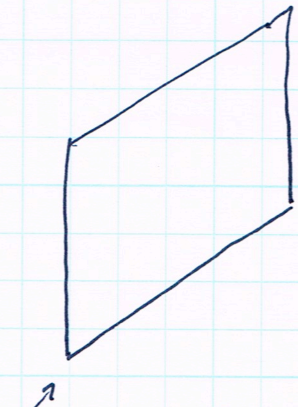
location — corresponding location for $\hat{x}$, found using ϕ

- Notice that if $\underline{x}, \hat{\underline{x}}$ are parametrizations of the same surface S, then some ϕ with this property must exist.

- Now consider $\psi : S_1 \longrightarrow S_2$ (maps surfaces to surfaces).

- if ψ does not change lengths (and so angles) it is an <u>isometry</u>

- ~~Not~~ <u>Example</u>: Rotation + Translation



flat sheet $= S_1$

roll up
without stretch

$= S_2$

· There is no stretch

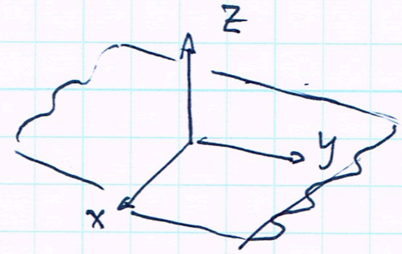
∴ all lengths on surface are unchanged
(it follows all angles are unchanged, too)

∴ an isometry must exist

in coordinates:

$$S_1 = [s, t, 0]$$

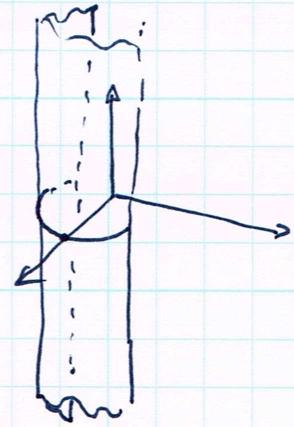
$$s \in [-1, 1]$$



$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{at all points}$$

$S_2 = [\cos, \sin s, t]$

$s \in [-1, 1]$



$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ at <u>all points</u>

So an <u>isometry</u> must exist

~~(ϕ = identity works)~~.

Q: when does an isometry exist?

~~Q:~~ Geometric properties ~~p~~ invariant under isometry are referred to as

<u>Intrinsic</u>

Clearly, H <u>isn't</u> intrinsic.

Lemma: assume Ψ is an isometry.

$\Psi : S_1 \to S_2$.

$\underline{x} : (U_1 \subset \mathbb{R}^2) \longrightarrow S_1$

$\hat{\underline{x}} : (U_2 \subset \mathbb{R}^2) \longrightarrow S_2$

then there is a parametrization of S_2 such that

$$\hat{E} = E \qquad \hat{F} = F \qquad \hat{G} = G$$

coord functions of I on S_1 for $\underline{x}$

coord functions of I on S_2 for new param.

Proof:

$$\begin{tikzcd} S_1 \arrow[r, "\Psi"] & S_2 \\ U_1 \arrow[u, "\underline{x}"] & U_2 \arrow[u, "\hat{\underline{x}}"'] \end{tikzcd}$$

Now parametrize S_2 by $\Psi \circ S_1$

→ lengths, angles are the same

→ ~~same~~ a in U_1 refers to CSP pts in S_1, S_2

→ $\hat{E} = E$, etc.

Important consequence:

- Any property that can be expressed in terms of E, F, G (and their derivatives) is intrinsic

Theorema egregium: Gaussian curvature is intrinsic

Proof: (not super enlightening – see scanned pages) Manipulate formula for K to produce expression in E, F, G and derivatives.

Corollary: Isometric surfaces have the same Gaussian curvature at csp points

We now have two threads to look at:

- We can abstract away embeddings and study only <u>intrinsic</u> properties (given by E, F, G). To do this, we think about I as a function on 2D, and don't really worry about embedding. This leads to Riemannian geometry; we'll do some of this.

- We can look at extrinsic (+ intrinsic, on occasion) properties, where the embedding matters (often a lot). We'll do lots of this, next.