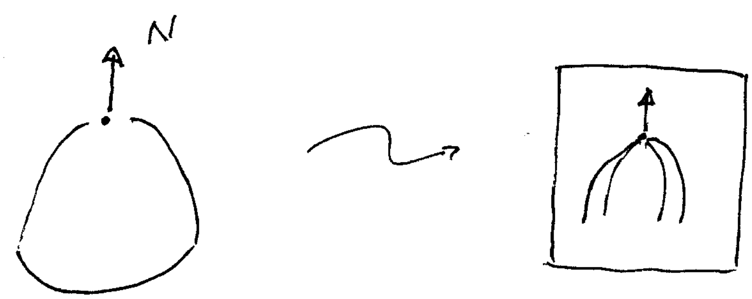
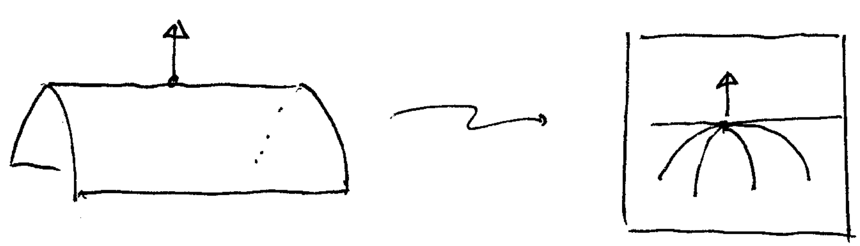
Local (Differential) Geometry of surfaces:

Choose a point on a surf.

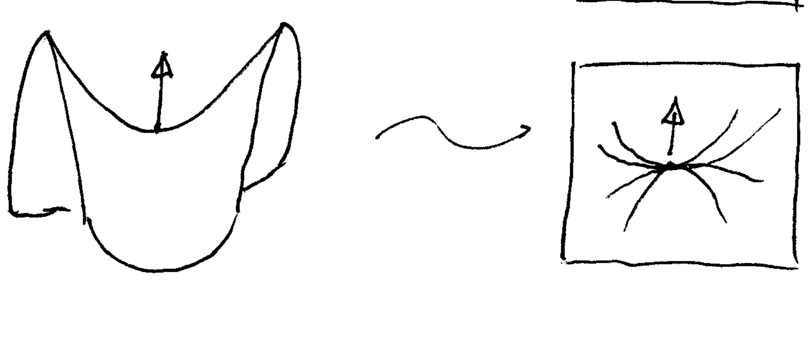
- compute normal
- Build family of planes thru pt, normal
- consider these X-sections of surf
- 3 cases



Elliptic



Parabolic



Hyperbolic

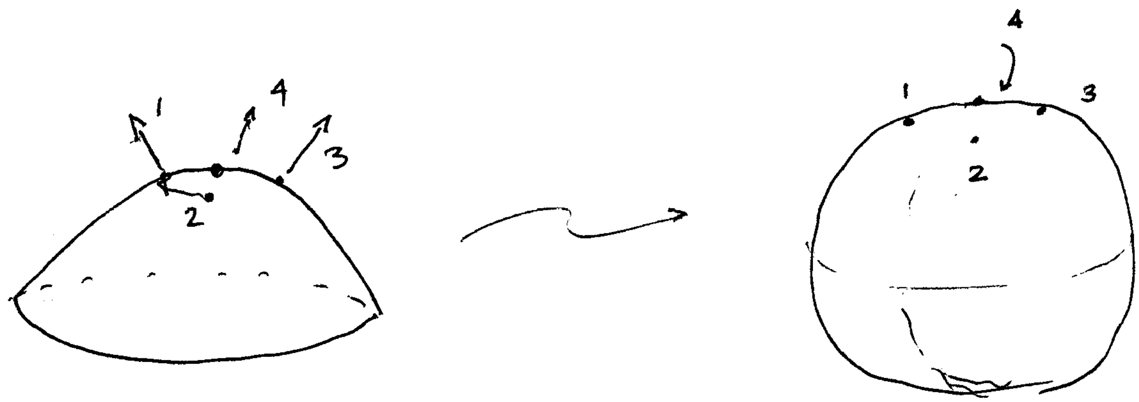
A finer classification would be helpful

Both Elliptic

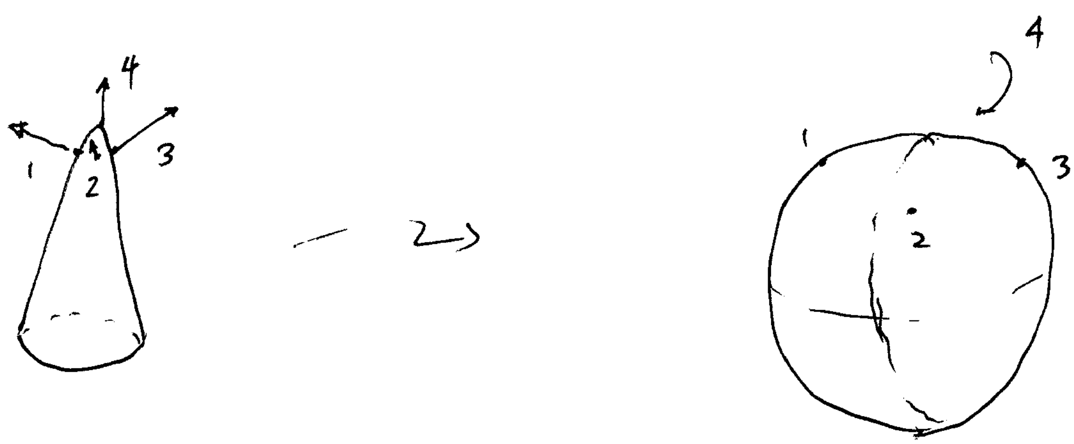
We get this from the <u>Gauss map</u>.

$$\underset{\text{pt. on surface}}{\underline{x}} \longrightarrow \underset{\text{pt on sphere given by normal}}{N(\underline{x})}$$

I

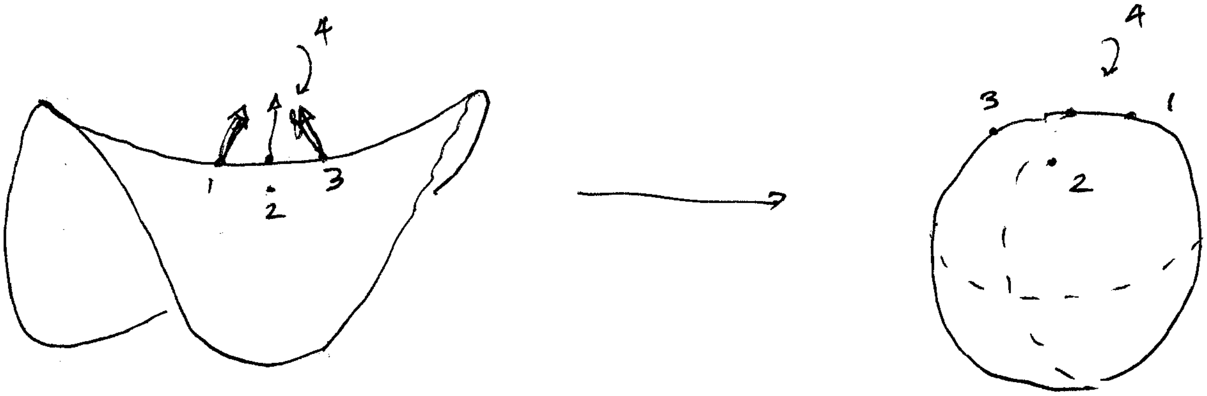


II

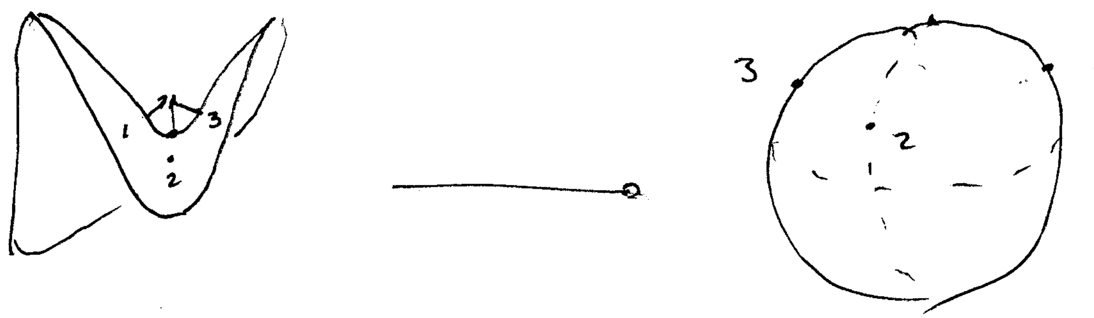


- map a small circle round p to sphere
- case I: small circle
  $\downarrow$
  small "
- case II: small $\longrightarrow$ big.

I



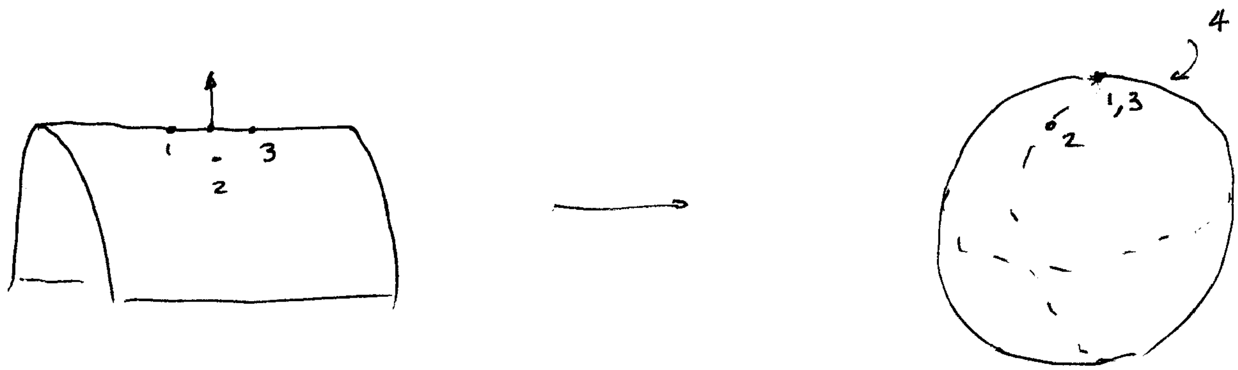
II



- Notice <u>direction reverses</u>

I : small $\longrightarrow$ small

II : small $\longrightarrow$ big



· small → area zero.

<u>Defn</u>

K = Gaussian curvature

$$= \lim_{\text{radius} \to 0} \left\{ \frac{\text{Area on Gauss map}}{\text{Area on Surf}} \right\}$$

$$K = \begin{cases} < 0 & \text{Hyperbolic} \\ 0 & \text{Parabolic} \\ > 0 & \text{Elliptic} \end{cases}$$

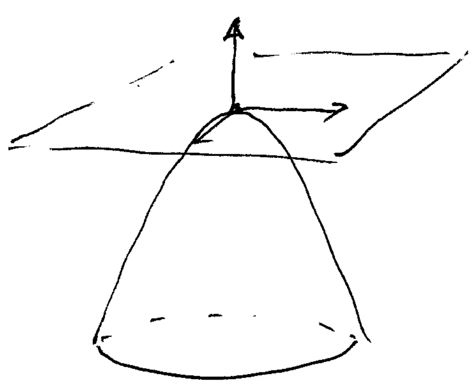
<u>Bending</u> does not change K

- You must $\left\{\begin{matrix} \text{add} \\ \text{subtract} \end{matrix}\right\}$ area.

(So there must be another description to add detail.)

- Take a point on a surface.
- Construct a coord system ~~in~~

(x,y) in tangent plane, with z normal



- IN THIS COORD SYSTEM, near this pt, write Taylor Series.

Surface is

$$(x, y, z(x,y))$$

$$\approx \left(x, y, z_0 + (\nabla z)\cdot(x,y) + \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix} + O(x,y)^3\right)$$

but $z_0 = 0$

$\nabla z = 0$

so $$(x, y, z(x,y)) = \left(x, y, \underbrace{\frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix}} + O(3)\right)$$

- this is a quadratic form
- symmetric

$\therefore$ rotate coord sys

$$\left(u, v, z(u,v)\right) = \left(u, v, \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right) + O(3)\right)$$

Now recall a curve

$(u, \frac{1}{2}au^2)$ has curvature a at $u = 0$

- So the curvature of the u section is K_1

v " is K_2

~~$S = u\cos\theta + v\sin\theta$~~ "

$S = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ " is $K_1 \cos^2\theta + K_2 \sin^2\theta$

$\therefore$ The directional curvature has

maximum $\max(K_1, K_2)$

min $\min(K_1, K_2)$