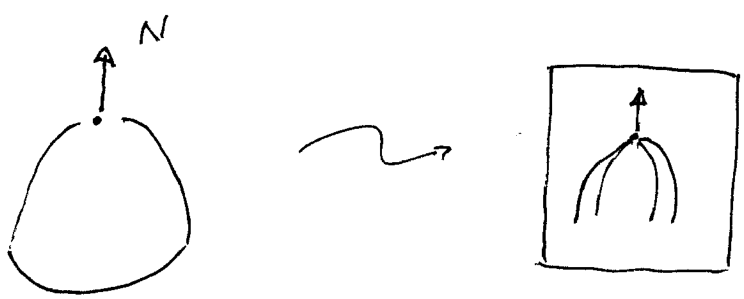
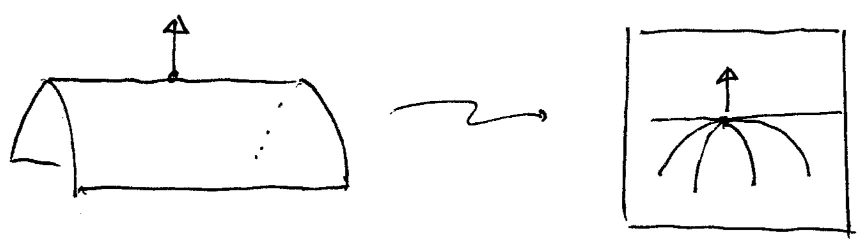
Local (Differential) Geometry of surfaces:

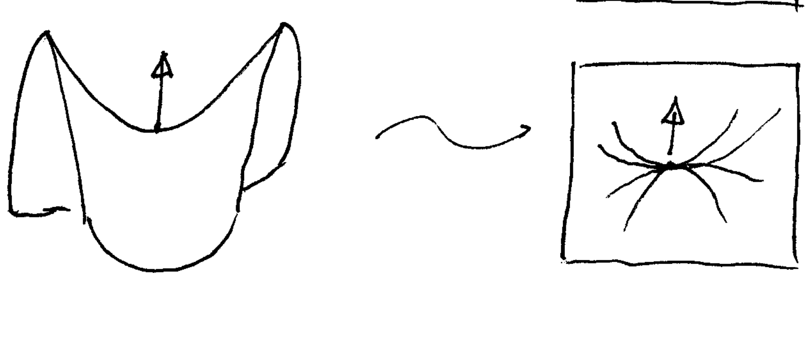
Choose a point on a surf.
- compute normal
- Build family of planes thru pt, normal
- consider these X-sections of surf
- 3 cases



Elliptic



Parabolic



Hyperbolic

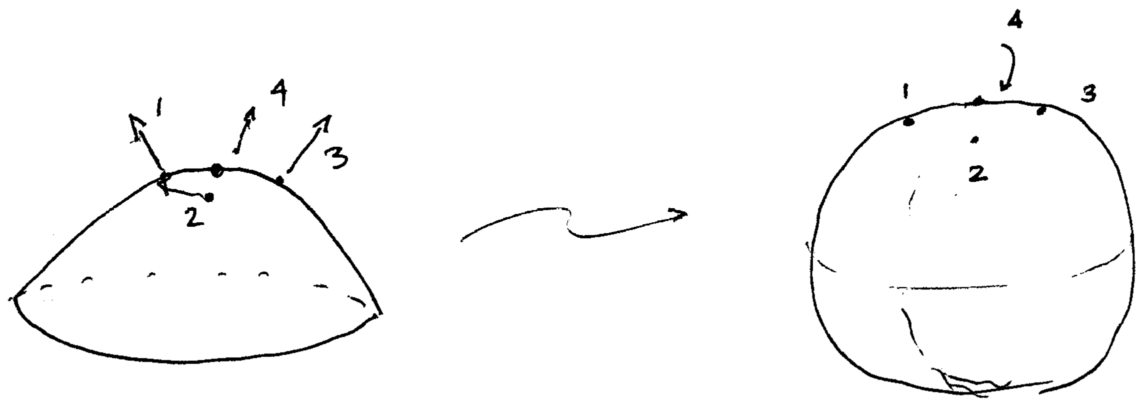
A finer classification would be helpful

Both Elliptic

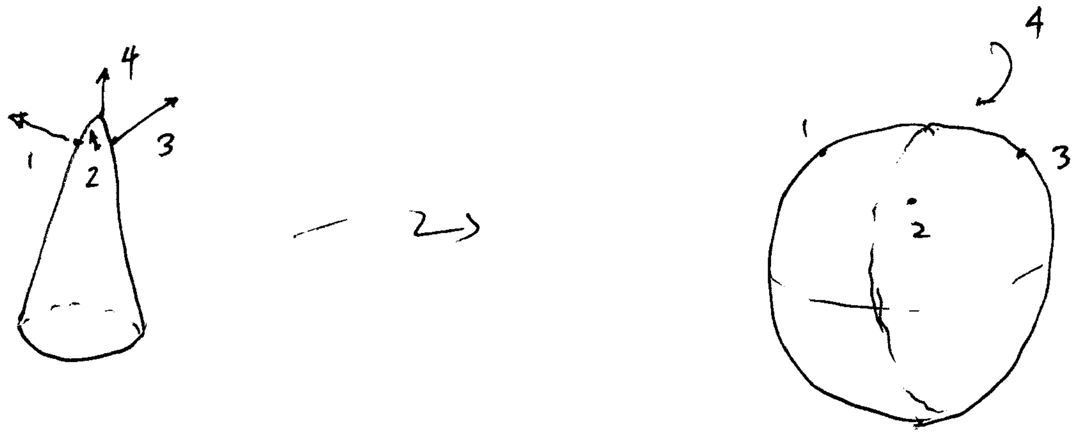
We get this from the <u>Gauss map</u>.

$$\underset{\text{p.t. on surface}}{\underline{x}} \longrightarrow \underset{\text{pt on sphere given by normal}}{N(\underline{x})}$$

I

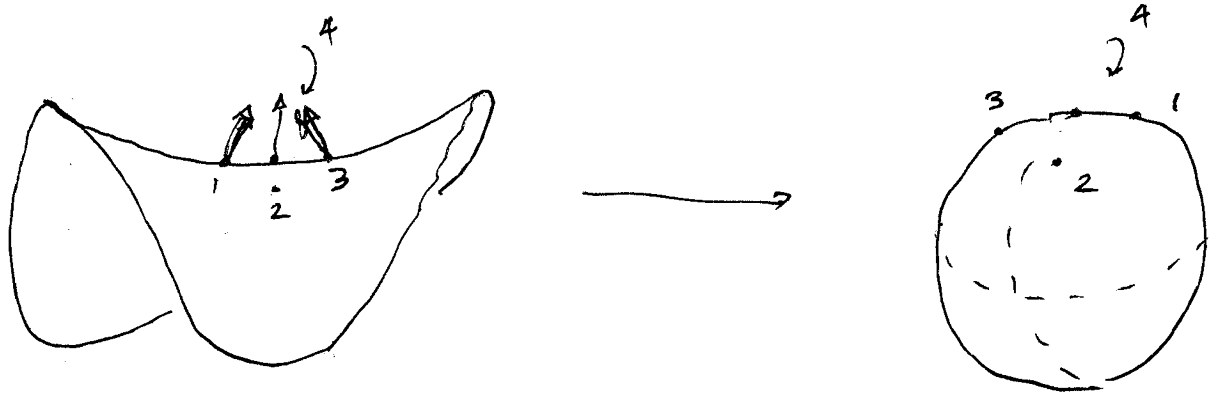


II

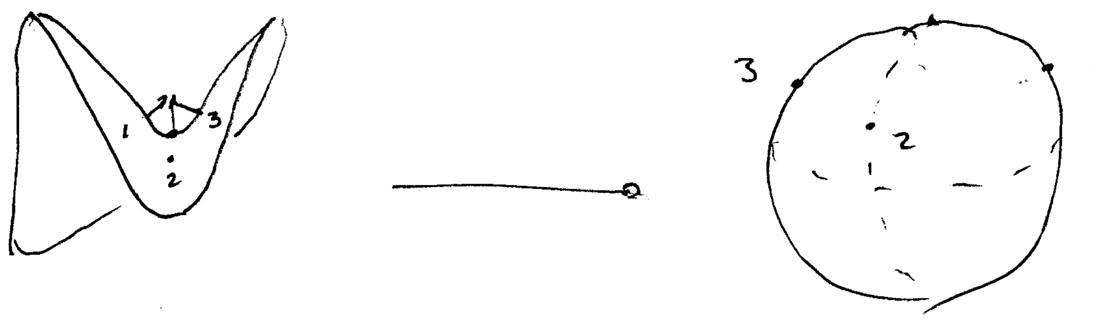


- map a small circle round p to sphere
- case I : small circle $\downarrow$ small "
- case II : small $\longrightarrow$ big.

I



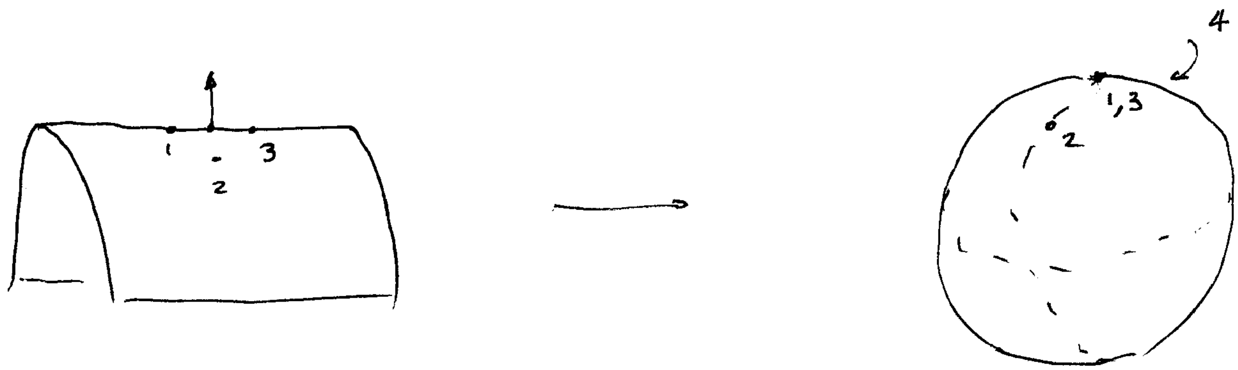
II



- Notice <u>direction reverses</u>

I : small $\longrightarrow$ small

II : small $\longrightarrow$ big



· small $\longrightarrow$ area zero.

<u>Defn</u>

K = Gaussian curvature

$$= \lim_{\text{radius} \to 0} \left\{ \frac{\text{Area on Gauss map}}{\text{Area on surf}} \right\}$$

$$K = \begin{cases} < 0 & \text{Hyperbolic} \\ 0 & \text{Parabolic} \\ > 0 & \text{Elliptic} \end{cases}$$

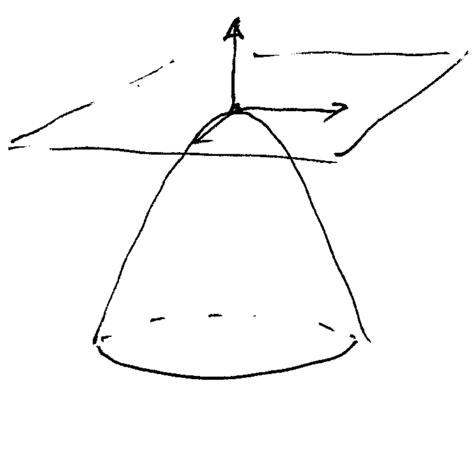
<u>Bending</u> does not change K

- You must $\left\{ \begin{array}{l} \text{add} \\ \text{subtract} \end{array} \right\}$ area.

(So there must be another Description to add detail.)

- Take a point on a surface.
- Construct a coord system ~~in~~

(x,y) in tangent plane, with z normal



- IN THIS COORD SYSTEM, near this pt, write Taylor Series.

Surface is

$$(x, y, z(x,y))$$

$$\approx \left(x, y, z_0 + (\nabla z)\cdot(x,y) + \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix} + O(x,y)^3\right)$$

but $z_0 = 0$

$\nabla z = 0$

so $$(x, y, z(x,y)) = \Big(x, y, \underbrace{\frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix}} + O(3)\Big)$$

- this is a quadratic form
- symmetric

$\therefore$ rotate coord sys

$$\Big(u, v, z(u,v)\Big) = \Big(u, v, \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right) + O(3)\Big)$$

Now recall a curve

$(u, \frac{1}{2}au^2)$ has curvature a at $u=0$

- So the curvature of the u section is K_1

v " is K_2

~~$s = u\cos\theta + v\sin\theta$~~ "

$s = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ " is $K_1\cos^2\theta + K_2\sin^2\theta$

$\therefore$ The directional curvature has

maximum $\max(K_1, K_2)$

min $\min(K_1, K_2)$

at each point, there are two orthonormal directions in which the directional curvature is extremal.

$$\boxed{\begin{array}{l}\text{principal directions} \\ \qquad\text{curvatures}\end{array}}$$

for a surface

$$\left(s, t, \tfrac{1}{2}\left(\kappa_1 s^2 + \kappa_2 t^2\right) + O(3) \right)$$

<u>Compute tangents:</u>

- think of surface as map from a piece of $\mathbb{R}^2 \to \mathbb{R}^3$
- $(s,t) \longrightarrow \underline{x}(s,t)$.
- Then $\dfrac{\partial \underline{x}}{\partial s}$ must be tangent, by the same argument as for curves

  $\dfrac{\partial \underline{x}}{\partial t}$ "
- and N is unit vector $\perp$ to tangents

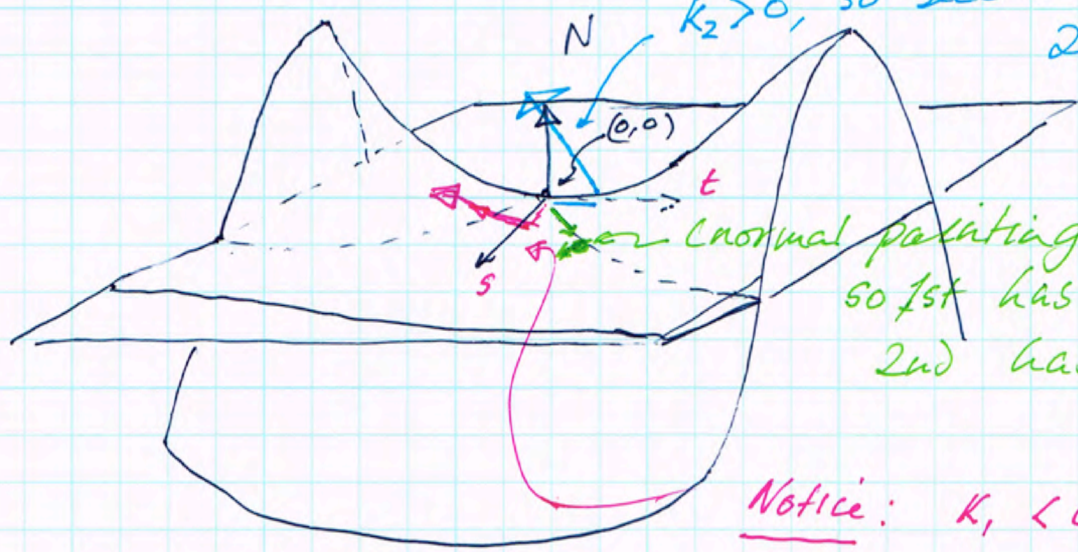
so

$$T_1: \quad (1, 0, K_1 s)(1 + O(2))$$

$$T_2: \quad (0, 1, K_2 t)(1 + O(2))$$

$$N: \quad (-K_1 s, -K_2 t, 1)(1 + O(2))$$

A small step away from $(0,0)$ to $(\Delta u, \Delta v)$ in the tangent plane causes the normal to <u>swing</u> to $(-K_1 \Delta u, -K_2 \Delta v, 1)$



- Now consider a "box" $(0,0) \to (\varepsilon, 0) \to (\varepsilon, \varepsilon)$
$\to (0, \varepsilon)$
$\to (0,0)$

- To first order, 3rd normal component doesn't change
  - on gauss map, we get "box"

$(0,0,1) \to (-K_1\varepsilon, 0, 1) \to (-K_1\varepsilon, -K_2\varepsilon, 1) \to (0, -K_2\varepsilon, 1) \to (0,0,1)$

- i.e ratio of areas is

$$\boxed{\text{Gaussian curvature} = K_1 K_2}$$

- Notice that rotating the coordinate system in the Tangent plane will get us non-zero st terms in the quadratic form — this expression applies only in the right coordinate system.

But . consider a new coordinate system in tangent plane

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

rotation

then

$$\underbrace{[s \quad t] \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}_{Q} = \underbrace{[u \quad v] \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} k_1 & \\ & k_2 \end{bmatrix} \begin{bmatrix} R^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}}_{R Q R^T}$$

we say: the action of the rotation on the quadratic form takes $Q \rightarrow RQR^T$

Notice

$$\det(Q) = \det(RQR^T) = k_1 k_2$$

– it's invariant to the rotation!

Notice that there is a <u>second</u> invariant

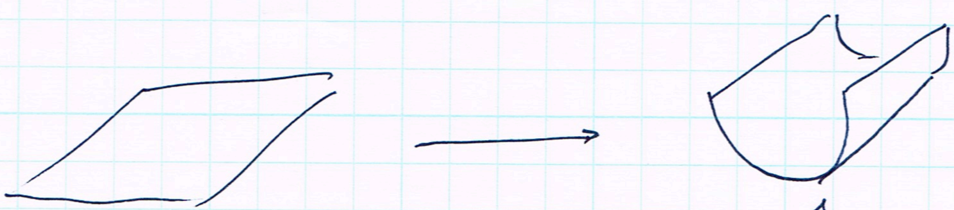
$$\mathrm{Trace}(Q) = \mathrm{Trace}(R^{T} Q R^{T}) = \kappa_1 + \kappa_2$$

$$\boxed{\text{Mean curvature} = \frac{\kappa_1 + \kappa_2}{2}}$$

AARGH!

Gaussian curvature has to do with <u>area</u>
(demo w/ piece of paper)

Mean curvature with <u>bending</u>



Gaussian 0
Mean 0

Gaussian 0
Mean non-zero.

Clearly, there are surfaces w/ non-zero G.C. and zero M.C.

$\Rightarrow$ here G.C. is <u>always</u> -ve.

At this point, we need more powerful machinery — we don't want to constantly reparametrize.

- we are interested in surfaces **away** from singular points

OK

NOT OK

formally

$$\underline{x} : U \in \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

- Such that $\underline{D\underline{x}}$ has full rank (=2)

[Jacobian of $\underline{x}$
Derivative of $\underline{x}$]

- Notice that anything smooth can (locally) be reparametrized to be like this

- In $\mathbb{R}^3$, we have a metric we're used to working with — we can tell the length of a vector, or the angle between 2 vectors, easily.

- Define

$$x_s = \frac{\partial \underline{x}}{\partial s} \cdot \cancel{\frac{1}{\left\| \frac{\partial \underline{x}}{\partial s} \right\|}} \quad \leftarrow \text{tangent in } s \text{ direction}$$

$$x_t = \frac{\partial \underline{x}}{\partial t} \quad \leftarrow \text{tangent in } t \text{ direction}$$

→ these two <u>aren't</u> necessarily unit, <u>and</u> they're not necessarily orthogonal either.

→ They span a <u>tangent plane</u> at each point p, often T_p

→ There's one, usually different, Tangent plane at each point.

→ They're <u>not</u> parallel, because $D\underline{x}$ has full rank.

- Now at any point p, we can specify any tangent vector by using $\underline{x}_s$, $\underline{x}_t$ as basis elements

$$\underbrace{V}_{\text{tangent vector}} = a\,\underline{x}_s + b\,\underline{x}_t$$

and

$$V \cdot V = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_s \cdot \underline{x}_t \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and if we have

$$U = c\,\underline{x}_s + d\,\underline{x}_t$$

$$V \cdot U = \begin{pmatrix} a \\ b \end{pmatrix} \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_t \cdot \underline{x}_s \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

- this object allows us to measure lengths and angles in the given parametrization — ~~change param, and then change~~,

- Quadratic form — $\boxed{\text{The first fundamental form}}$

- often write $I(u, v)$

for $\begin{pmatrix} a \\ b \end{pmatrix} \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_s \cdot \underline{x}_t \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$

- We can now consider

N = the unit normal.

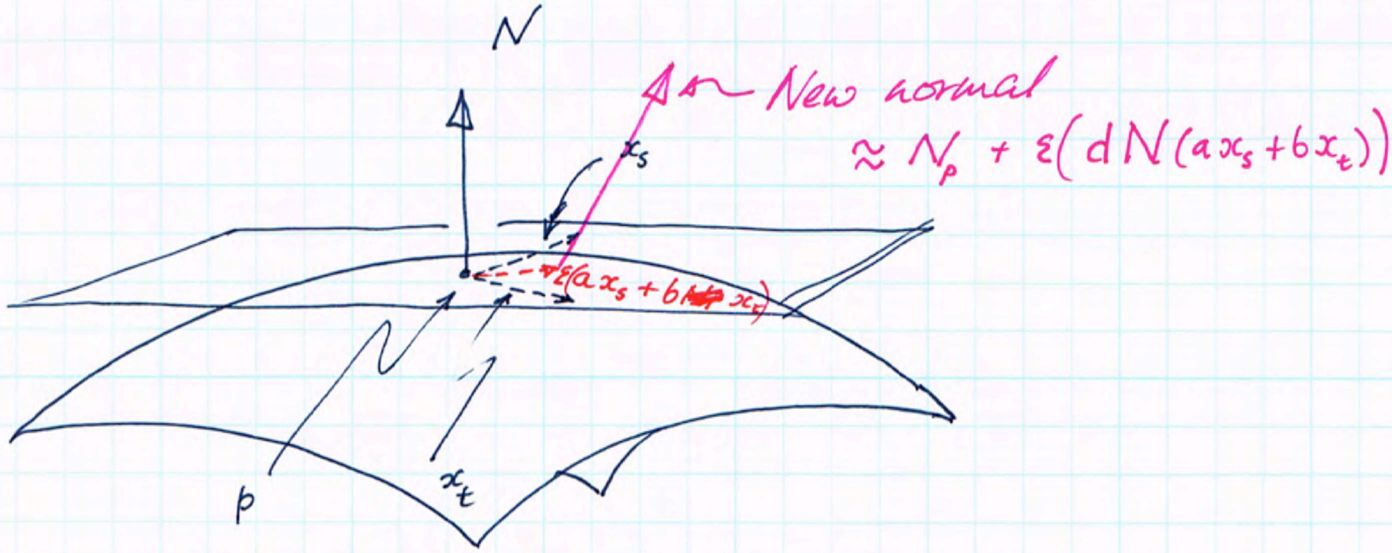
$$= \frac{\underline{x}_s \times \underline{x}_t}{\| \underline{x}_s \times \underline{x}_t \|}$$

defined for our patch, because Dx has full rank.

- because N is unit normal, we have

$$N \cdot N = 1 \quad , \quad N \cdot x_s = 0 \, , \; N \cdot x_t = 0$$

- now consider moving away from $\underline{p}$, by a small ~~amount, a~~ step in the tangent plane at p.
  - represent as $\varepsilon(a \, \underline{x}_s + b \, \underline{x}_t)$



- if we think of N as a <u>map</u> from 2D to 2D [this is the Gauss map]

(s, t)

points on the sphere

- it must have a derivative

dN

- which is a linear map from plane to plane

tangent plane to surf

tangent plane to sphere.

This may worry you – why doesn't the normal swing "in 3D"?

$$N \cdot N = 1 \qquad \text{so } N_s \cdot N = 0, \quad N_t \cdot N = 0$$

and the derivative in some new dir'n (in T_p) u has $\frac{\partial}{\partial u} = a\frac{\partial}{\partial s} + b\frac{\partial}{\partial t}$ so $N_u \cdot N = 0$

· the normal only moves on T_p for infinitesimal steps.

· particularly interesting is

$$\boxed{-\underbrace{I}_{\text{first fundamental form}}(\underbrace{dN(u)}_{\text{another tangent vector}}, \underbrace{V}_{\text{tangent vector}}) = \underbrace{II(u, V)}_{\text{SECOND FUNDAMENTAL FORM}}}$$

this is easy to work out in coords:

let $\quad V = v_0 \underline{x}_s + v_1 \underline{x}_t$

$$U = u_0 \underline{x}_s + \underline{u}_1 \underline{x}_t$$

$$dN(u) = u_0 \cancel{x_s} N_s + u_1 \cancel{x_t} N_t$$

$$I(dN(u), V) = \begin{pmatrix} u_0 & u_1 \end{pmatrix} \begin{pmatrix} \underline{N}_s \cdot \underline{x}_s & \underline{N}_s \cdot \underline{x}_t \\ \underline{N}_t \cdot \underline{x}_s & \underline{N}_t \cdot \underline{x}_t \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

this might worry you, because it looks like it isn't symmetric

→ untidy

→ you have to remember order of arguments

BUT

$$N \cdot x_s = 0 \qquad\qquad N \cdot x_t = 0$$

$$\therefore \quad N_t \cdot x_s = -N \cdot x_{st} = N_s \cdot x_t$$

$$N_s \cdot x_s = -N \cdot x_{ss} \quad ; \qquad N_t \cdot x_t = -N \cdot x_{tt}$$

So

II is symmetric

Key result

$$\boxed{\begin{aligned} K &= \text{Gaussian curvature} = \det(-\mathrm{I}^{-1}\,\mathrm{II}) \\ H &= \text{Mean curvature} = \operatorname{trace}(-\mathrm{I}^{-1}\,\mathrm{II}) \end{aligned}}$$

which we can establish a bunch of different ways

Advantage: · don't need to compute taylor series at some location

Elegant way to see that K and H are as given.

① for our surfaces

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(t) \qquad \text{at } (0,0)$$

$$II = -\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} + O(t) \qquad \text{at } (0,0)$$

So at $(0,0)$ $\det(I^{-1} II) = k_1 k_2$

$$\operatorname{trace}(I^{-1} II) = k_1 + k_2$$

② rotating and translating a surface <u>cannot</u> change I or II

- entries are dot products of vectors that also rotate; translation doesn't change the vectors

③ What about change of parametrization? we could think about

$x(s(u,v), t(u,v))$

New parameters — but parametrizing THE SAME geometry.

$$\begin{bmatrix} \underline{x}_u^T \\ \underline{x}_v^T \end{bmatrix} = \begin{bmatrix} s_u & t_u \\ s_v & t_v \end{bmatrix} \begin{bmatrix} x_s^T \\ \underline{x}_t^T \end{bmatrix}$$

I haven't been careful about rows + cols till now, because there wasn't a need; but vectors are col. vectors, and this is 2 × 3

Derivative of repar map, transp

write $J = \begin{bmatrix} s_u & s_v \\ t_u & t_v \end{bmatrix}$

So $$I^{(u,v)} = \begin{bmatrix} \underline{x}_u^T \\ \underline{x}_v^T \end{bmatrix} \begin{bmatrix} \underline{x}_u & \underline{x}_v \end{bmatrix} = J^T I^{(st)} J$$

$$II^{(u,v)} = \begin{bmatrix} N_u^T \\ N_v^T \end{bmatrix} \begin{bmatrix} \underline{x}_u & \underline{x}_v \end{bmatrix} = J^T II^{(st)} J$$

So $$\left[I^{(u,v)}\right]^{-1}\left[II^{(u,v)}\right] = J^{-1}\left[I^{(st)}\right]^{-1} II^{(st)} \cdot J$$

now $\det(AB) = \det(A)\det(B)$ and $\text{trace}(ABC) = \text{trace}(BCA)$

So $$\det\left(-\left[I^{(u,v)}\right]^{-1} II^{(u,v)}\right) = \det\left(-\left[I^{(st)}\right]^{-1} II^{(st)}\right)$$

and $$\text{trace}\left(-I^{-1} II\right)^{(st)} = \text{trace}\left(-I^{-1} II\right)^{(u,v)}$$

NOTE: I have used a standard, superpowerful, geometric argument.

$$\boxed{\text{· Prove something in an easy coordinate system, then show that change of coords doesn't matter.}}$$

NOTE: We think of K, H as local geometric properties of surfaces BECAUSE they're invariant to rigid motion and reparametrization

- Sometimes, we are interested in other groups (rigid motion + scale; affine tx; projective tx).
- All the above has analogous, much fiddlier, constructions for these cases
  - can be looked up, or worked out; not usually worth it.

Some notation:

it is traditional to write

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x_s \cdot x_s & x_s \cdot x_t \\ x_s \cdot x_t & x_t \cdot x_t \end{pmatrix} \quad \mathrm{I}$$

and

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} N \cdot x_{ss} & N \cdot x_{st} \\ N \cdot x_{st} & N \cdot x_{tt} \end{pmatrix} \quad \mathrm{II}$$

now we know that

$$\begin{pmatrix} \underline{N}_s^T \\ \underline{N}_t^T \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} = a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \underline{x}_s^T \\ \underline{x}_t^T \end{pmatrix} \quad \text{for some } a_{ij}$$

(because $N_s \in T_p$, etc) but we don't know a_{ij}

$\rightarrow$ easy to get

$$\begin{bmatrix} \underline{N}_s^T \\ \underline{N}_t^T \end{bmatrix} [\underline{x}_s \ \underline{x}_t] = -\mathrm{II} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \cdot \mathrm{I}$$

so $$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = -\mathit{II}\;\mathit{I}^{-1}$$

and so $$K = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \quad H = \mathrm{tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

Now $\{x_u, x_v, N\}$ is a basis for <u>any</u> vector at p (not just tangent.

Recall the situation w/ curves — we had a frame, and much geometry was revealed by what happened if we took a small step

$$\underline{x}_{ss} = \Gamma^1_{11}\,\underline{x}_s + \Gamma^2_{11}\,\underline{x}_t + L_1\,\underline{N}$$

$$\underline{x}_{st} = \Gamma^1_{12}\,\underline{x}_s + \Gamma^2_{12}\,x_t + L_2\,\underline{N} \qquad (= x_{ts})$$

$$\underline{x}_{tt} = \Gamma^1_{22}\,\underline{x}_s + \Gamma^2_{22}\,\underline{x}_t + L_3\,\underline{N}$$

$$\underline{N}_s = a_{11}\,\underline{x}_s + a_{12}\,\underline{x}_t$$

$$\underline{N}_t = a_{12}\,\underline{x}_s + a_{22}\,\underline{x}_t$$

above is basically just notation ($\{x_u, x_v, N\}$ is a complete basis, so there must be an expansion)

We can fill in some detail.

$$x_{ss} \cdot N = L_1 (N \cdot N) = L_1 = e$$

Similarly $\cancel{x_{st}}$ $L_2 = f$, $L_3 = g$

which basis vector

$$\Gamma^{i}_{jk} \leftarrow \text{Christoffel symbols}$$

which derivative

<u>Notice</u> $E_s = 2(x_{ss} \cdot x_s)$ $\quad E_t = 2(x_{st} \cdot x_s)$

$$F_s = (x_{ss} \cdot x_t) + (x_s \cdot x_{st})$$

$$F_t = (x_{st} \cdot x_t) + (x_s \cdot x_{st})$$

$G_s = 2(x_{st} \cdot x_t)$ $\quad G_t = 2(x_{tt} \cdot x_t)$

linear algebra yields

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_s \\ F_s - \frac{1}{2} E_t \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_t \\ \frac{1}{2} G_s \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} F_t - \frac{1}{2} G_s \\ \frac{1}{2} G_t \end{pmatrix}$$

(you can look these up, or derive them; the form isn't that significant)

SIGNIFICANT: Christoffel symbols can be recovered from I and its derivatives. — no use of the embedding.

Some more on Christoffel symbols

notice we have

$$(x_{uu} \cdot x_u) = \Gamma_{11}^{1} \cdot (x_u \cdot x_u) + \Gamma_{11}^{2} (x_v \cdot x_u)$$

etc.

now we need to simplify notation. I'll write ~~x_1 for~~ g_{11} for $(x_u \cdot x_u)$ etc.

then

$$\begin{pmatrix} (x_{uu} \cdot x_u) \\ (x_{uu} \cdot x_v) \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{pmatrix}$$

but this has full rank.

now notice $\frac{\partial}{\partial u}(x_u \cdot x_v) = (x_{uu} \cdot x_v) + (x_{uv} \cdot x_u)$

etc

now this will allow me to extract an expression for Γ's.

<u>let</u> $\partial_1 \equiv \frac{\partial}{\partial u}$; $\partial_2 \equiv \frac{\partial}{\partial v}$

(<u>Note</u>: we are looking forward to when there are many params.

then $(x_{uu} \cdot x_u) = \frac{1}{2}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right)$

$$(x_{uv} \cdot x_u) = \frac{1}{2}\left(\partial_1 g_{21} + \partial_2 g_{11} - \partial_1 g_{12}\right)$$

just write it out

This brings us to:

$$\boxed{\sum_i g_{ri} \Gamma^i_{jk} = \frac{1}{2}\left(\partial_j g_{ke} + \partial_k g_{ej} - \partial_e g_{jk}\right)}$$

<u>So</u>: Christoffel symbols depend on metric, its derivatives

Notice something here:

- We can now differentiate vector fields on surfaces in a meaningful way

- You may not have noticed, but previously we couldn't — derivative wasn't necessarily on surface, which often doesn't make sense.

    - e.g. a ball is moving on a surface w/ velocity $\underline{v}$; what acceleration occurs w.r.t. the surface?

A: compute acceleration in the usual way THEN project to ~~not~~ tangent plane.

So consider

$$V = a(u,v)\,\underline{x}_u + b(u,v)\,\underline{x}_v$$

↑ vector field on surface.

write X for some Tangent vector at a point (or T. vec. field)

$$\left[\begin{array}{l}\text{Directional derivative of } V \\ \quad \text{in Dir } X\end{array}\right] \leftarrow$$

may not be tangent

$\boxed{\text{Example next page}}$

$$\parallel$$

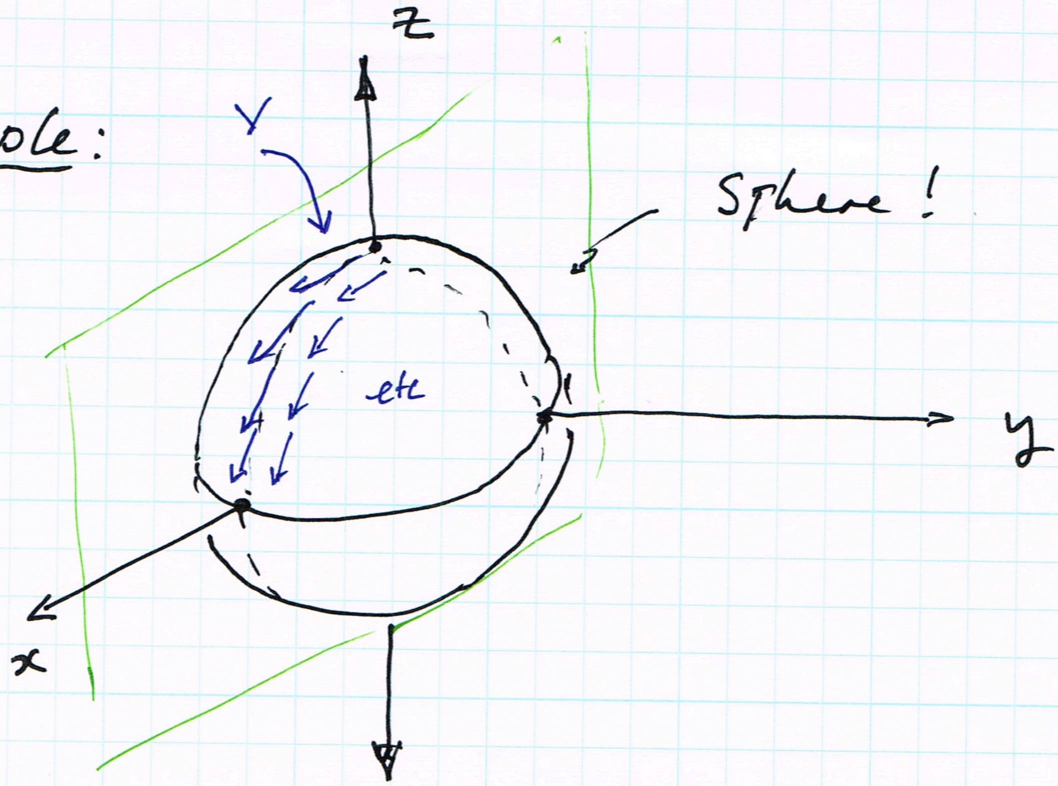
$$D_X V$$

<u>But</u>

$$\nabla_X V = \left[\begin{array}{l}\text{take directional derivative then} \\ \text{PROJECT to tangent space}\end{array}\right]$$

<u>is</u> Tangent.

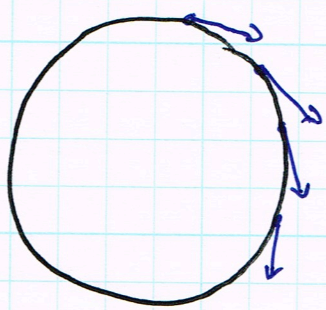
Example:



V is tangent, ~~also tangent~~ also rot. sym. around y.

Now think about $D_V V$

V is plane, so can draw section (green plane)



$D_V V$ cant be tangent
– V is swinging in Normal dir.

We should now start thinking of vectors as <u>differential operators</u> on <u>surf's</u>

e.g. $X = m\underline{x}_u + n\underline{x}_v$

$$\left[\begin{array}{c}\text{Directional Derivative of} \\ f \text{ in } X \text{ dir.n}\end{array}\right] = Xf$$

$$= \lim_{\varepsilon \to 0}\left(\frac{f(\underline{x}+\varepsilon X) - f(\underline{x})}{\varepsilon}\right) = m\sum_i \frac{\partial f}{\partial x_i}\cdot\frac{\partial x_i}{\partial u} + n\sum_i \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial v}$$

$$= m\frac{\partial f}{\partial u} + n\frac{\partial f}{\partial v}$$

$$= \left(m\frac{\partial}{\partial u} + n\frac{\partial}{\partial v}\right)\cdot f$$

→ it is often convenient to write basis vectors

$\underline{x}_u$, $\underline{x}_v$ as $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$

Now I want to differentiate vector fields

$$\nabla_X V = \begin{bmatrix} \text{Directional Derivative of} \\ V\text{, proj onto } \cancel{\text{surf}} \text{ tangent plane} \end{bmatrix}$$

$$= \Pi \underbrace{\left[m \frac{\partial}{\partial u} \left[a \underline{x}_u + b \underline{x}_v \right] + n \frac{\partial}{\partial v} \left[a \underline{x}_u + b \underline{x}_v \right] \right]}$$

Π = project to surface tangent plane

$$= \begin{bmatrix} m a_u + m a \Gamma^1_{11} + m b \Gamma^1_{12} \\ + n a_v + n a \Gamma^1_{12} + n b \Gamma^1_{22} \end{bmatrix} \underline{x}_u +$$

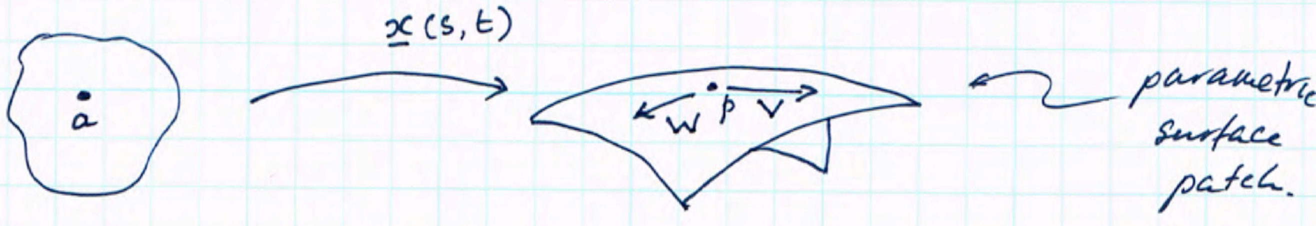
$$\begin{bmatrix} m b_u + m a \Gamma^2_{11} + m b \Gamma^2_{12} \\ + n b_v + n a \Gamma^2_{12} + n b \Gamma^2_{22} \end{bmatrix} \underline{x}_v$$

We can cook up a cleaner notation for this mess, but for the moment, nice to know can do; operation is called:

$$\boxed{\text{COVARIANT DERIVATIVE}}$$

I as a metric:

- recall what I does



U = parametric domain
(s,t)

- Given tangent vectors V, W on surface at p, I can measure lengths, angles ~~at~~ by

$$\text{length}(V) = (V\cdot V)^{1/2}, \quad \text{etc}, \text{ using}$$

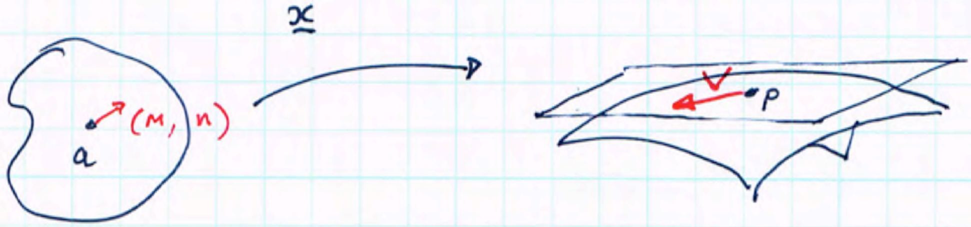
dot product in 3-space

- there is a natural rep'n of V, W at p in terms of basis $\underline{x}_s$, $\underline{x}_t$

$$V = m\,\underline{x}_s + n\,\underline{x}_t \qquad \text{etc.}$$

- Now I could represent V (on T_p) as (m,n) (at a, $\underline{x}: a \to p$, in par domain).

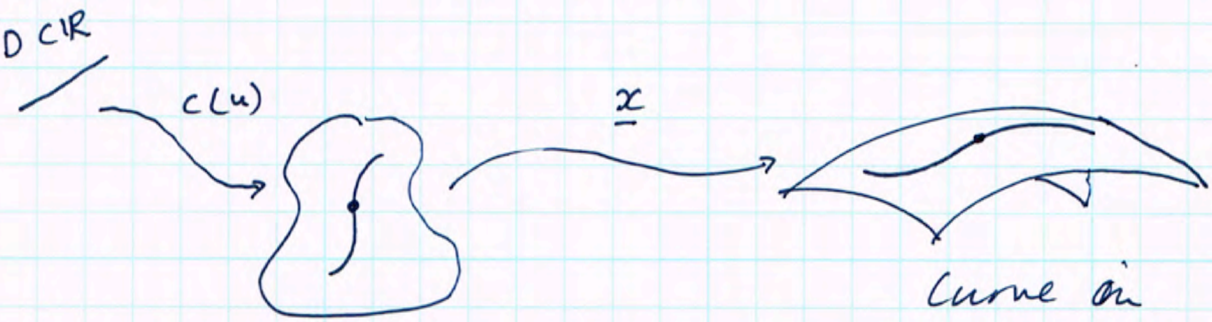
I allows me to measure lengths, angles <u>there</u>



$I\left(\begin{pmatrix} m \\ n \end{pmatrix}, \begin{pmatrix} m \\ n \end{pmatrix}\right) = V \cdot V$

etc for angles

So I can measure the length of a curve on the surface



$U \in \mathbb{R}^2$

Curve in par domain

$s(u), t(u)$

Curve on Surface

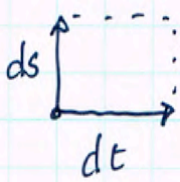
$\underline{x}(s(u), t(u))$.

So length $= \displaystyle\int_D \sqrt{I\left(\begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}, \begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}\right)}\, du$

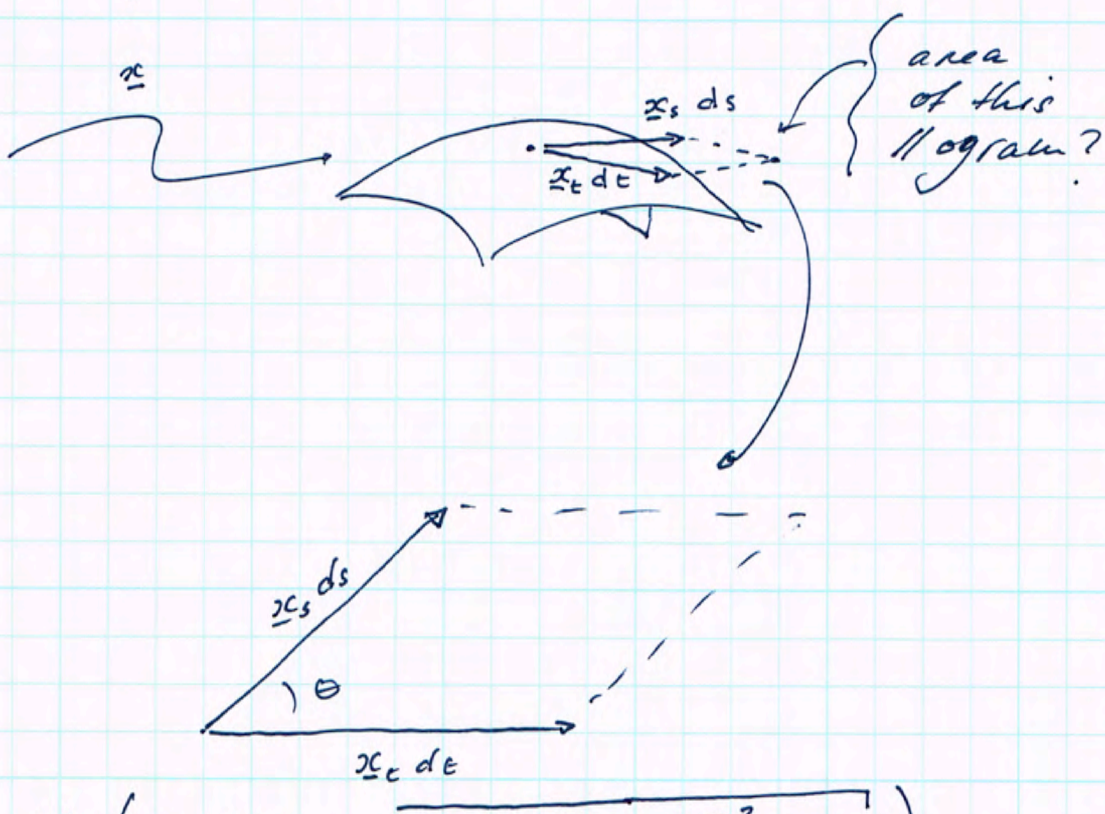
tangent to curve in surface pars

Area:

- I get the area of a patch on a surface by adding up the area of elementary patches
- Divide relevant piece of parametrisation domain into infinitesimal quads ds, dt. and add up. areas



$\mathcal{U}$ = param domain



<u>Area</u>: $\quad (ds\,dt)\left(\|x_s\|\,\|x_t\| \sqrt{1 - \left[\frac{(\underline{x}_s \cdot \underline{x}_t)}{\|x_s\|\,\|x_t\|}\right]^2} \right)$

So

$$\underline{\text{Area}} = ds\,dt \left[\sqrt{\|x_s\|^2 \|x_t\|^2 - (x_s \cdot x_t)^2} \right]$$

$$= ds\,dt \left[\sqrt{EF - G^2} \right]$$

this is $\det I$

So area cut out by $U \subset$ domain

$$\int_U \det(I)\, ds\, dt$$

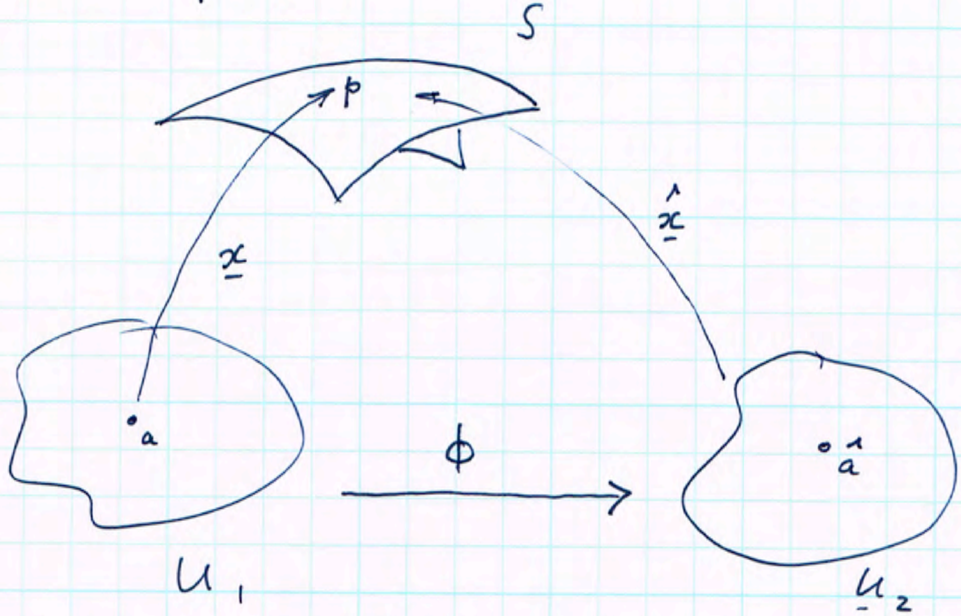
(not usually easy!)

Notice that in these calculations, we used I, $\underline{\text{not}}$ $\underline{x}$.

- i.e. if I have a I over a domain, I can complete length, area w/o knowing $\underline{x}$

Reparametrizing a Surface

· the same surface can have many different parametrizations



· consider two parametrizations, $\underline{x}$, $\hat{\underline{x}}$ of a surface S; there must be some ϕ, 1-1 linking them. In this picture

$$\hat{\underline{x}} \circ \phi = \underline{x}$$

Now since the parametrization can't affect lengths, angles on surf, we must have

$$\overset{\leftarrow \text{ for } \underline{x}, \underline{\hat{x}} \text{ case } \rightarrow}{} \qquad \text{Derivative of } \phi \text{ at } a.$$

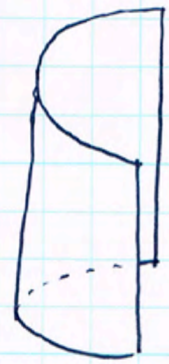
$$\underbrace{I_a^{(x)}}_{\text{location}}(u,v) = \underbrace{I_{\phi(a)}^{(\hat{x})}}_{\text{corresponding location for } \hat{x}\text{, found using } \phi}(d\phi(u), d\phi(v))$$

- Notice that if $\underline{x}, \underline{\hat{x}}$ are parametrizations of the same surface S, then some ϕ with this property must exist.

- Now consider $\psi: S_1 \longrightarrow S_2$ (maps surfaces to surfaces).

- if ψ does not change lengths (and so angles) it is an <u>isometry</u>

- ~~Not~~ <u>Example</u>: Rotation + Translation



roll up
<u>without</u> stretch

$= S_2$

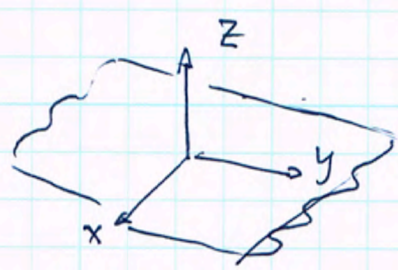
flat sheet $= S_1$

· There is no stretch

∴ all lengths on surface are unchanged
(it follows all angles are unchanged, too)

∴ an isometry <u>must</u> exist

<u>in coordinates:</u>



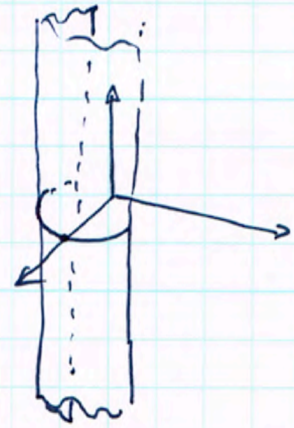
$$S_1 = [s, t, 0]$$

$$s \in [-1, 1]$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{at } \underline{\text{all points}}$$

$S_2 = [\cos, \sin s, t]$

$s \in [-1, 1]$



$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ at <u>all points</u>

So an <u>isometry</u> must exist

~~(ϕ = identity works)~~.

Q: when does an isometry exist?

Q: Geometric properties invariant under isometry are referred to as

<u>intrinsic</u>

<u>Clearly</u>, H <u>isn't</u> intrinsic.

<u>Lemma:</u> assume Ψ ~~ϕ~~ ~~G~~ is an isometry.

Ψ ~~G~~ $: S_1 \to S_2$. $\quad \underline{x} : (U_1 \subset \mathbb{R}^2) \longrightarrow S_1$

$\hat{\underline{x}} : (U_2 \subset \mathbb{R}^2) \longrightarrow S_2$

then ~~on~~ there is a parametrization of S_2 such that

$$\hat{E} = E \qquad \hat{F} = F \qquad \hat{G} = G$$

coord functions of I on S_1 for $\underline{x}$

coord functions of I on S_2 for new param.

<u>Proof:</u>

$$\begin{tikzcd} S_1 \arrow[r, "\Psi"] & S_2 \\ U_1 \arrow[u, "\underline{x}"] & U_2 \arrow[u, "\hat{\underline{x}}"'] \end{tikzcd}$$

Now parametrize S_2 by $\Psi \circ S_1$

→ lengths, angles are the same

→ ~~same~~ a in U_1 refers to CSP pts in S_1, S_2

→ $\hat{E} = E$, etc.

Important consequence:

- Any property that can be expressed in terms of E, F, G (and their derivatives) is intrinsic

Theorema egregium: Gaussian curvature is intrinsic

Proof: (not super enlightening – see scanned pages) Manipulate formula for K to produce expression in E, F, G and derivatives.

Corollary: Isometric surfaces have the same Gaussian curvature at csp points

Devious proof of T.E.

- I claim that K is a fn of g_{ij} and derivatives only
- rotation and translation cannot change this property
- we have seen K is invar. to reparam.

$\Rightarrow$ OK to work in "good" frame w/ "good" param

- at Point of interest, $N = (0,0,1)$, $\underline{x} = (0,0,0)$

Surf is $\underline{x}(u,v) = (u, v, f(u,v))$

$$\therefore \; g = \begin{pmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{pmatrix} \qquad \text{AND } f_x, f_y = 0 \text{ at } u$$

<u>NOW</u> we <u>can't</u> write $\boxed{\sqrt{g_{11} - 1} = f_u}$

WRONG

because at $0,0$, $f_x = 0$ and we don't know how the sign changes

but some differentiation, etc
gets

$$\begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix} \begin{pmatrix} f_u & f_v \\ f_v & f_u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(g_{11u} + g_{22u}) & g_{12u} \\ \frac{1}{2}(g_{11v} + g_{22v}) & g_{12v} \end{pmatrix}$$

__Now__ in __this__ C-sys, param, we have

$$K\Big|_{0,0} = \det\begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}\Bigg|_{0,0}$$

(which you should check!)

$$= \left. \frac{\det\begin{pmatrix} \frac{1}{2}(g_{11u} + g_{22u}) & g_{12u} \\ \frac{1}{2}(g_{11v} + g_{22v}) & g_{12v} \end{pmatrix}}{f_u^2 - f_v^2} \right|_{0,0}$$

$$f_u^2 - f_v^2 \longrightarrow g_{11} - g_{22}$$

So we are __done__ !

We now have two threads to look at:

- We can abstract away embeddings and study only intrinsic properties (given by E, F, G). To do this, we think about I as a function on 2D, and don't really worry about embedding. This leads to Riemannian geometry; we'll do some of this.

- We can look at extrinsic (+ intrinsic, on occasion) properties, where the embedding matters (often a lot). We'll do lots of this, next.