Local (Differential) Geometry of Surfaces:

Choose a point on a sur.
- Compute normal
- Build family of planes through normal
- Consider these X-sections of surf
- 3 cases

Elliptic
Parabolic
Hyperbolic
A finer classification would be

Both Elliptic

We get this from the Gauss map.

\[ x \]
\[ p.t. \text{ on surface} \rightarrow p.t. \text{ on sphere given by normal} \]
map a small circle round P to sphere

- case I: small circle
  small \rightarrow \text{big}

- case II: small \rightarrow \text{big}
Notice direction reverses

I: small → small

II: small → big
small $\rightarrow$ area Zero.

\textbf{Defn}

\[ K = \text{Gaussian curvature} \]

\[ = \lim_{\text{radius} \to 0} \left\{ \frac{\text{Area of Gauss map}}{\text{Area on surf}} \right\} \]

\[ K = \begin{cases} < 0 & \text{Hyperbolic} \\ 0 & \text{Parabolic} \\ > 0 & \text{Elliptic} \end{cases} \]
Bending does not change \( K \) - you must \{ add \} area.

(So there must be another description to add detail.)

1. Take a point on a surface.
2. Construct a coord system \((x, y)\) in tangent plane, \( z \) is normal.

IN THIS COORD SYSTEM, near this pt, write Taylor Series.
Surface is

\[(x, y, z(x, y))\]

\[z(x, y, z_0 + (\nabla z) \cdot (x, y) + \frac{1}{2} (x^T H(x) y) + O(x^3)\]

but \(z_0 = 0\)

\[\nabla z = 0\]

So \((x, y, z(x, y)) = (x, y, \frac{1}{2} (y^T H(x) y) + O(x^3))\)

\(\hline\)

- This is a quadratic form
- Symmetric
- Rotate coord sys

\((u, v, z(u,v)) = (u, v, \frac{1}{2} (k_1 u^2 + k_2 v^2) + O(x))\)
Now recall a curve

\[ (u, \frac{1}{2} au^3) \]

has curvature \( a \) at \( u = 0 \).

So the curvature of the \( u \) section is \( K_1 \),

\[ v \]

is \( K_2 \).

\[ s = u \cos \theta + v \sin \theta \]

\[ s = \left( \begin{array}{c} \cos \theta \\ \sin \theta \end{array} \right) \]

is \( K_1 \cos^2 \theta + K_2 \sin^2 \theta \).

Therefore, the directional curvature has

maximum \( \max(K_1, K_2) \)

minimum \( \min(K_1, K_2) \)
at each point, there are two directions in which the directional curvature is extremal.

principal directions curvatures

for a surface
\[ (s, t, \frac{1}{2}(K_1 s^2 + K_2 t^2) + O(3)) \]

Compute tangents:

- think of surface as map from a piece of \( \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)
  
  \[ (s, t) \rightarrow \mathbf{x}(s, t) \]

- Then \( \frac{\partial \mathbf{x}}{\partial s} \) must be tangent, by the same argument as for curves

  \[ \frac{\partial \mathbf{x}}{\partial t} \]

- and \( N \) is unit vector normal to tangents
So

\[ T_1 : (1, 0, k_1 s) \left( 1 + O(2) \right) \]

\[ T_2 : (0, 1, k_2 t) \left( 1 + O(2) \right) \]

\[ N : (-k_1 s, -k_2 t, 1) \left( 1 + O(2) \right) \]

A small step away from \((0, 0)\) to \((\Delta u, \Delta v)\) in the tangent plane causes the normal to swing to \((-k_1 \Delta u, -k_2 \Delta v, 1)\)

Notice: \(k_1 < 0\), so first component increases, \(k_2 > 0\), so second component decreases.
Now consider a "box" $$(0,0) \rightarrow (\frac{3}{2}, 0) \rightarrow (\frac{3}{2}, \frac{3}{2}) \rightarrow (0, \frac{3}{2}) \rightarrow (0,0)$$

To first order, 3rd normal component doesn't change.

On Gauss map, we get "box" 
$$(0,0,1) \rightarrow (0, -k_3, 1) \rightarrow (0, -k_2, 1) \rightarrow (0, -k_2, 1) \rightarrow (0,0,1)$$

This ratio of areas is Gaussian curvature $= k_1 k_2$

Notice that rotating the coordinate system will get us non-zero st terms in the quadratic form—this expression applies only in the right coordinate system.
But consider a new coordinate system in tangent plane

\[
\begin{bmatrix}
u \\
v\end{bmatrix} = \begin{bmatrix} R \\ t \end{bmatrix}
\]

then

\[
\begin{bmatrix} s & t \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} s \\
t \end{bmatrix} = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} R \\ k_1 & k_2 \end{bmatrix} \begin{bmatrix} R^T \\ u \end{bmatrix}
\]

\[
\begin{bmatrix} k_1 \\
k_2 \end{bmatrix}
\]

\[
Q \quad RQRT
\]

we say: the action of the rotation on the quadratic form takes \( Q \rightarrow RQRT \)

Notice

\[
det(Q) = det(RQRT) = k_1k_2
\]

- it's invariant to the rotation!
Notice that there is a second invariant

\[ \text{Trace}(Q) = \text{Trace}(R Q R^T) = K_1 + K_2 \]

**Mean curvature** = \( K_1 + K_2 \)

Gaussian curvature has to do with area (demo w/ piece of paper)

Mean curvature with bending

\[ \text{Gaussian} \rightarrow \text{Gaussian} \]
\[ \text{Mean} \rightarrow \text{Mean no-zero} \]

Clearly, there are surfaces of non-zero G.C. and zero M.C.

\[ \Rightarrow \text{here G.C. is always -ve} \]

At this point, we need more powerful machinery - we don't want to constantly reparametrize.
we are interested in surfaces away from singular points

OK

NOT OK

formally

\[ x: U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]

such that \( \frac{Dx}{\det} \) has full rank (\( = 2 \))

Notice that anything smooth can (locally) be reparametrized to be like this
In $\mathbb{R}^3$, we have a metric we're used to working with — we can tell the length of a vector, or the angle between 2 vectors, easily.

Define

$$ x_s = \frac{\partial x}{\partial s} \quad \rightarrow \quad \text{tangent in } s \text{ direction} $$

$$ \frac{\partial x}{\partial t} \quad \rightarrow \quad \text{tangent in } t \text{ direction} $$

These two aren't unit, and they're not necessarily orthogonal either.

They span a tangent plane at each point $p$, often $T_p$.

There's one, usually different, tangent plane at each point.

They're not parallel, because $\partial x$ has full rank.
Now at any point $p$, we can specify any tangent vector by using $x_s, x_t$ as basis elements.

$$\mathbf{v} = a \mathbf{x}_s + b \mathbf{x}_t$$

A tangent vector

and

$$\mathbf{v} \cdot \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{bmatrix} x_s \cdot x_s & x_s \cdot x_t \\ x_s \cdot x_t & x_t \cdot x_t \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and if we have

$$\mathbf{u} = c \mathbf{x}_s + d \mathbf{x}_t$$

$$\mathbf{v} \cdot \mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{bmatrix} x_s \cdot x_s & x_t \cdot x_s \\ x_s \cdot x_t & x_t \cdot x_t \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

This object allows us to measure lengths and angles in the given parametrization — change param, and then change.

Quadratic form — The first fundamental form
Often write \( I(u, v) \)

\[
\begin{align*}
\text{(a)} & \left[ \begin{array}{c} x_s \\ x_t \\
\end{array} \right] \\
\text{(b)} & \left[ \begin{array}{ccc} x_s \cdot x_s & x_s \cdot x_t & 1 \\
\end{array} \right]
\end{align*}
\]

\( \text{(c)} \)

\( \text{(d)} \)

We can now consider

\( N = \text{the unit normal}. \)

\[
= \frac{x_s \times x_t}{\| x_s \times x_t \|}
\]

Defined for our patch, because \( \partial x \) has full rank.

Because \( N \) is unit normal, we have

\( N \cdot N = 1 \), \( N \cdot x_s = 0 \), \( N \cdot x_t = 0 \)

Now consider moving away from \( p \) by a small amount, a step in the tangent plane at \( p \).

Represent as \( \varepsilon (a \ x_s + b \ x_t) \)
If we think of $N$ as a map from 2D to 2D (this is the Gauss map), it must have a derivative $dN$. Which is a linear map from plane to plane (tangent plane to surf) to tangent plane to sphere.
This may worry you - why doesn't the normal swing "in 3D"?

\[ N \cdot N = 1 \quad \text{so} \quad N_5 \cdot N = 0, \quad N_6 \cdot N = 0 \]

and the derivative in some new dirn \((u, v)\)
\[ u \text{ has } \frac{\partial u}{\partial s} = a \frac{\partial}{\partial s} + b \frac{\partial}{\partial t} \quad \text{so} \quad N_u \cdot N = 0 \]

- the normal only moves on \(T_p\) for infinitesimal steps.

- particularly interesting is

\[ -I(dN(u), v) = II(u, v) \]

**First Fundamental Form**

**Tangent Vector**

**Another Tangent Vector**

**Second Fundamental Form**
This is easy to work out in coords:

let \( V = v_0 x_s + v_1 x_t \)

\( U = u_0 x_s + u_1 x_t \)

\( dN(u) = u_0 x_s N_s + u_1 x_t N_t \)

\[
I(\,dNC(u), V) = (u_0 \quad u_1) \begin{pmatrix}
N_s \cdot x_s & N_s \cdot x_t \\
N_t \cdot x_s & N_t \cdot x_t
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]

This might worry you, because it looks like it isn't symmetric

\[ \Rightarrow \text{untidy} \]

\[ \Rightarrow \text{you have to remember order of arguments} \]

**But**

\[ N \cdot x_s = 0 \]

\[ \therefore N_t \cdot x_s = -N \cdot x_{st} = N_s \cdot x_t \]

\[ N_s \cdot x_s = -N \cdot x_{ss}; \quad N_t \cdot x_t = -N \cdot x_{tt} \]
So, \( \mathbb{I} \) is symmetric

**Key result**

\[
K = \text{Gaussian curvature} = \det(-I' \mathbb{I}) \\
H = \text{Mean curvature} = \text{trace}(-I' \mathbb{I})
\]

which we can establish a bunch of different ways.

**Advantage:** don't need to compute Taylor series at some location.
Elegant way to see that $K$ and $H$ are as given.

1. for our surfaces

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(t) \quad \text{at } (0,0) \]
\[ II = \begin{pmatrix} -k & 0 \\ 0 & k_2 \end{pmatrix} + O(t) \quad \text{at } (0,0) \]

So at $(0,0)$ \( \det(I^{-1}II) = K_1K_2 \)

\[ \text{trace } (I^{-1}II) = K_1 + K_2 \]

2. rotating and translating a surface cannot change I or II

- entries are dot products of vectors that also rotate; translation doesn't change the vectors

3. What about change of parametrization?

we could think about \( x(s(u,v), t(u,v)) \)

\[ \text{New parameters} \quad \text{but parametrizing THE SAME geometry.} \]
\[
\begin{bmatrix}
\frac{\partial c}{\partial u} \\
\frac{\partial c}{\partial v} \\
\end{bmatrix} =
\begin{bmatrix}
S_u & t_u \\
S_v & t_v \\
\end{bmatrix}
\begin{bmatrix}
\frac{x_s}{x_t} \\
\frac{x_s}{x_t} \\
\end{bmatrix}
\]

I haven't been careful about rows + cols till now, because there wasn't a need, but vectors are \(\text{col. vecs}\), and this is \(2 \times 3\).

So
\[
I^{(u,v)} = [x_u^T] [x_u x_v] = \mathbf{J}^T I^{(cst)} \mathbf{J}
\]

\[
\Pi^{(u,v)} = [N_u^T] [x_u x_v] = \mathbf{J}^T \Pi^{(cst)} \mathbf{J}
\]

So
\[
[I^{(u,v)}]^{-1} \Pi^{(u,v)} = \mathbf{J}^{-1} [I^{(cst)}]^{-1} \Pi^{(cst)} \mathbf{J}^{-1}
\]

now \(\det(AB) = \det(A) \det(B)\) and \(\text{trace}(ABC)\)

so
\[
\det([-I^{-1} \Pi^{(cst)}]) = \det([I^{(cst)}]^{-1} \Pi^{(cst)})
\]

and \(\text{trace}(-I^{-1} \Pi^{(cst)}) = \text{trace}(-I^{-1} \Pi^{(cst)})\)
Note: I have used a standard, superpowerful, geometric argument.

- Prove something in an easy coordinate system, then show that change of coords doesn't matter.

Note: We think of $k, h$ as local geometric properties of surfaces because they're invariant to rigid motion and reparametrization.

- Sometimes, we are interested in other groups (rigid motion + scale; affine + $\times$; projective $\times$).

- All the above has analogous, much fiddlier, constructions for these cases. Can be looked up, or worked out; not usually worth it.
Some notation:

It is traditional to write

\[
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix} = \begin{pmatrix}
x_s \cdot x_s & x_s \cdot x_t \\
x_s \cdot x_t & x_t \cdot x_t
\end{pmatrix}
\]

I

and

\[
\begin{pmatrix}
e & f \\
f & g
\end{pmatrix} = \begin{pmatrix}
N \cdot x_s & N \cdot x_t \\
N \cdot x_t & N \cdot x_t
\end{pmatrix}
\]

II

Now we know that

\[
\begin{pmatrix}
N_s^T \\
N_t^T
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{pmatrix}\begin{pmatrix}
x_s^T \\
x_t^T
\end{pmatrix}
\]

for some \( a_{ij} \)

(because \( N_s \in T_p, \) etc.) but we don't know \( a_{ij} \)

\( \rightarrow \) easy to get

\[
\begin{pmatrix}
N_s^T \\
N_t^T
\end{pmatrix} \begin{pmatrix}
x_s & x_t
\end{pmatrix} = -I \begin{pmatrix}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{pmatrix} \cdot I
\]
so \[ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = -II \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

and so \( K = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \), \( H = \operatorname{tr} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \)

Now \( \{x_u, x_v, N \} \) is a basis for any vector at \( p \) (not just tangent).

Recall the situation with curves — we had a frame \( p \), and much geometry was revealed by what happened if we took a small step

\[
\begin{align*}
\frac{x_{ss}}{s^2} &= \Gamma_{11}^1 x_s + \Gamma_{11}^2 x_t + L_1 N \\
\frac{x_{st}}{s} &= \Gamma_{12}^1 x_s + \Gamma_{12}^2 x_t + L_2 N \\
\frac{x_{tt}}{t} &= \Gamma_{22}^1 x_s + \Gamma_{22}^2 x_t + L_3 N \\
N_s &= a_{11} x_s + a_{12} x_t \\
N_t &= a_{12} x_s + a_{22} x_t
\end{align*}
\]
above is basically just notation
(\mathbf{E}_u, x_u, N) is a complete basis, so there must be an expansion)

We can fill in some detail.

\[ x_{\mathbf{E}_u} \cdot N = \lambda_1, \quad (N \cdot N) = \lambda_1 = e \]

Similarly \[ x_{\mathbf{E}_2} \cdot f = \lambda_2 = f, \quad \lambda_3 = g \]

\[ \text{Notice} \quad \mathbf{E}_s = 2(x_s \cdot x_s) \quad \mathbf{E}_t = 2(x_t \cdot x_s) \]

\[ F_s = (x_s \cdot x_t) + (x_s \cdot x_{st}) \]

\[ F_t = (x_t \cdot x_s) + (x_s \cdot x_{st}) \]

\[ G_s = 2(x_s \cdot x_t) \quad C_{t_t} = 2(x_t \cdot x_t) \]
Linear algebra yields
\[
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}
\begin{pmatrix}
\Gamma^1_{11} \\
\Gamma^2_{11}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{2} E_{ss} \\
F_{ss} - \frac{1}{2} E_t
\end{pmatrix}
\]
\[
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}
\begin{pmatrix}
\Gamma^1_{12} \\
\Gamma^2_{12}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{2} E_t \\
\frac{1}{2} G_s
\end{pmatrix}
\]
\[
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}
\begin{pmatrix}
\Gamma^1_{22} \\
\Gamma^2_{22}
\end{pmatrix}
= 
\begin{pmatrix}
F_t - \frac{1}{2} G_s \\
\frac{1}{2} G_t
\end{pmatrix}
\]

(you can look these up, or derive them; the form isn't that significant)

**significant**: Christoffel symbols can be recovered from $I$ and its derivatives. No use of the embedding.
I as a metric:

- recall what I do.

\[ x(s, t) \]

\[ U = \text{parametric domain} \]
\[ (s, t) \]

- Given tangent vectors \( V, W \) on surface at \( p \), I can measure lengths, angles by

\[ \text{length}(V) = (V \cdot V)^{1/2}, \text{etc., using} \]

\[ \text{dot product in 3-space} \]

- If there is a natural rep of \( V, W \) at \( p \) in terms of basis \( x_s, x_t \)

\[ V = m x_s + n x_t, \text{etc.} \]

- Now I could represent \( V \) (on \( T_p \)) as

\[ (m, n) \oplus (at \ a, x: a \rightarrow p, \text{in parametric domain}) \]
I allows me to measure lengths, angles there

\[ I((m), (n)) = V \cdot V \]

etc for angles

So I can measure the length of a curve on the surface

\[ \int_D \sqrt{I \left( \begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}, \begin{pmatrix} \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial v} \end{pmatrix} \right)} \, du \]

So length = \[ \int_D \sqrt{I \left( \begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}, \begin{pmatrix} \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial v} \end{pmatrix} \right)} \, du \]

\text{tangent to curve on surface pars}
Area:

- I get the area of a patch on a surface by adding up the area of elementary patches.

- Divide relevant piece of parameterisation domain into infinitesimal quads $ds, dt$ and add up areas.

\[
\text{Area: } (ds \, dt) \left( \frac{1}{\|x_s\| \|x_t\|} \sqrt{1 - \left( \frac{x_s \cdot x_t}{\|x_s\| \|x_t\|} \right)^2} \right)
\]
\[ \text{Area} = \int \det \left[ \begin{array}{cc} \|x_s\| & \|x_t\| \\ (x_s \cdot x_t) & \end{array} \right] \\
= \int \det \left[ \sqrt{EF - C^2} \right] \left( \text{this is } \det I \right) \]

so area cut out by \( U \) \( \in \) domain

\[ \int_U \det(I) \, ds \, dt \] (not usually easy!)

Notice that in these calculations we used \( I \), not \( x \).

i.e., if I have a \( I \) over a domain, I can compute length, area w/o knowing \( x \).
Reparameterizing a Surface

- The same surface can have many different parametrizations

Consider two parametrizations, \( x \), \( \hat{x} \) of a surface \( S \); there must be some \( \phi \), 1-1 linking them. In this picture, \( \hat{x} \circ \phi = x \)

Now since the parametrization can't affect lengths, angles on surf,
We must have for all \( x, \hat{x} \) case:

\[
I_a(x) (u, v) = I_{\phi(a)} (d\phi(u), d\phi(v))
\]

Derivative of \( \phi \) at \( a \).

\( \text{location} \)

\( \text{corresponding location for } \hat{x} \)

\( \text{found using } \phi \)

- Notice that if \( x, \hat{x} \) are parametrizations of the same surface \( S \), then some \( \phi \) with this property must exist.

- Now consider \( \psi \) (maps surfaces to surfaces).

- If \( \psi \) does not change lengths (and so angles) it is an isometry.

- Total Example: Rotation + Translation
There is no stretch.

**all lengths on surface are unchanged**

(if follows all angles are unchanged, too)

**an isometry must exist**

In coordinates:

\[ S_1 = \begin{bmatrix} s & t & 0 \end{bmatrix} \]

\[ s \in [-1, 1] \]

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

at all points
$$s_2 = [\cos, \sin, t] \quad \text{se} [-1, 1]$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{at all points}$$

So an \underline{isometry} must exist \( \phi = \text{identity works} \).

Q: \underline{When does an isometry exist?}

\[ \Box \] Geometric properties \( \phi \) invariant under isometry are referred to as \underline{intrinsic}

Clearly, \( H \) isn't intrinsic.
Lemma: Assume \( \Psi \) is an isometry.

\[ \Psi : S_1 \rightarrow S_2, \quad \chi : (u_1 \mathbb{R}^2) \rightarrow S_1, \quad \hat{\chi} : (u_2 \mathbb{R}^2) \rightarrow S_2 \]

Then there is a parametrization of \( S_2 \) such that

Proof:

Now parametrize \( S_2 \) by \( \Psi \circ S_1 \);

\[ \rightarrow \text{lengths, angles are the same} \]
\[ \rightarrow \text{same a in } U_1 \text{ refers to CSP pts in } S_1, S_2 \]
\[ \rightarrow \hat{E} = E, \text{ etc.} \]
Important consequence:

- Any property that can be expressed in terms of $E, F, G$ (and their derivatives) is intrinsic.

**Theorema Egregium**: Gaussian curvature is intrinsic.

**Proof**: (not super enlightening — see scanned pages) manipulate formula for $K$ to produce expression in $E, F, G$ and derivatives.

**Corollary**: Isometric surfaces have the same Gaussian curvature at corresponding points.
We now have two threads to look at:

- We can abstract away embeddings and study only intrinsic properties (given by $E,F,G$). To do this, we think about $I$ as a function on $2D$, and don't really worry about embedding. This leads to Riemannian geometry; we'll do some of this.

- We can look at extrinsic (or intrinsic, on occasion) properties, where the embedding matters (often a lot). We'll do lots of this next.