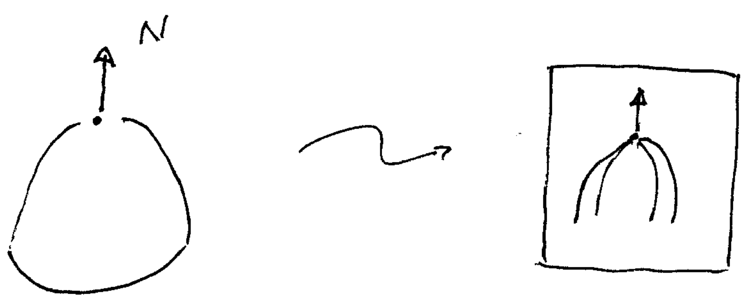
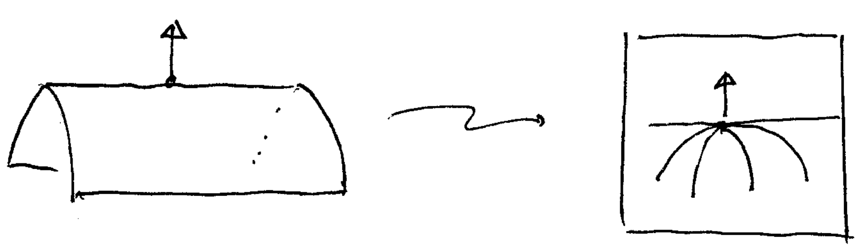
# Local (Differential) Geometry of surfaces:

Choose a point on a surf.

- compute normal
- Build family of planes thru pt, normal
- consider these X-sections of surf
- 3 cases




Elliptic



Parabolic

Hyperbolic

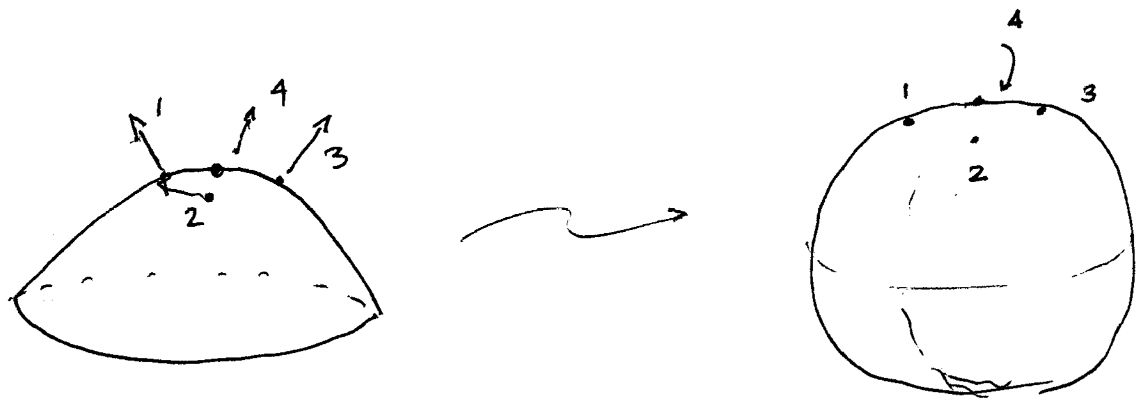
A finer classification would be helpful

Both Elliptic

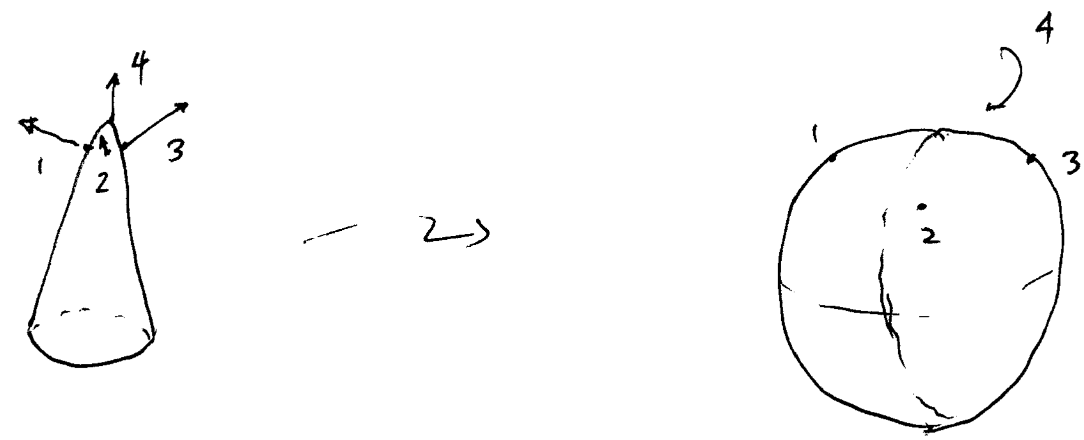
We get this from the **Gauss map**.

$$\underset{\text{p.t. on surface}}{\underline{x}} \longrightarrow \underset{\text{pt on sphere given by normal}}{N(\underline{x})}$$

I

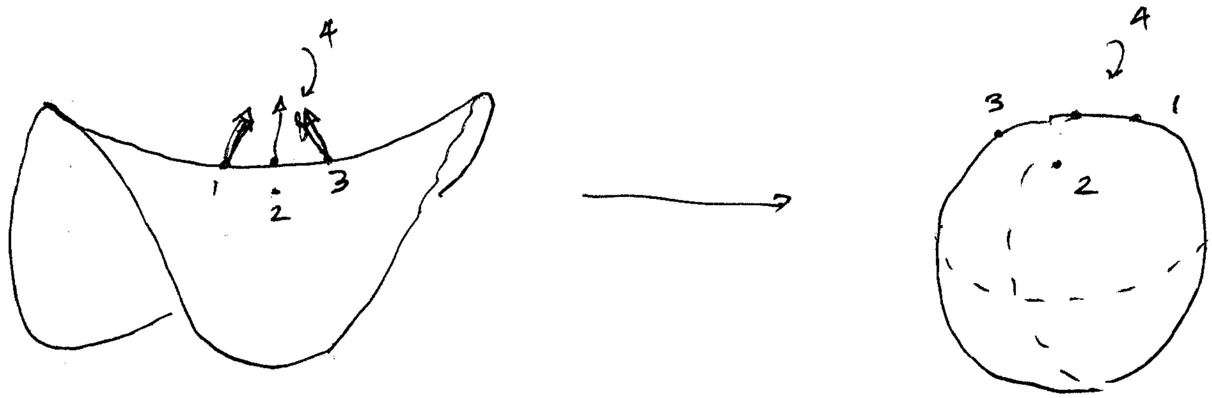



II

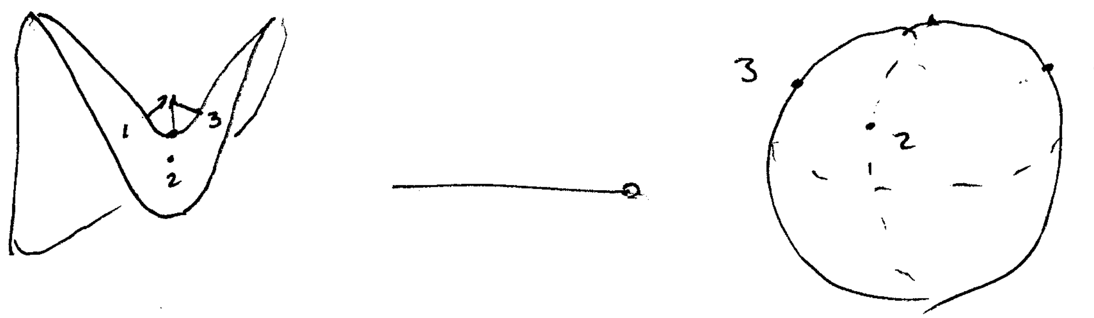



- map a small circle round $p$ to sphere
- case I: small circle
  $\downarrow$
  small "
- case II: small $\longrightarrow$ big.

I



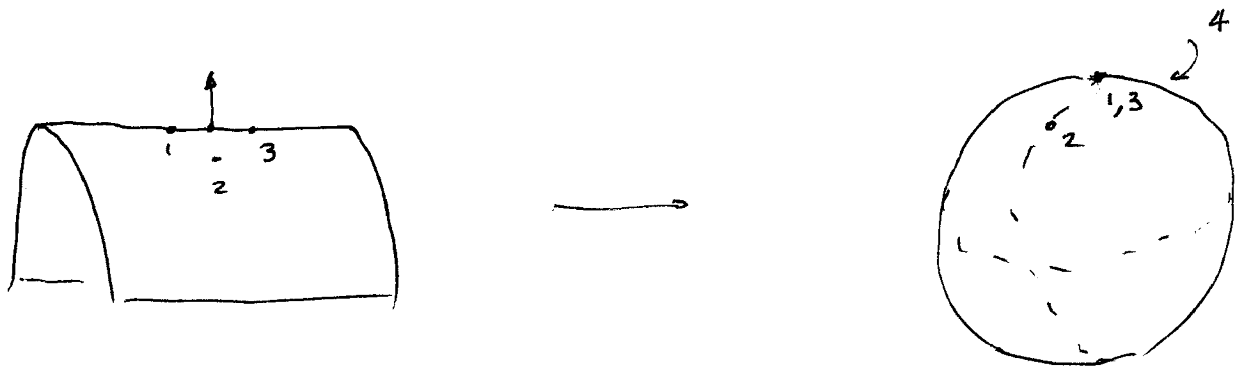

II




- Notice <u>direction reverses</u>

I : small $\longrightarrow$ small

II : small $\longrightarrow$ big




. small $\longrightarrow$ area zero.

<u>Defn</u>

$K$ = Gaussian curvature

$$= \lim_{radius \to 0} \left\{ \frac{\text{Area on Gauss map}}{\text{Area on Surf}} \right\}$$

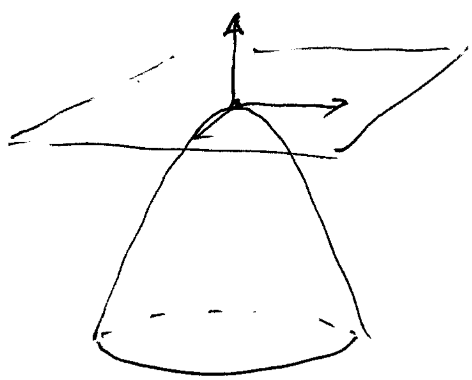
$$K = \begin{cases} < 0 & \text{Hyperbolic} \\ 0 & \text{Parabolic} \\ > 0 & \text{Elliptic} \end{cases}$$

<u>Bending</u> does not change $K$

– You must $\left\{ \begin{array}{l} \text{add} \\ \text{subtract} \end{array} \right\}$ area.

( So there must be another
Description to add detail. )

- Take a point on a surface.
- Construct a coord system ~~in~~
$(x,y)$ in tangent plane, with $z$ normal



- IN THIS COORD SYSTEM, near this pt, write Taylor Series.

Surface is

$$(x, y, z(x,y))$$

$$\approx \left(x, y, z_0 + (\nabla z)\cdot(x,y) + \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix} + O(x,y)^3\right)$$

but

$$z_0 = 0$$
$$\nabla z = 0$$

so

$$(x, y, z(x,y)) = \Big(x, y, \underbrace{\frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}^T H \begin{pmatrix}x\\y\end{pmatrix}} + O(3)\Big)$$

- this is a quadratic form
- symmetric

$\therefore$ rotate coord sys

$$\Big(u, v, z(u,v)\Big) = \Big(u, v, \frac{1}{2}\left(k_1 u^2 + k_2 v^2\right) + O(3)\Big)$$

Now recall a curve

$(u, \frac{1}{2} a u^2)$ has curvature $a$ at $u = 0$

- So the curvature of the $u$ section is $K_1$

$v$ " is $K_2$

~~$s = u\cos\theta + v\sin\theta$~~ "

$s = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ " is $K_1 \cos^2\theta + K_2 \sin^2\theta$

$\therefore$ The directional curvature has

maximum $\max(K_1, K_2)$

min $\min(K_1, K_2)$

at each point, there are two orthonormal directions in which the directional curvature is extremal.

$$\boxed{\begin{array}{l}\text{principal directions} \\ \qquad\quad \text{curvatures}\end{array}}$$

for a surface

$$\left(s, t, \frac{1}{2}\left(\kappa_1 s^2 + \kappa_2 t^2\right) + O(3)\right)$$

Compute tangents:

- think of surface as map from a piece of $\mathbb{R}^2 \to \mathbb{R}^3$
- $(s, t) \longrightarrow \underline{x}(s, t)$.
- Then $\dfrac{\partial \underline{x}}{\partial s}$ must be tangent, by the same argument as for curves

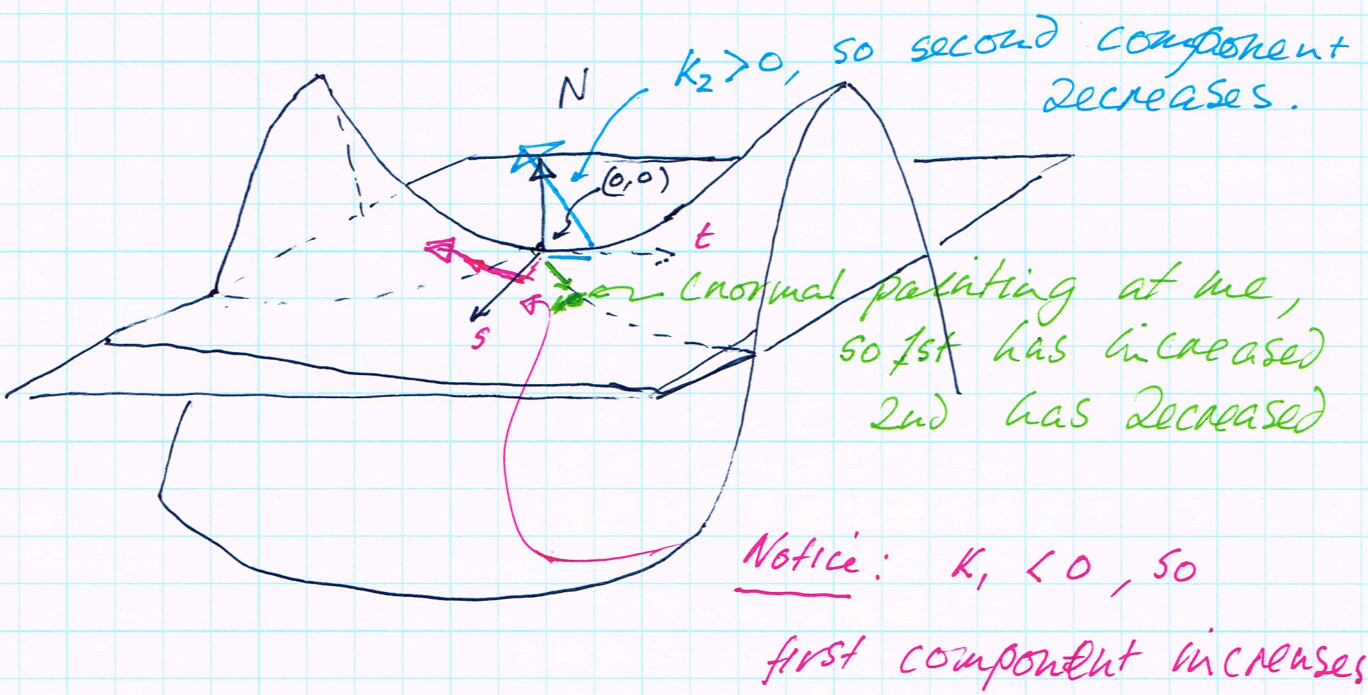
  $\dfrac{\partial \underline{x}}{\partial t}$ "
- and $N$ is unit vector $\perp$ to tangents

so $T_1: \quad (1,\ 0,\ k_1 s)\,(1 + O(2))$

$T_2: \quad (0,\ 1,\ k_2 t)\,(1 + O(2))$

$N: \quad (-k_1 s,\ -k_2 t,\ 1)\,(1 + O(2))$

A small step away from $(0,0)$ to $(\Delta u, \Delta v)$ in the tangent plane causes the normal to <u>swing</u> to $(-k_1 \Delta u,\ -k_2 \Delta v,\ 1)$



- Now consider a "box" $(0,0) \to (\varepsilon, 0) \to (\varepsilon, \varepsilon) \to (0, \varepsilon) \to (0,0)$

- To first order, 3rd normal component doesn't change
  - on gauss map, we get "box"

$$(0,0,1) \to (-\kappa_1\varepsilon, 0, 1) \to (-\kappa_1\varepsilon, -\kappa_2\varepsilon, 1) \to (0, -\kappa_2\varepsilon, 1) \to (0,0,1)$$

- i.e. ratio of areas is

$$\boxed{\text{Gaussian curvature} = \kappa_1 \kappa_2}$$

- Notice that rotating the coordinate system in the tangent plane will get us non-zero $st$ terms in the quadratic form — this expression applies <u>only</u> in the right coordinate system.

<u>But</u> • consider a new coordinate system in tangent plane

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

rotation

then

$$\underbrace{[s \;\; t] \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}_{Q} = \underbrace{[u \;\; v] \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} k_1 & \\ & k_2 \end{bmatrix} \begin{bmatrix} R^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}}_{R Q R^T}$$

<u>we say</u>: the action of the rotation on the quadratic form takes $Q \rightarrow RQR^T$

<u>Notice</u>

$$\det(Q) = \det(RQR^T) = k_1 k_2$$

– it's <u>invariant</u> to the rotation!

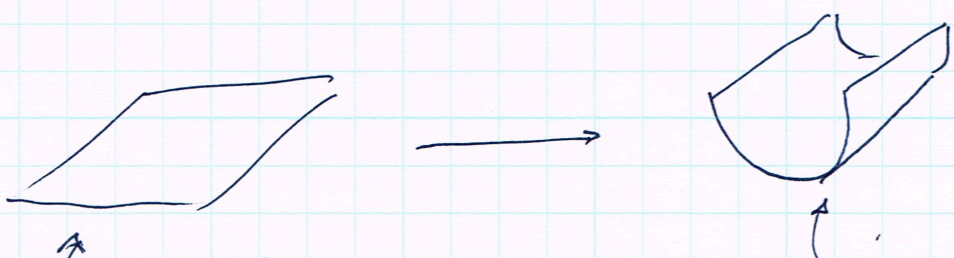
Notice that there is a <u>second</u> invariant

$$\mathrm{Trace}(Q) = \mathrm{Trace}(R Q R^T) = \kappa_1 + \kappa_2$$

$$\boxed{\text{Mean curvature} = \kappa_1 + \kappa_2}$$

Gaussian curvature has to do with <u>area</u>
(demo w/ piece of paper)

Mean curvature with <u>bending</u>



Gaussian 0
Mean 0

Gaussian 0
Mean non-zero.

Clearly, there are surfaces w/ non-zero G.C. and zero M.C.

$\Rightarrow$ here G.C. is <u>always</u> −ve.

At this point, we need more powerful machinery – we don't want to constantly reparametrize.

- we are interested in surfaces <u>away</u> from singular points

<u>OK</u>

<u>NOT OK</u>


<u>formally</u>

$$\underline{x} : U \in \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

- Such that $\underline{D\underline{x}}$ has full rank $(=2)$

[Jacobian of $\underline{x}$
Derivative of $\underline{x}$]

- Notice that anything smooth can (locally) be reparametrized to be like this

- In $\mathbb{R}^3$, we have a metric we're used to working with — we can tell the length of a vector, or the angle between 2 vectors, easily.

- Define

$$x_s = \frac{\partial x}{\partial s} \cancel{\cdot \frac{1}{\left\| \frac{\partial x}{\partial s} \right\|}} \quad \leftarrow \text{ ~~unit~~ tangent in } s \text{ direction}$$

$$x_t = \frac{\partial x}{\partial t} \quad \leftarrow \text{ tangent in } t \text{ direction}$$

→ these two <u>aren't</u> necessarily unit, <u>and</u> they're not necessarily orthogonal either.

→ They span a <u>tangent plane</u> at each point $p$, often $T_p$

→ There's one, usually different, Tangent plane at each point.

→ They're <u>not</u> parallel, because $Dx$ has full rank.

- Now at any point $p$, we can specify any tangent vector by using $\underline{x}_s$, $\underline{x}_t$ as basis elements

$$V = a\,\underline{x}_s + b\,\underline{x}_t$$

tangent vector

and

$$V \cdot V = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_s \cdot \underline{x}_t \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and if we have

$$u = c\,\underline{x}_s + d\,\underline{x}_t$$

$$V \cdot u = \begin{pmatrix} a \\ b \end{pmatrix} \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_t \cdot \underline{x}_s \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

- this object allows us to measure lengths and angles in the given parametrization — ~~change param, and then change~~

- Quadratic form — $\boxed{\text{The first fundamental form}}$

- often write $I(u,v)$

for $\begin{pmatrix} a \\ b \end{pmatrix} \begin{bmatrix} \underline{x}_s \cdot \underline{x}_s & \underline{x}_s \cdot \underline{x}_t \\ \underline{x}_s \cdot \underline{x}_t & \underline{x}_t \cdot \underline{x}_t \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$

- we can now consider

$$N = \text{the } \underline{\text{unit}} \text{ normal.}$$

$$= \frac{\underline{x}_s \times \underline{x}_t}{\| \underline{x}_s \times \underline{x}_t \|}$$

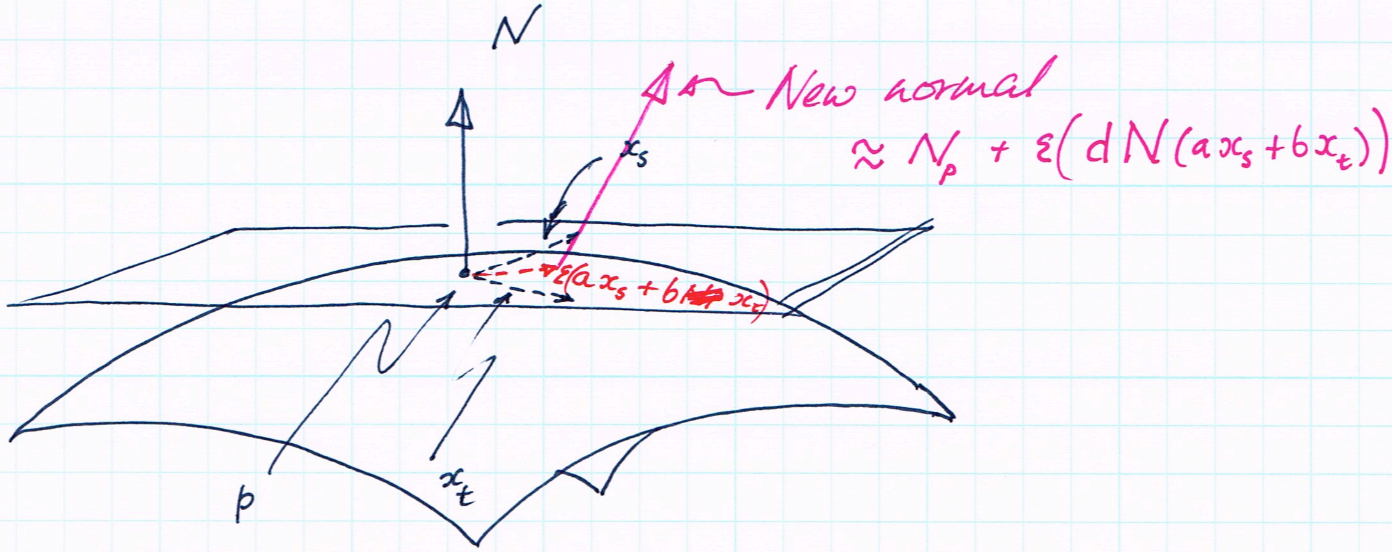
defined for our patch, because $Dx$ has full rank.

- because $N$ is <u>unit normal</u>, we have

$$N \cdot N = 1 \quad , \quad N \cdot x_s = 0 \, , \quad N \cdot x_t = 0$$

- now consider moving away from $\underline{p}$, by a small ~~amount, a~~ step in the tangent plane at $p$.

  - represent as $\varepsilon(a \underline{x}_s + b \underline{x}_t)$




- if we think of N as a map from 2D to 2D [this is the Gauss map]

$(s, t)$

points on the sphere

- it must have a derivative

$$dN$$

- which is a linear map from plane to plane

tangent plane to surf

tangent plane to sphere.

This may worry you – why doesn't the normal swing "in 3D"?

$N \cdot N = 1 \quad$ so $N_s \cdot N = 0$, $N_t \cdot N = 0$

and the derivative in some new dirn (in $T_p$) $u$ has $\frac{\partial}{\partial u} = a\frac{\partial}{\partial s} + b\frac{\partial}{\partial t}$ so $N_u \cdot N = 0$

- the normal only moves on $T_p$ for infinitesimal steps.
- particularly interesting is

$$\boxed{-\underset{\text{first fundamental form}}{I}(\underset{\text{another tangent vector}}{dN(u)}, \underset{\text{tangent vector}}{V}) = \underset{\text{SECOND FUNDAMENTAL FORM}}{II(u,V)}}$$

this is easy to work out in coords:

let $V = v_0 \underline{x}_s + v_1 \underline{x}_t$

$u = u_0 \underline{x}_s + \underline{u}_1 \underline{x}_t$

$$dN(u) = u_0 \cancel{x_s} N_s + u_1 \cancel{x_t} N_t$$

$$I(dN(u), V) = (u_0 \quad u_1) \begin{pmatrix} \underline{N}_s \cdot \underline{x}_s & \underline{N}_s \cdot \underline{x}_t \\ \underline{N}_t \cdot \underline{x}_s & \underline{N}_t \cdot \underline{x}_t \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

this might worry you, because it looks like it isn't symmetric

→ untidy

→ you have to remember order of arguments

BUT

$$N \cdot x_s = 0 \qquad\qquad N \cdot x_t = 0$$

$$\therefore \quad N_t \cdot x_s = -N \cdot x_{st} = N_s \cdot x_t$$

$$N_s \cdot x_s = -N \cdot x_{ss} \quad ; \quad N_t \cdot x_t = -N \cdot x_{tt}$$

So

$\mathrm{II}$ is symmetric

Key result

$$\boxed{\begin{aligned} K &= \text{Gaussian curvature} = \det(-\mathrm{I}^{-1}\mathrm{II}) \\ H &= \text{Mean curvature} = \operatorname{trace}(-\mathrm{I}^{-1}\mathrm{II}) \end{aligned}}$$

which we can establish a bunch of different ways

Advantage: · don't need to compute taylor series at some location

Elegant way to see that $K$ and $H$ are as given.

① for our surfaces

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(\cancel{t}) \qquad \text{at } (0,0)$$

$$II = -\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} + O(\cancel{t}) \qquad \text{at } (0,0)$$

So at $(0,0)$ $\det(I^{-1} II) = k_1 k_2$

$\operatorname{trace}(I^{-1} II) = k_1 + k_2$

② rotating and translating a surface <u>cannot</u> change $I$ or $II$

- entries are dot products of vectors that also rotate; translation doesn't change the vectors

③ What about change of parametrization? we could think about

$$x(s(u,v), t(u,v))$$

New parameters – but parametrizing THE SAME geometry.

$$\begin{bmatrix} \underline{x}_u^T \\ \underline{x}_v^T \end{bmatrix} = \begin{bmatrix} s_u & t_u \\ s_v & t_v \end{bmatrix} \begin{bmatrix} \underline{x}_s^T \\ \underline{x}_t^T \end{bmatrix}$$

I haven't been careful about rows + cols till now, because there wasn't a need; but vectors are <u>col</u>. vectors, and this is $2 \times 3$

Derivative of repar map, transp

write $J = \begin{bmatrix} s_u & s_v \\ t_u & t_v \end{bmatrix}$

So
$$\mathrm{I}^{(u,v)} = \begin{bmatrix} \underline{x}_u^T \\ \underline{x}_v^T \end{bmatrix} \begin{bmatrix} \underline{x}_u & \underline{x}_v \end{bmatrix} = J^T \, \mathrm{I}^{(st)} \, J$$

$$\mathrm{II}^{(u,v)} = \begin{bmatrix} N_u^T \\ N_v^T \end{bmatrix} \begin{bmatrix} \underline{x}_u & \underline{x}_v \end{bmatrix} = J^T \, \mathrm{II}^{(st)} \, J$$

So
$$\left[\mathrm{I}^{(u,v)}\right]^{-1} \left[\mathrm{II}^{(u,v)}\right] = J^{-1} \left[\mathrm{I}^{(st)}\right]^{-1} \mathrm{II}^{(st)} \cdot J$$

now $\det(AB) = \det(A)\det(B)$ and $\operatorname{trace}(ABC) = \operatorname{trace}(BCA)$

So
$$\det\left(-\left[\mathrm{I}^{(u,v)}\right]^{-1} \mathrm{II}^{(u,v)}\right) = \det\left(-\left[\mathrm{I}^{(st)}\right]^{-1} \mathrm{II}^{(st)}\right)$$

and
$$\operatorname{trace}\left(-\mathrm{I}^{-1}\mathrm{II}\right)^{(st)} = \operatorname{trace}\left(-\mathrm{I}^{-1}\mathrm{II}\right)^{(u,v)}$$

NOTE: I have used a standard, superpowerful, geometric argument.

$$\boxed{\begin{array}{l}\text{• Prove something in an easy coordinate}\\ \text{system, then show that change of coords}\\ \text{doesn't matter.}\end{array}}$$

NOTE: We think of K, H as local geometric properties of surfaces BECAUSE they're invariant to rigid motion and reparametrization

- Sometimes, we are interested in other groups (rigid motion + scale; affine tx; projective tx).
- All the above has analogous, much fiddlier, constructions for these cases
  - can be looked up, or worked out; not usually worth it.

Some notation:

it is traditional to write

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} x_s \cdot x_s & x_s \cdot x_t \\ x_s \cdot x_t & x_t \cdot x_t \end{pmatrix}$$

I

and

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} N \cdot x_{ss} & N \cdot x_{st} \\ N \cdot x_{st} & N \cdot x_{tt} \end{pmatrix}$$

II

now we know that

$$\begin{pmatrix} \underline{N}_s^T \\ \underline{N}_t^T \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} = a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \underline{x}_s^T \\ \underline{x}_t^T \end{pmatrix} \quad \text{for some } a_{ij}$$

(because $N_s \in T_p$, etc) but we don't know $a_{ij}$

$\rightarrow$ easy to get

$$\begin{bmatrix} \underline{N}_s^T \\ \underline{N}_t^T \end{bmatrix} [\underline{x}_s \; \underline{x}_t] = -\mathrm{II} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \cdot \mathrm{I}$$

so $\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = -\dot{\mathrm{II}}\ \mathrm{I}^{-1}$

and so $K = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, $H = \mathrm{tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$

Now $\{x_u, x_v, N\}$ is a basis for <u>any</u> vector at $p$ (not just tangent.

Recall the situation w/ curves — we had a frame, and much geometry was revealed by what happened if we took a small step

$$\underline{x}_{ss} = \Gamma_{11}^{1}\,\underline{x}_s + \Gamma_{11}^{2}\,\underline{x}_t + L_1\,\underline{N}$$

$$\underline{x}_{st} = \Gamma_{12}^{1}\,\underline{x}_s + \Gamma_{12}^{2}\,x_t + L_2\,\underline{N} \qquad (= x_{ts})$$

$$\underline{x}_{tt} = \Gamma_{22}^{1}\,\underline{x}_s + \Gamma_{22}^{2}\,\underline{x}_t + L_3\,\underline{N}$$

$$\underline{N}_s = a_{11}\,\underline{x}_s + a_{12}\,\underline{x}_t$$

$$\underline{N}_t = a_{12}\,\underline{x}_s + a_{22}\,\underline{x}_t$$

above is basically just notation
( $\{x_u, x_v, N\}$ is a complete basis, so there must be an expansion)

We can fill in some detail.

$$x_{ss} \cdot N = L_1 (N \cdot N) = L_1 = e$$

Similarly $\cancel{x_{st}}$ $L_2 = f$ , $L_3 = g$

which basis vector

$$\Gamma^{i}_{jk} \leftarrow \text{Christoffel symbols}$$

which derivative

<u>Notice</u> $E_s = 2(x_{ss} \cdot x_s)$ $E_t = 2(x_{st} \cdot x_s)$

$$F_s = (x_{ss} \cdot x_t) + (x_s \cdot x_{st})$$

$$F_t = (x_{st} \cdot x_t) + (x_s \cdot x_{st})$$

$$G_s = 2(x_{st} \cdot x_t) \qquad G_t = 2(x_{tt} \cdot x_t)$$

linear algebra yields

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_s \\ F_s - \frac{1}{2} E_t \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{12}^{1} \\ \Gamma_{12}^{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_t \\ \frac{1}{2} G_s \end{pmatrix}$$

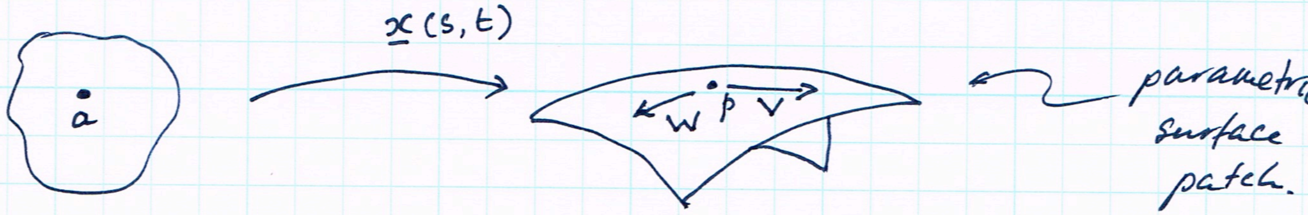
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{22}^{1} \\ \Gamma_{22}^{2} \end{pmatrix} = \begin{pmatrix} F_t - \frac{1}{2} G_s \\ \frac{1}{2} G_t \end{pmatrix}$$

(you can look these up, or derive them; the form isn't that significant)

SIGNIFICANT: Christoffel symbols can be recovered from $I$ and its derivatives. — no use of the embedding.

## I as a metric:

- recall what I does.




$U$ = parametric domain

$(s,t)$

- Given tangent vectors $V, W$ on surface at $p$, I can measure lengths, angles ~~at~~ by

$$\text{length}(V) = (V\cdot V)^{1/2}, \quad \text{etc}, \text{ using}$$

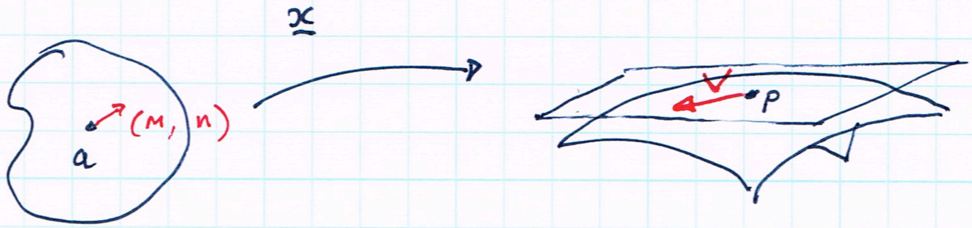
dot product in 3-space

- ~~I~~ there is a natural rep'n of ~~the~~ $V, W$ at $p$ in terms of basis $\underline{x}_s, \underline{x}_t$

$$V = m\,\underline{x}_s + n\,\underline{x}_t \qquad \text{etc.}$$

- Now I could represent $V$ (on $T_p$) as $(m,n)$ ~~a~~ (at $a$, $\underline{x}: a \to p$, in par domain).

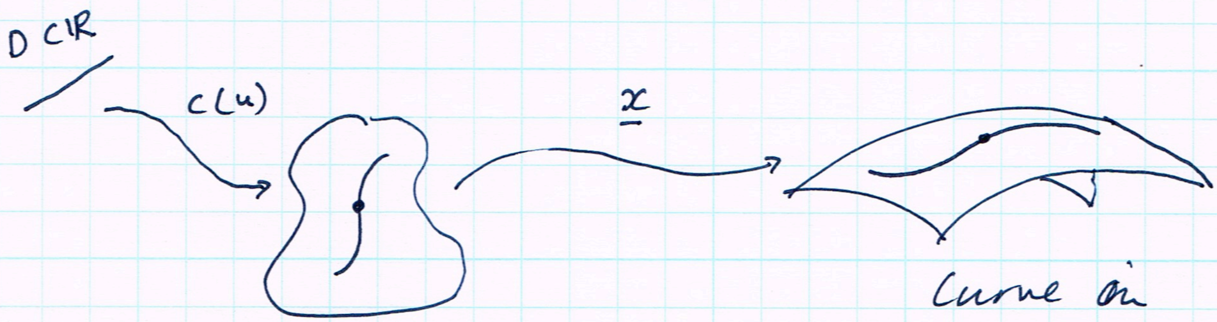
I allows me to measure lengths, angles <u>there</u>




$$I\left(\begin{pmatrix} m \\ n \end{pmatrix}, \begin{pmatrix} m \\ n \end{pmatrix}\right) = V \cdot V$$

etc for angles

So I can measure the length of a curve on the surface




$u \in \mathbb{R}^2$

curve in par domain

$s(u), t(u)$

Curve in Surface

$\underline{x}(s(u), t(u))$.

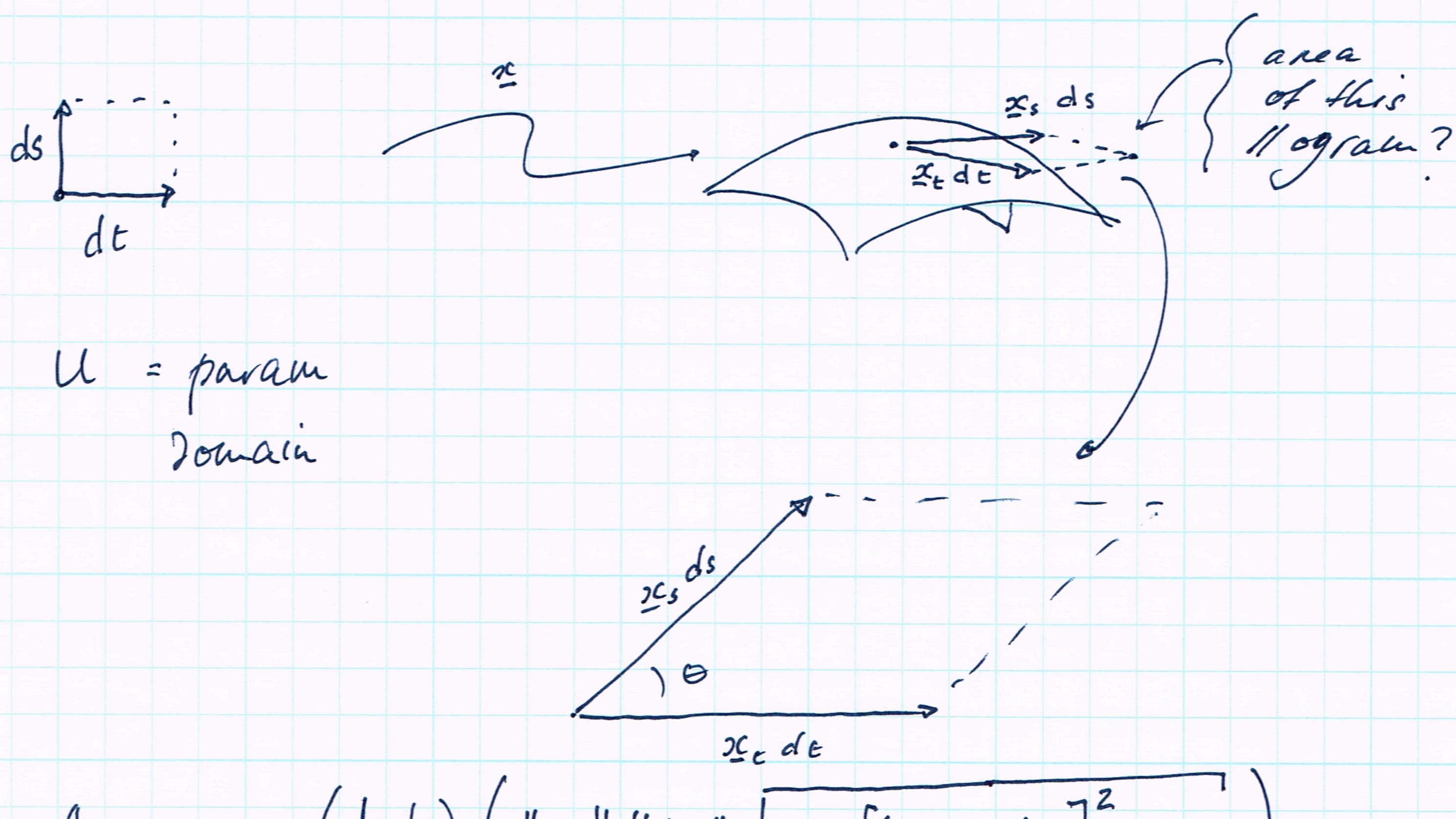
So length $= \displaystyle\int_D \sqrt{I\left(\begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}, \begin{pmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial t}{\partial u} \end{pmatrix}\right)}\, du$

↑ tangent to curve in surface pars

## Area:

- I get the area of a patch on a surface by adding up the area of elementary patches

- Divide relevant piece of parametrisation domain into infinitesimal quads $ds, dt$. and add up. areas




**Area:** $$(ds\,dt)\left(\|x_s\|\,\|x_t\|\sqrt{1-\left[\frac{(\underline{x}_s\cdot\underline{x}_t)}{\|x_s\|\,\|x_t\|}\right]^2}\right)$$

So

$$\underline{\text{Area}} = ds\,dt \left[ \sqrt{\|x_s\|^2 \|x_t\|^2 - (x_s \cdot x_t)^2} \right]$$

$$= ds\,dt \left[ \sqrt{EF - G^2} \right]$$

this is det I

So area cut out by $U \in$ domain

$$\int_U \det(I)\, ds\, dt$$

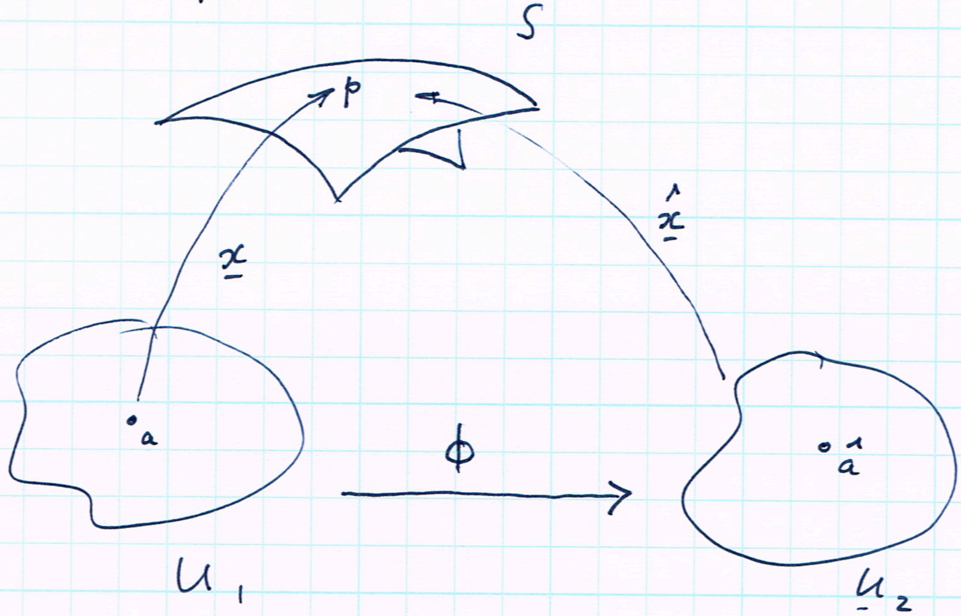
(not usually easy!)

Notice that in these calculations, we used $I$, $\underline{\text{not}}$ $\underline{x}$.

- i.e. if I have a $I$ over a domain, I can complete length, area w/o knowing $\underline{x}$

# Reparametrizing a Surface

- the same surface can have many different parametrizations




- consider two parametrizations, $\underline{x}$, $\hat{\underline{x}}$ of a surface S; there must be some $\varphi$, 1-1 linking them. In this picture

$$\hat{\underline{x}} \circ \phi = \underline{x}$$

Now since the <u>parametrization</u> can't affect lengths, angles on surf, we must have

for $\underline{x}, \hat{\underline{x}}$ case

Derivative of $\phi$ at $a$.

$$I_a^{(x)}(u,v) = I_{\phi(a)}^{(\hat{x})}(d\phi(u), d\phi(v))$$

location

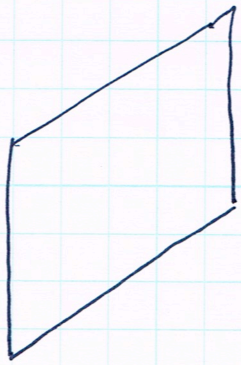
corresponding location for $\hat{x}$, found using $\phi$

- Notice that if $\underline{x}, \hat{\underline{x}}$ are parametrizations of the same surface $S$, then some $\phi$ with this property must exist.

- Now consider $\psi: S_1 \longrightarrow S_2$ (maps surfaces to surfaces).

  - if $\psi$ does not change lengths (and so angles) it is an <u>isometry</u>

- ~~Not~~ <u>Example</u>: Rotation + Translation



flat sheet $= S_1$

roll up <u>without</u> stretch

$= S_2$

· There is no stretch

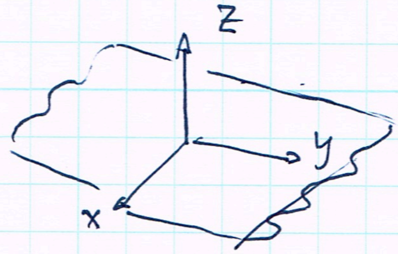
∴ all lengths on surface are unchanged (it follows all angles are unchanged, too)

∴ an isometry <u>must</u> exist

<u>in coordinates</u>:

$$S_1 = [s, t, 0]$$

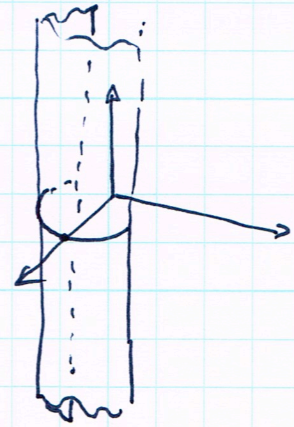
$$s \in [-1, 1]$$




$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ at <u>all points</u>

$S_2 = [\cos, \sin s, t]$

$s \in [-1, 1]$



$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ at <u>all points</u>

So an <u>isometry</u> must exist

~~( $\phi$ = identity works )~~.

Q: when does an isometry exist?

~~Q:~~ Geometric properties ~~f~~ invariant under isometry are referred to as

<u>intrinsic</u>

Clearly, H <u>isn't</u> intrinsic.

**Lemma:** assume $\Psi$ is an isometry.

$\Psi : S_1 \to S_2$.

$\underline{x} : (U_1 \subset \mathbb{R}^2) \longrightarrow S_1$

$\hat{\underline{x}} : (U_2 \subset \mathbb{R}^2) \longrightarrow S_2$

then there is a parametrization of $S_2$ such that

$$\hat{E} = E \qquad \hat{F} = F \qquad \hat{G} = G$$

coord functions of I on $S_1$ for $\underline{x}$

coord functions of I on $S_2$ for new param.

**Proof:**

$$\begin{tikzcd} S_1 \arrow[r, "\Psi"] & S_2 \\ U_1 \arrow[u, "\underline{x}"] & U_2 \arrow[u, "\hat{\underline{x}}"'] \end{tikzcd}$$

Now parametrize $S_2$ by $\Psi \circ S_1$

→ lengths, angles are the same

→ ~~same~~ a in $U_1$ refers to CSP pts in $S_1$, $S_2$

→ $\hat{E} = E$, etc.

Important consequence:

- Any property that can be expressed in terms of $E, F, G$ (and their derivatives) is intrinsic

Theorema egregium: Gaussian curvature is intrinsic

Proof: (not super enlightening – see scanned pages) Manipulate formula for $K$ to produce expression in $E, F, G$ and derivatives.

Corollary: Isometric surfaces have the same Gaussian curvature at csp points

We now have two threads to look at:

- We can abstract away embeddings and study only <u>intrinsic</u> properties (given by $E, F, G$). To do this, we think about $I$ as a function on 2D, and don't really worry about embedding. This leads to Riemannian geometry; we'll do some of this.

- We can look at extrinsic (+ intrinsic, on occasion) properties, where the embedding matters (often a lot). We'll do lots of this, next.