Very simple control, with PID

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We assume that everything is linear

- This creates huge mathematical simplifications
- Linear system:
  - accepts a signal $x(t)$
  - produces a signal $y(t)=K\,x(t)$
  - AND
    - $K\,(x(t) + y(t)) = K\,x(t) + K\,y(t)$
    - $K\,(a\,x(t)) = a\,K\,x(t)$
    - (notice this means $K\,0 = 0$)

K stands for a linear operator,
so that (for example) we could have
$K\,x(t) = a\,x(t)$
or
$K\,x(t) = dx/dt$

In fact, study only the response to a step

- You can approximate any function with a lot of steps
- Step is u(t)
  - this is 0 for t<=0, 1 otherwise
  - so u(t)-u(t+dt) is a bar
- Approximate f(t) by

$$\sum_i f(i\Delta t)(u(i\Delta t) - u(i\Delta t + \Delta t))$$

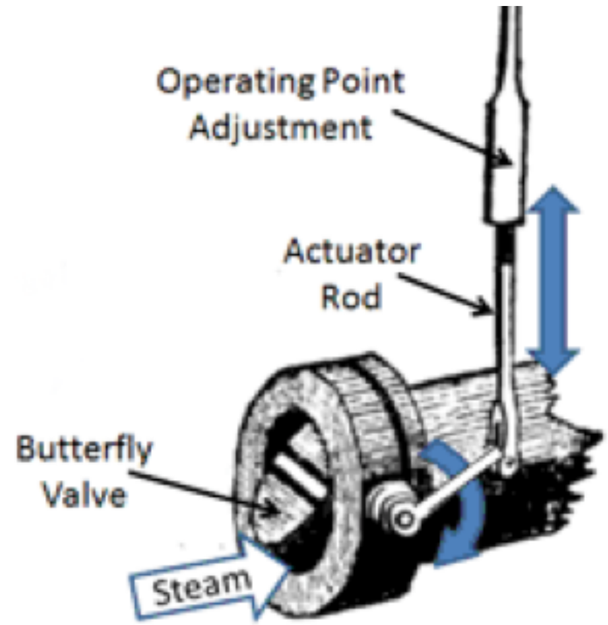
  - ex: simplify this expression
  - ex: we know K u(t) - what is K f(t)?

Ideas: plant/process, control

- Plant/process is the thing we wish to control
  - assume: 1 input, 1 output, linear
  - for simple examples, I'll write out the form of the plant
    - but very often, it isn't known exactly
      - System Identification
- Control:
  - supply the plant with the input needed to produce the output you want
  - Q: why is this hard?
    - A1: Plant may not be exactly known
    - A2: Plant may have dynamics
    - A3: Desired output may change

The very simplest control

- Plant: K x(t) = c x(t)
  - here c is a known constant
- We'd like the output to be 1
  - feed plant with 1/c
    - and go home early
- Example of open loop control
  - compute a fixed input and supply to plant
    - whatever the plant
- Advantages:
  - simple, sometimes works
- Disadvantages:
  - what if your model is wrong?

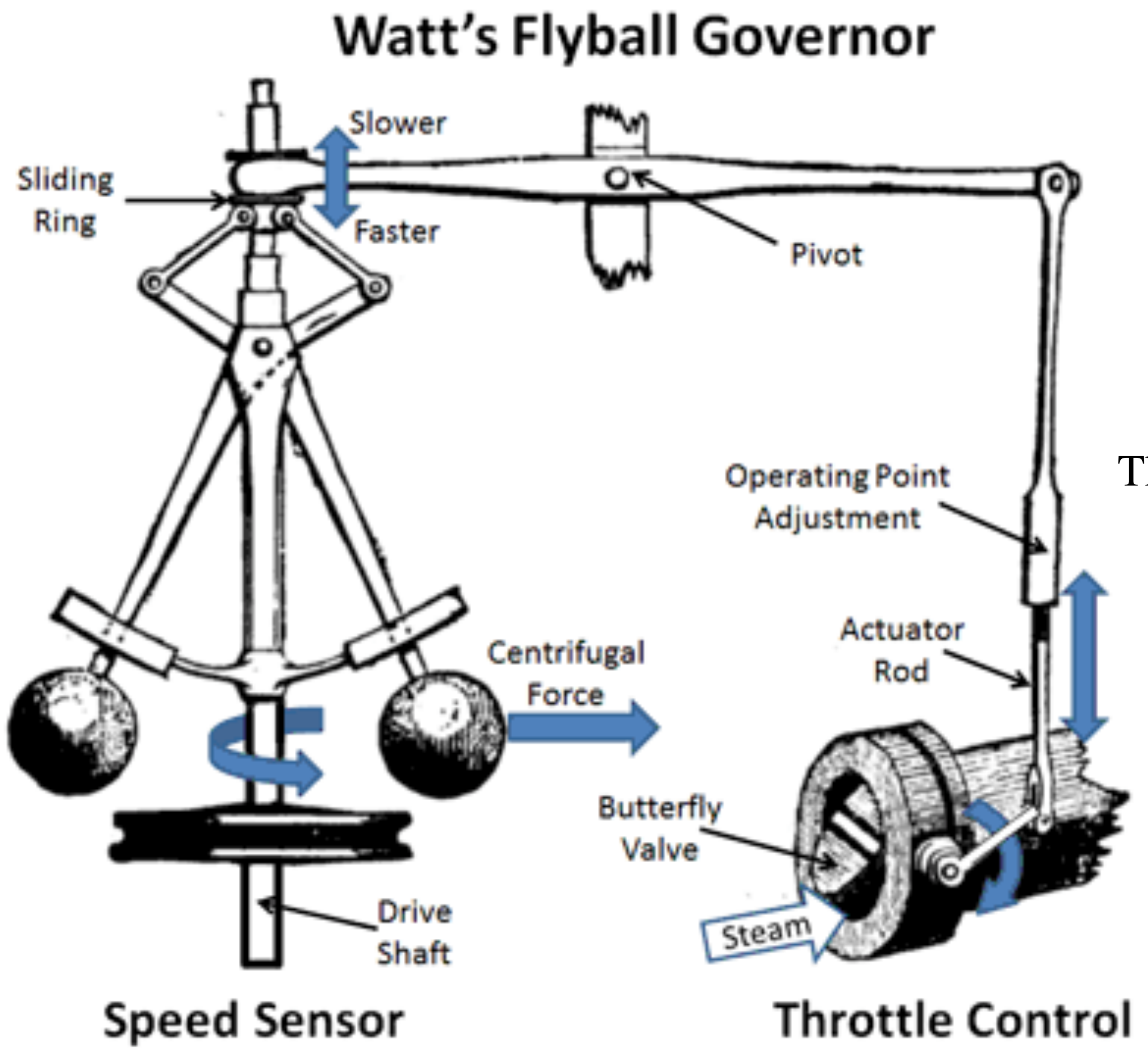


Throttle Control

Example: move car at constant velocity

- How?
  - supply accelerator input so that car moves at constant velocity
- Open loop:
  - Figure out how acceleration causes velocity
    - a=dv/dt-(some frictional loss)
  - Supply acceleration
    - a burst of acceleration to get to speed
    - then constant acceleration to cope with loss
- Doesn’t seem all that practical

History of feedback



Watt's flyball governor, C19

These were still in use in late C20!

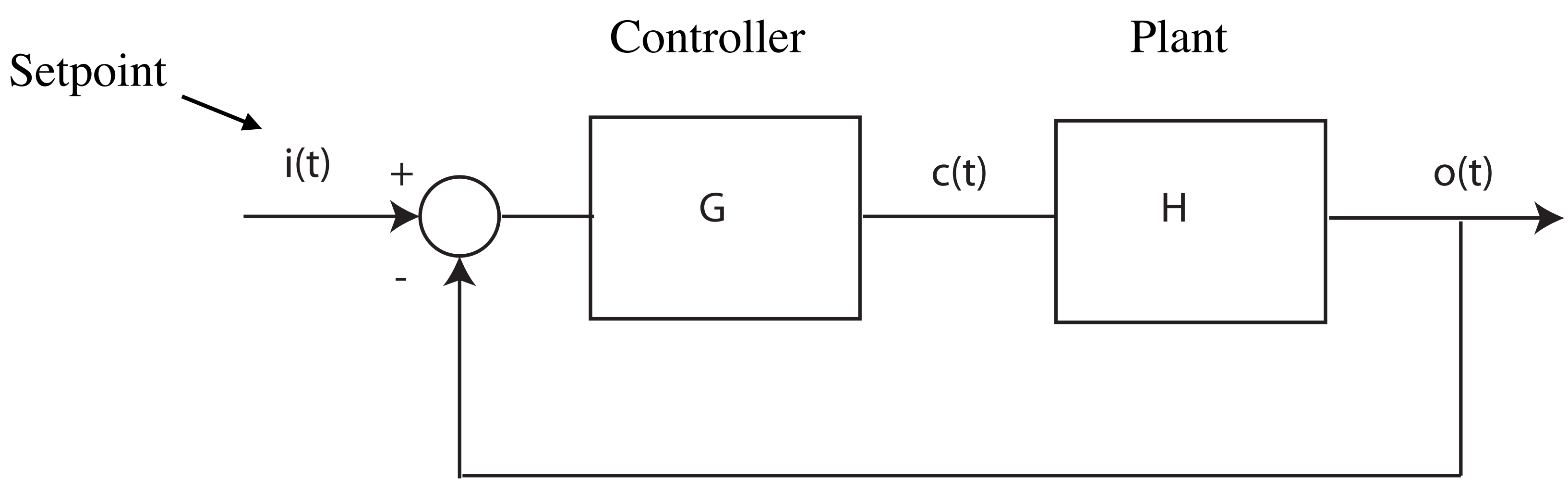
Example: move car at constant velocity

- How?
  - supply accelerator input so that car moves at constant velocity
- Closed loop:
  - Supply acceleration, measure speed
    - adjust acceleration so that car keeps speed
      - Q: how?

Closed loop control

- Derive an input to the plant from
  - setpoint (where you want the output to be)
  - current plant output

- The form we will discuss is:



We have

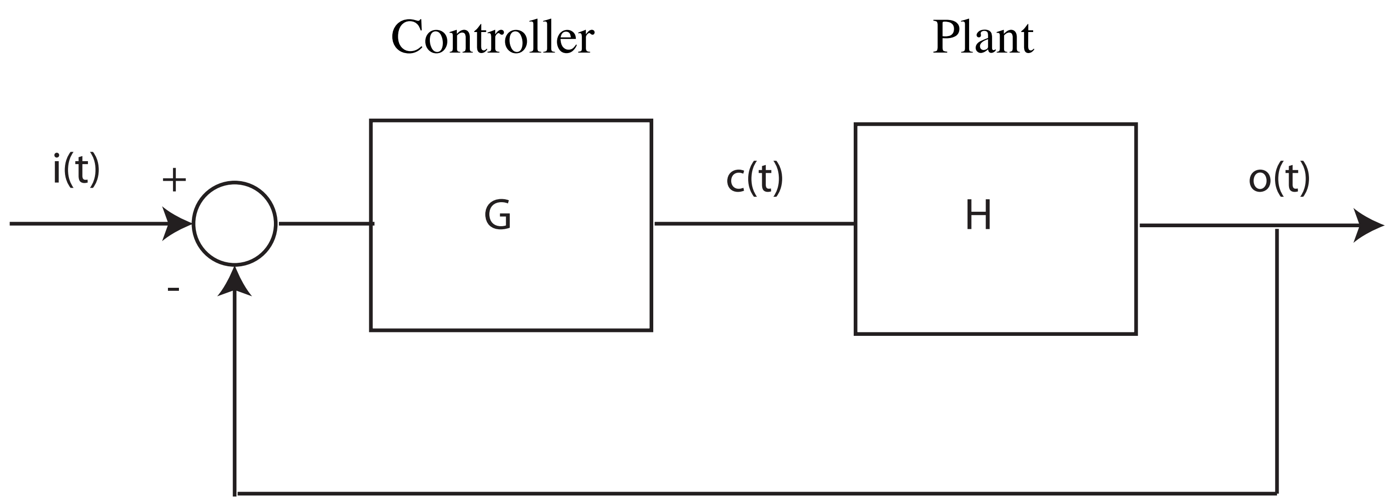
c(t)=G (i(t)-o(t))

o(t)=H c(t)

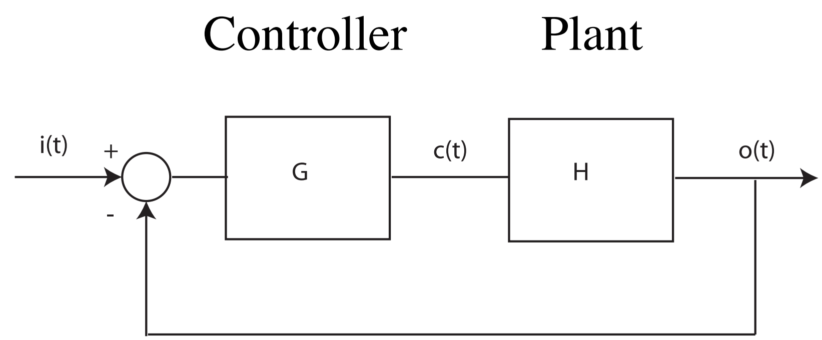
so

which you should remember

o(t)+H G o(t)=H G i(t)



Simple, worrying example



- H c(t) = a c(t)
- G x(t) = b x(t)

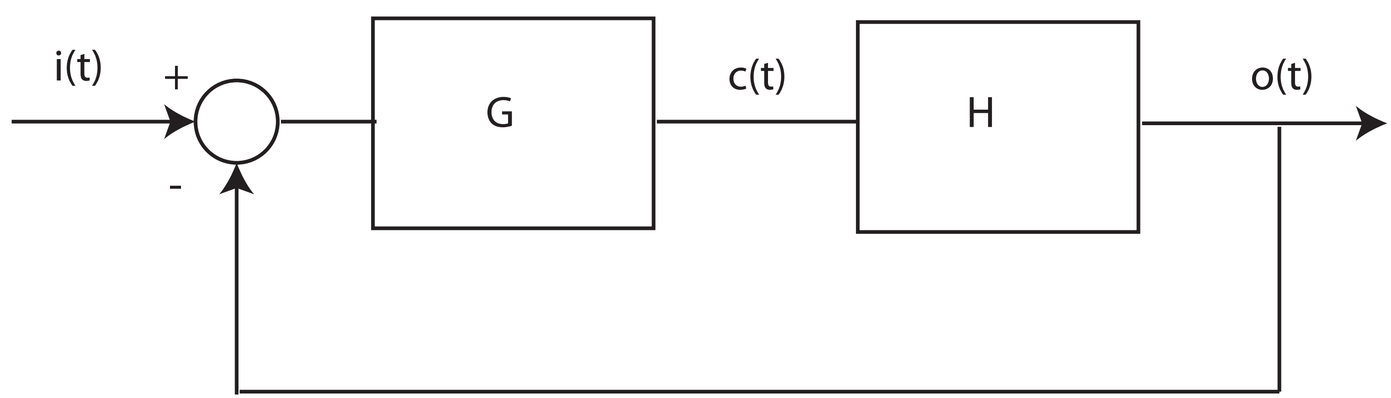
- o(t)+ab o(t)=ab i(t)

- Now imagine that i(t) is a step function
  - for t>0 we have
  - o(t)= ab/(1+ab)
    - which isn't what we wanted
      - (remember, i(t) is the output value we want)
  - steady state error is lim t->infinity (o(t)-i(t))

Fix with integral term

- Idea:
  - if (i(t)-o(t)) is not zero, there should be some control input
  - magnitude increases until it is zero
-

$$Gx(t) = bx(t) + c\int_0^t x(s)ds$$



Fixing with integral term

$$o(t) + abo(t) + ac\int_0^t o(s)ds = abi(t) + ac\int_0^t x(s)ds$$

Differentiate

$$(1+ab)\frac{do(t)}{dt} + aco(t) = ab\frac{di(t)}{dt} + aci(t)$$

BUT we're interested in t>0, and i(t) is a step at 0

$$(1+ab)\frac{do(t)}{dt} + aco(t) = aci(t)$$

Fixing with integral term

$$(1 + ab)\frac{do(t)}{dt} + aco(t) = ac$$

Assume that do/dt -> 0 as t-> infinity
(we'll see it does in a moment)

$$o(t) = 1$$

For large t, which is what we wanted

Fixing with integral term

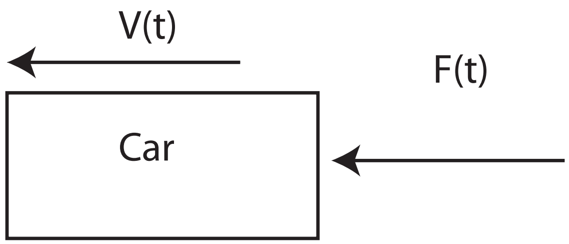
$$\frac{(1+ab)}{ac}\frac{do(t)}{dt} + o(t) = 1 \qquad\qquad o(0) = 0$$

$$o(t) = (1 - e^{\frac{-ac}{1+ab}t})$$

Example

- is it a good idea to get a faster response by making c bigger?

A more interesting plant



Force

$$v(t) = v(0) + \int_0^t \frac{F(s)}{m} dt$$

- Apply a force to the car to control its velocity
  - eg braking

Input

Output

$$v(t) = \int_0^t \frac{F(s)}{m} dt$$

Proportional control

o(t)+H G o(t)=H G i(t)

$$Gx(t) = bx(t)$$

$$o(t) + H\left[bo(t)\right] = H\left[bi(t)\right]$$

$$o(t) + \frac{b}{m}\int_0^t o(s)ds = \frac{b}{m}\int_0^t i(s)ds$$

$$\frac{do}{dt} + \frac{b}{m}o(t) = \frac{b}{m}$$

Recall that t>0, i(t)=1

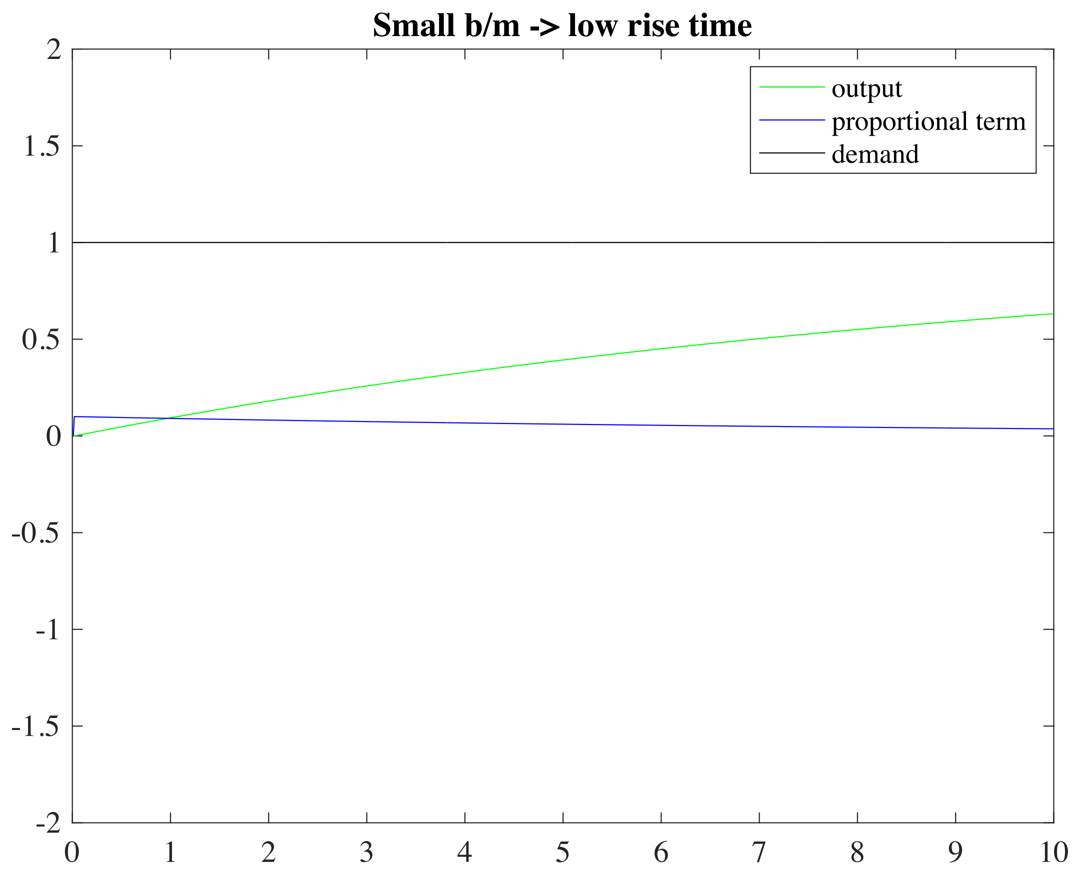
Notice

$$\frac{do}{dt} + \frac{b}{m} o(t) = \frac{b}{m}$$

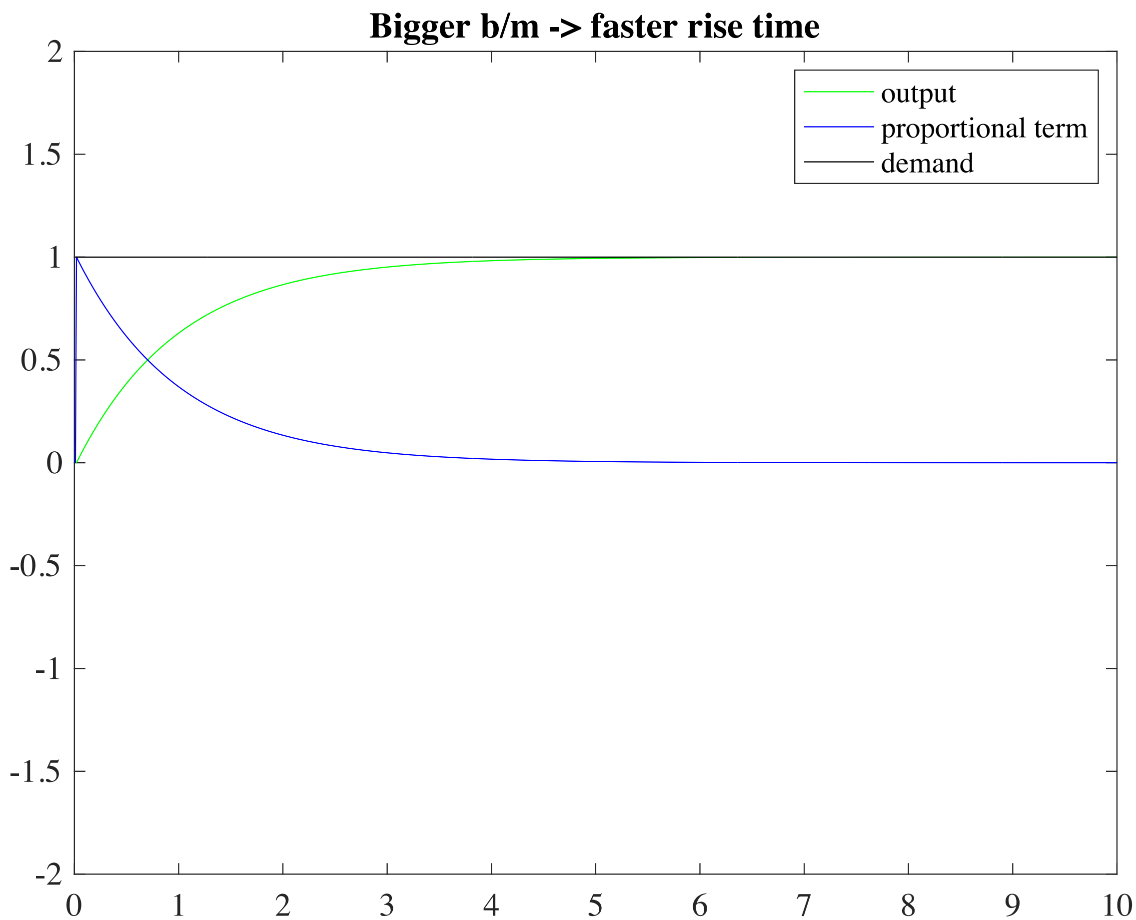
$$o(t) = (1 - e^{\frac{-bt}{m}})$$

- steady state error is now zero
- larger b/m -> faster response
  - BUT larger forces applied to car
- (obvious) b/m <0 -> unstable behavior
- Example

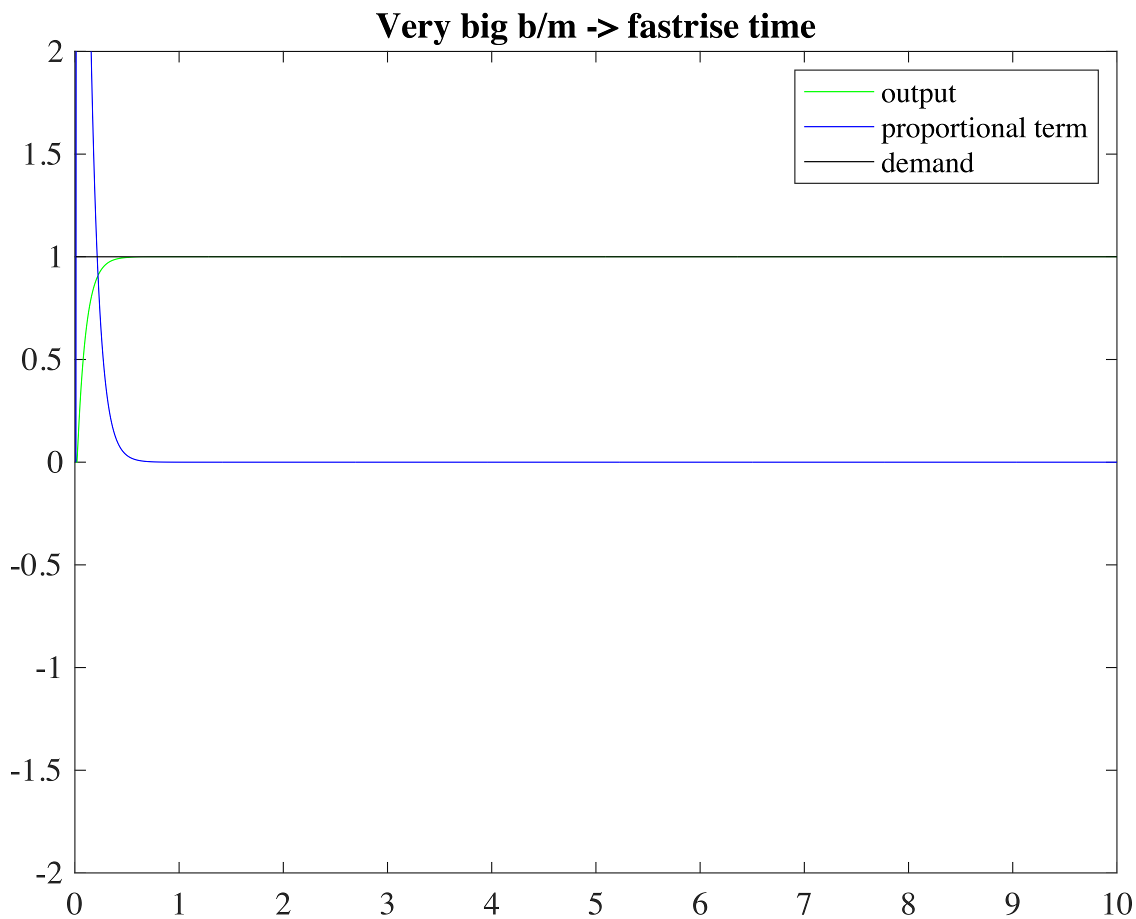
Examples


Small b/m -> low rise time
output
proportional term
demand
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10

Examples

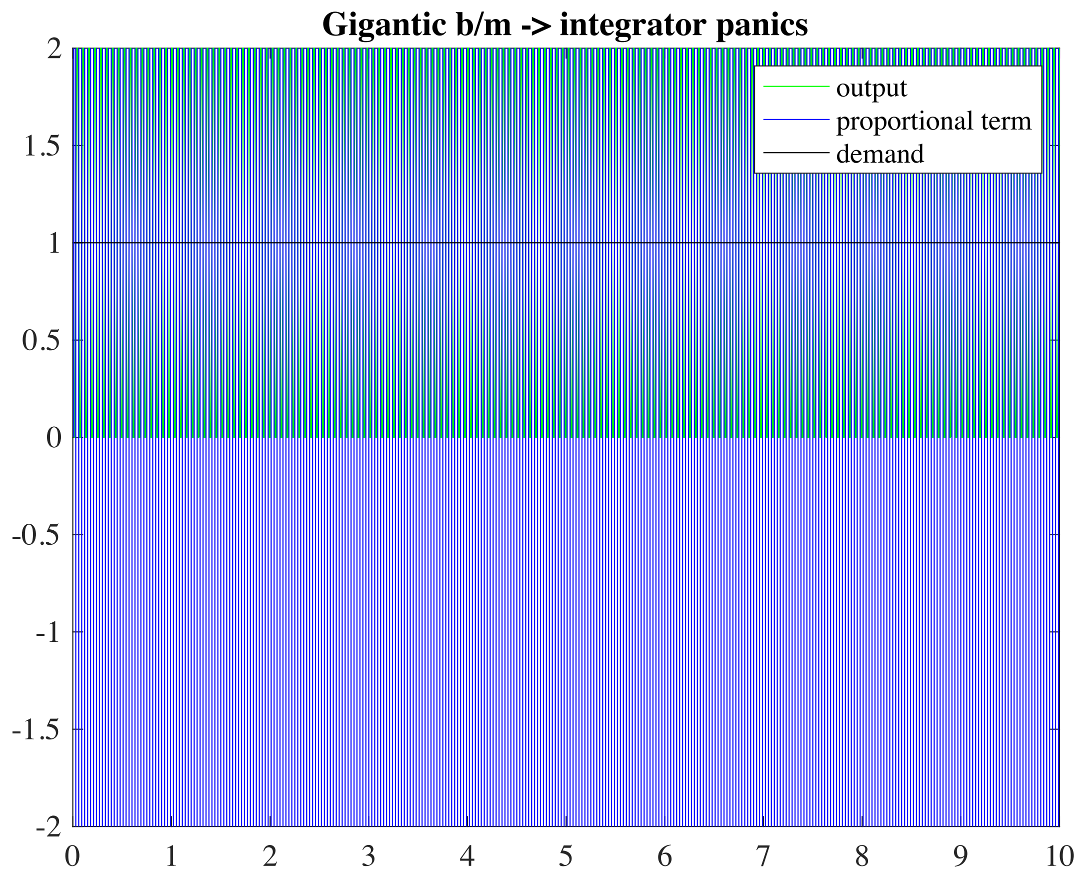

Bigger b/m -> faster rise time
output
proportional term
demand
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10

Examples

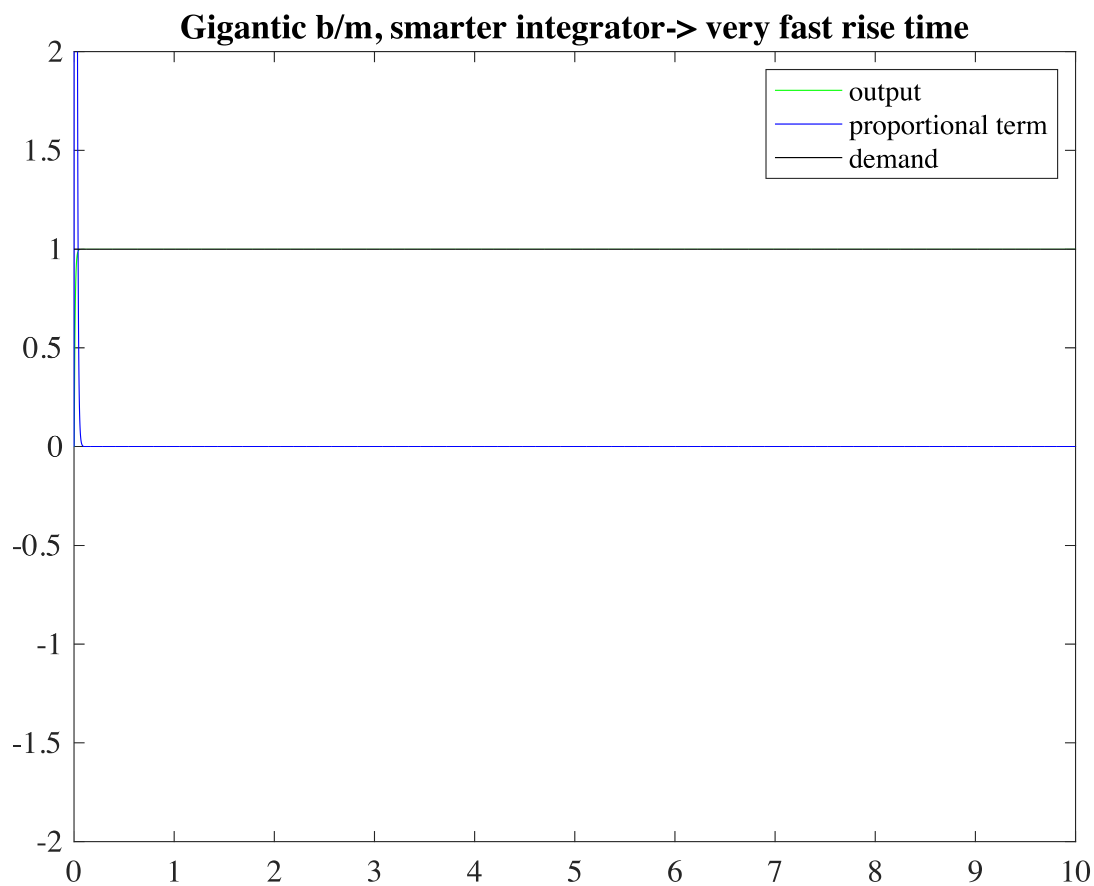


Very big b/m -> fastrise time
output
proportional term
demand
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10

Examples


Gigantic b/m -> integrator panics
output
proportional term
demand
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10

Examples



Gigantic b/m, smarter integrator-> very fast rise time
output
proportional term
demand
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
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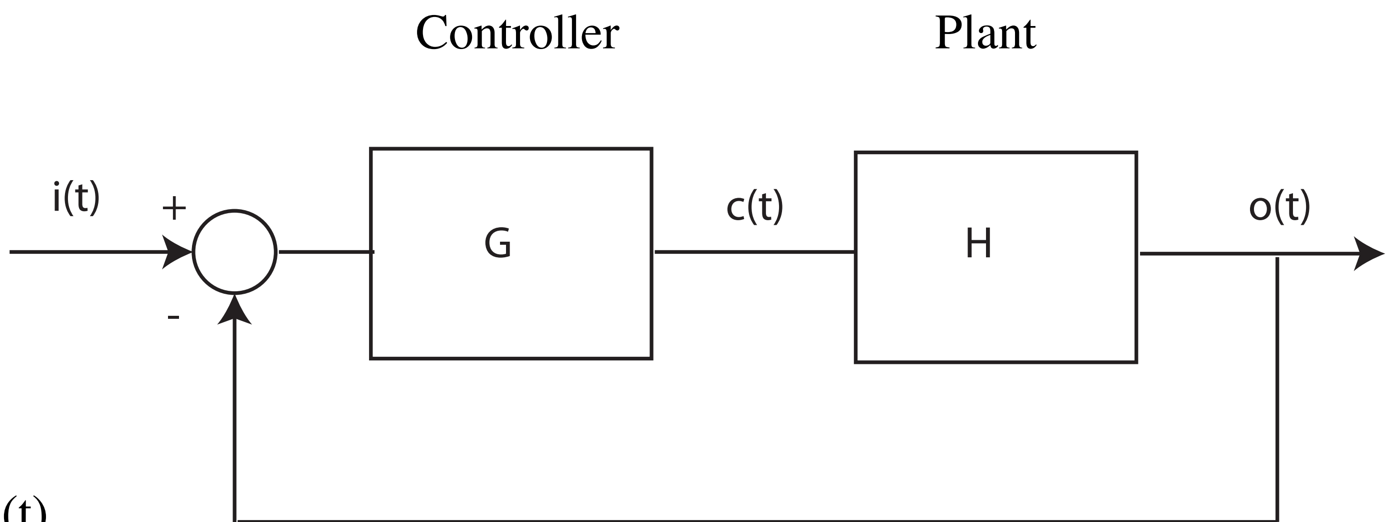
But…

- Controller has only one parameter - b
  - the weight of the proportional term
- Integral term was useful for the simplest plant
  - what about here?
  - we'd have another parameter to adjust..

Proportional - Integral (PI) control



o(t)+H G o(t)=H G i(t)

$$Gx(t) = bx(t) + c\int_0^t x(s)ds$$ ← Controller

Proportional - Integral (PI) control

o(t)+H G o(t)=H G i(t)

Plant is:

Input

Output

$$\boxed{v(t) = \int_0^t \frac{F(s)}{m} dt}$$

$$o(t) + H\left[bo(t) + c\int_0^t o(s)ds\right] = H\left[bi(t) + c\int_0^t i(s)ds\right]$$

Proportional - Integral (PI) control

o(t)+H G o(t)=H G i(t)

Plug in plant, controller to get

$$o(t) + \frac{1}{m}\int_0^t \left[bo(u) + c\int_0^u o(s)ds \right] = \frac{1}{m}\int_0^t \left[bi(u) + c\int_0^u i(s)ds \right]$$

Differentiate twice, to get

$$\frac{d^2 o}{dt^2} + \frac{b}{m}\frac{do}{dt} + \frac{c}{m}o(t) = \frac{c}{m}$$

(recall t>0, i(t)=1)

$$\frac{d^2o}{dt^2} + \frac{b}{m}\frac{do}{dt} + \frac{c}{m}o(t) = \frac{c}{m}$$

Steady state error is zero:

Assume derivatives ->0 as t-> infinity (we'll see they do)
then o(t) = 1 for very large t, which is what we wanted

$$\frac{d^2 o}{dt^2} + \frac{b}{m}\frac{do}{dt} + \frac{c}{m} o(t) = \frac{c}{m}$$

Solving:

Complex number!

Assume solution is of form:

$$A_1 e^{zt} + A_2 t + A_3$$

To get

$$A_1 e^{zt}\left(z^2 + \frac{b}{m} z + \frac{c}{m}\right) + A_2\left(\frac{b}{m} + t\frac{c}{m}\right) + A_3 \frac{c}{m} = \frac{c}{m}$$

What about A values?

$$A_1 e^{zt}\left(z^2 + \frac{b}{m}z + \frac{c}{m}\right) + A_2\left(\frac{b}{m} + t\frac{c}{m}\right) + A_3\frac{c}{m} = \frac{c}{m}$$

$$A_2 = 0$$

because there's no t on the right hand side

$$z^2 + \frac{b}{m}z + \frac{c}{m} = 0$$

because there's no t on the right hand side

$$A_3 = 1$$

to match c/m

$$A_1 = -1$$

because solution at time=0 is 0
and solution is

$$A_1 e^{zt} + A_2 t + A_3$$

Solution is:

$$(1 - e^{zt})$$

Where

$$z^2 + \frac{b}{m}z + \frac{c}{m} = 0$$

So

$$z = \frac{1}{2}\left[-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{c}{m}}\right]$$

Solution properties:

$$(1 - e^{zt})$$

$$z = \frac{1}{2}\left[-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{c}{m}}\right]$$

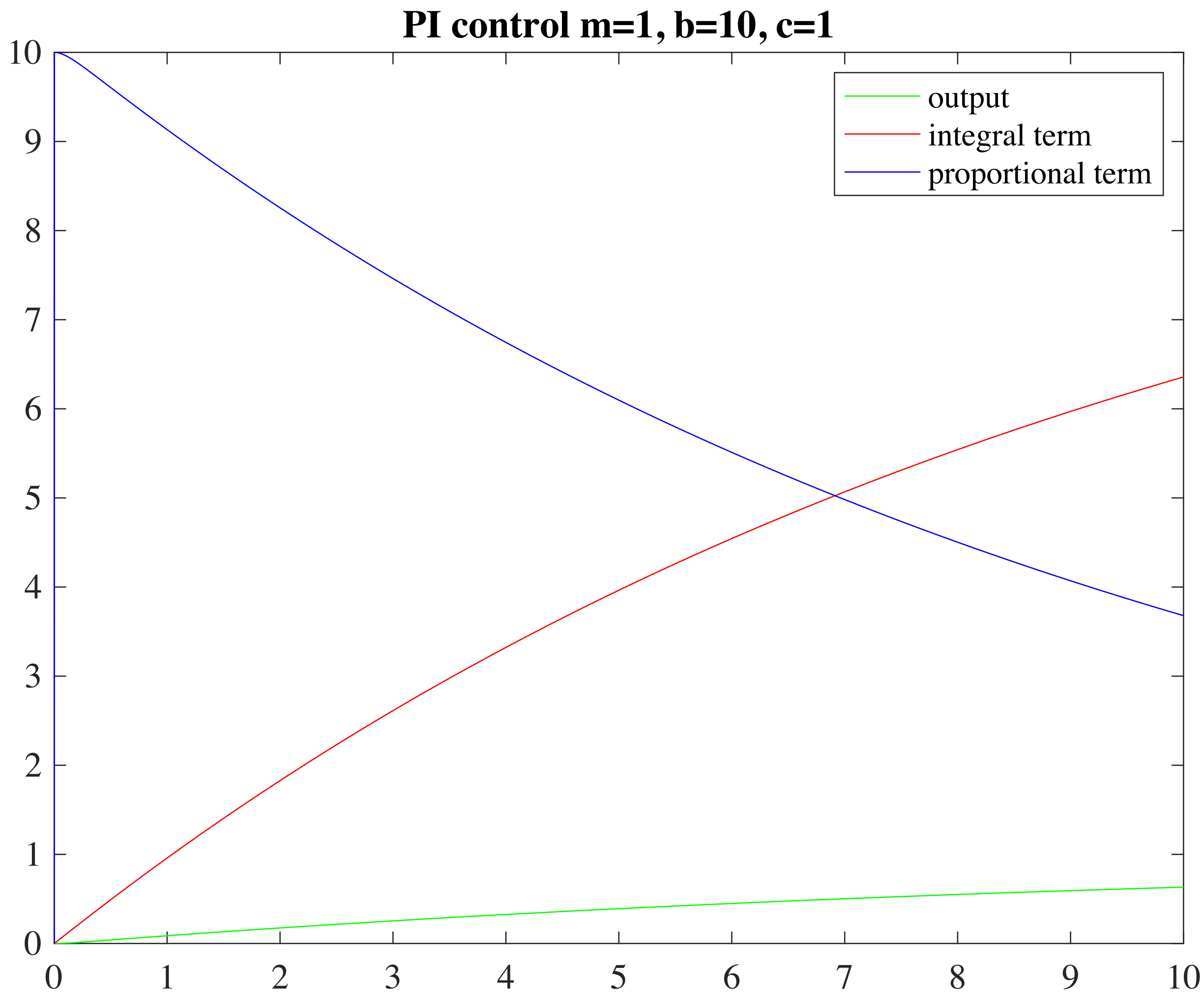
Cases:

b^2-4cm >0   (two real roots; sum of exponentials)

b^2-4cm=0 (two copies of the same root -
this is known as critical damping)

b^2-4cm<0 (sinusoid with exponential amplitude)

Stability:
-b/m >0 - soln GROWS with time,
otherwise OK

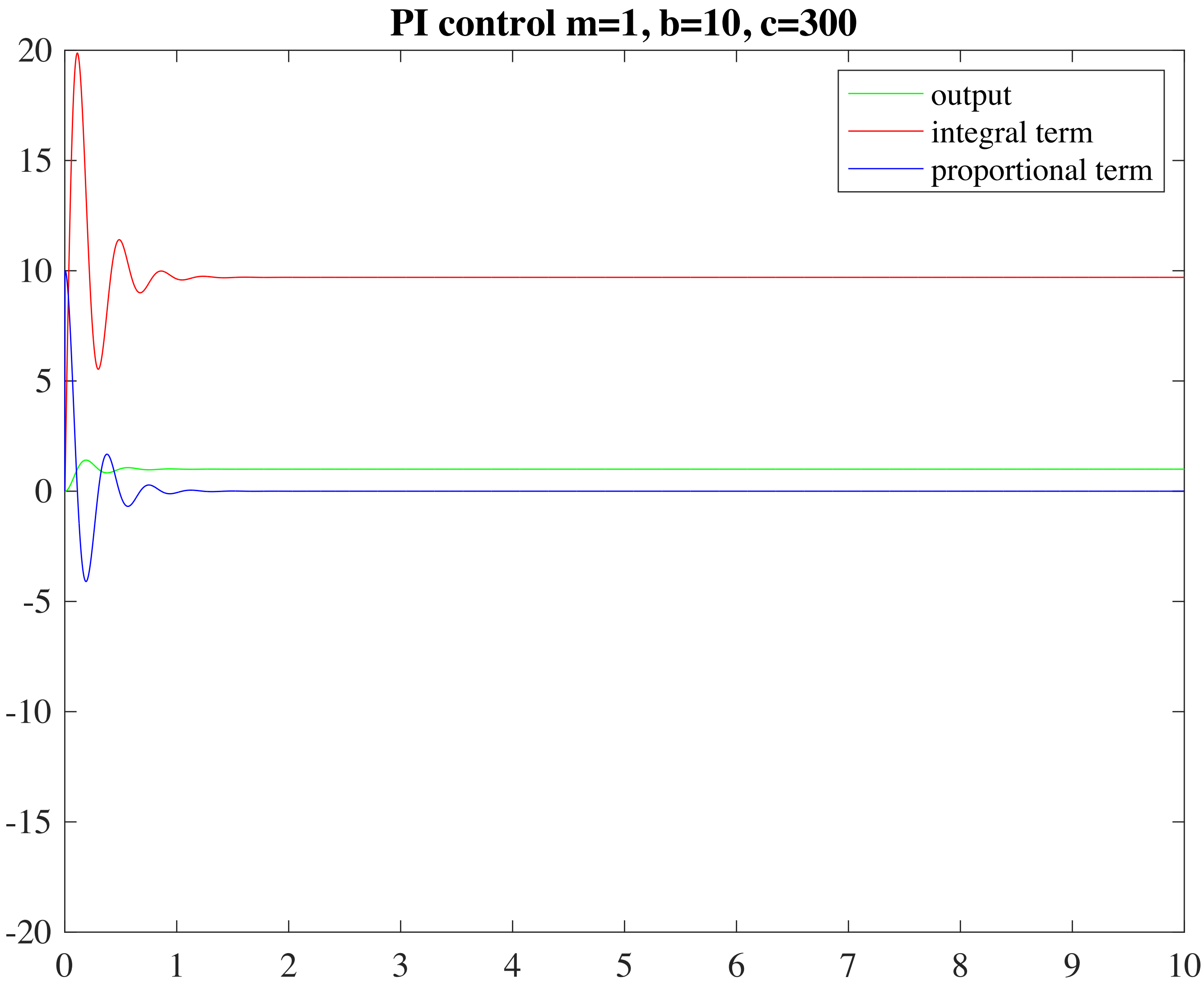


Cases:

I: two real roots; sum of exponentials

II: two copies of the same root

III: sinusoid with exponential amplitude



Cases:

I: two real roots; sum of exponentials

II: two copies of the same root

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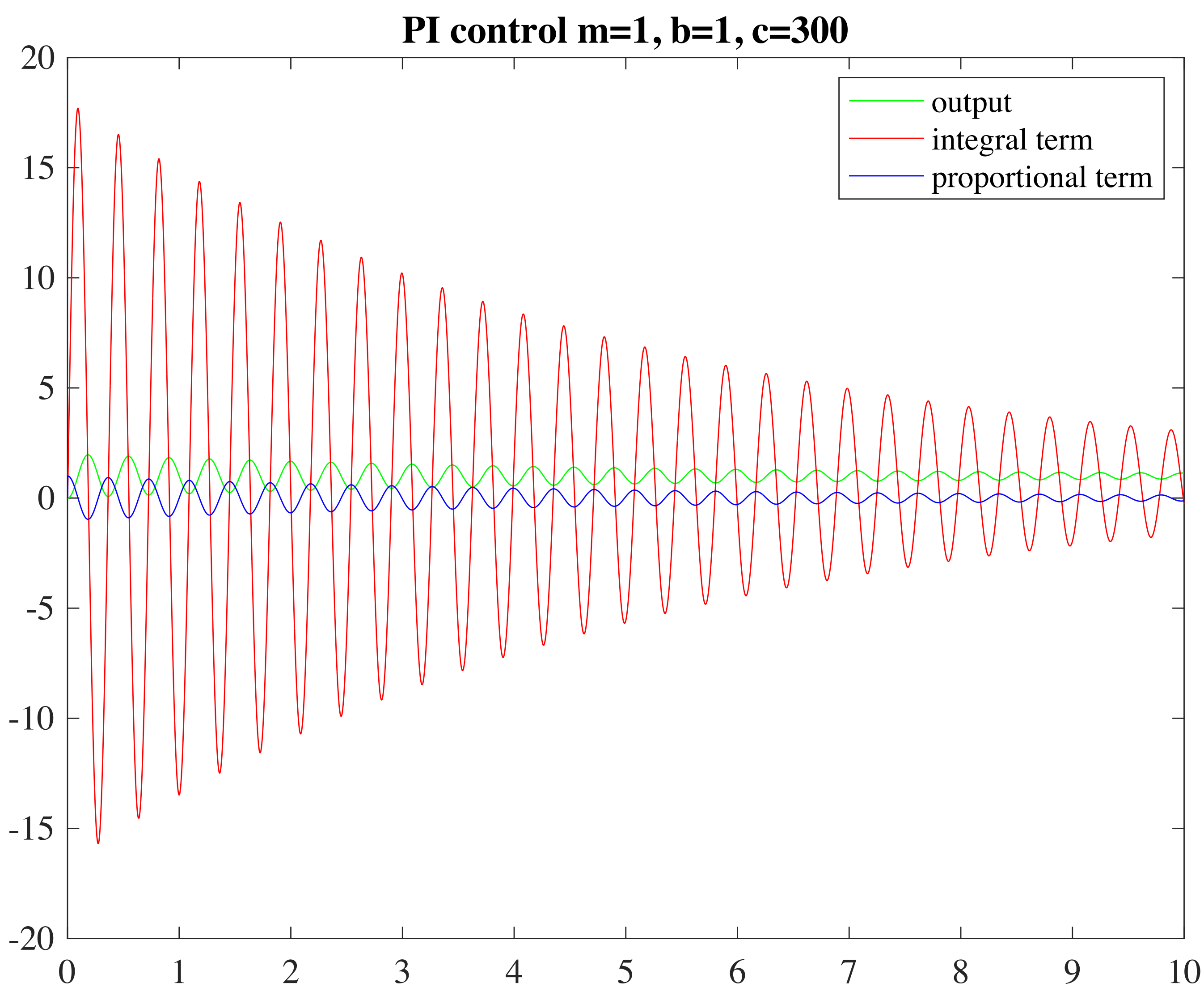
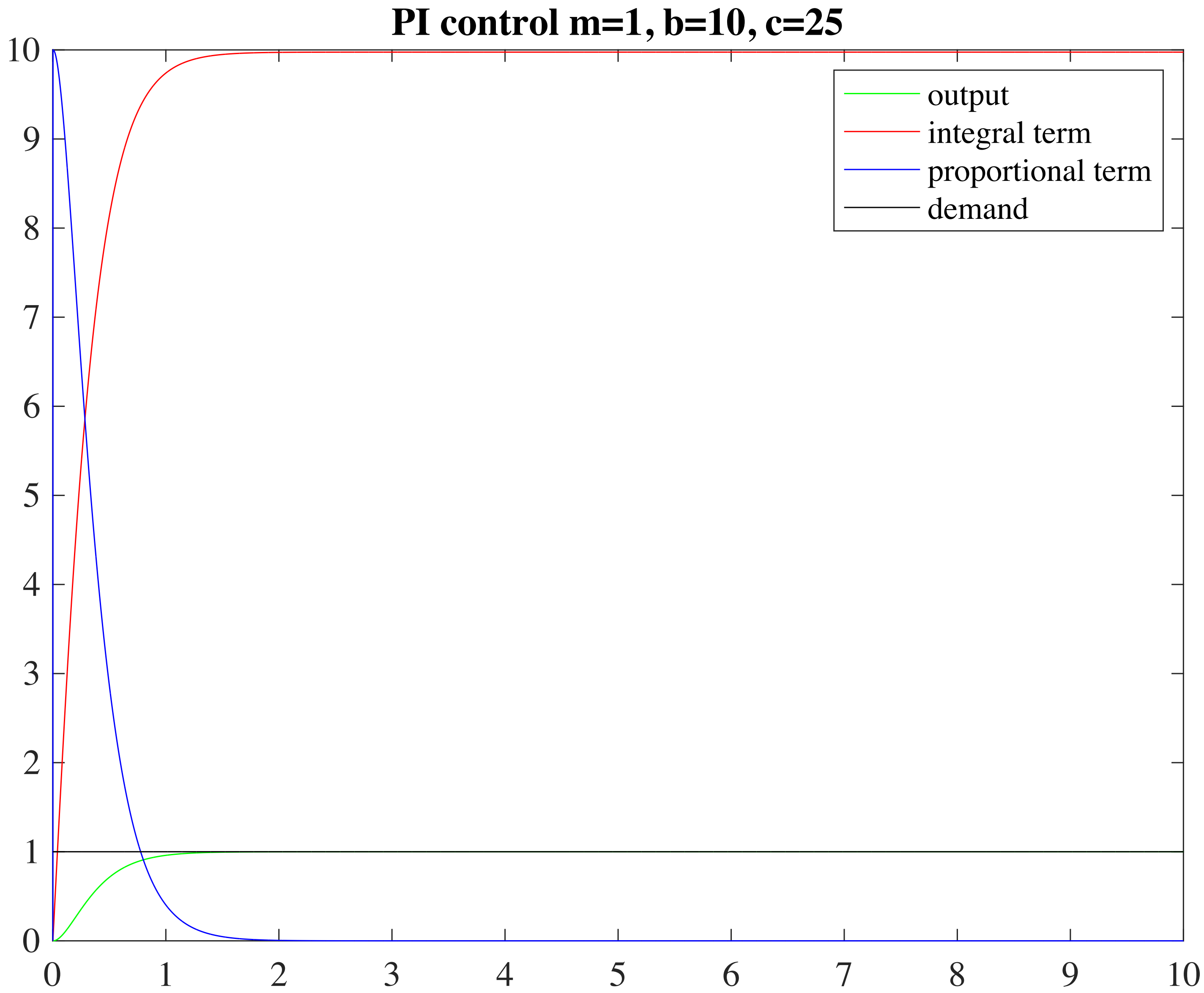

PI control m=1, b=1, c=300
output
integral term
proportional term
20
15
10
5
0
-5
-10
-15
-20
0
1
2
3
4
5
6
7
8
9
10

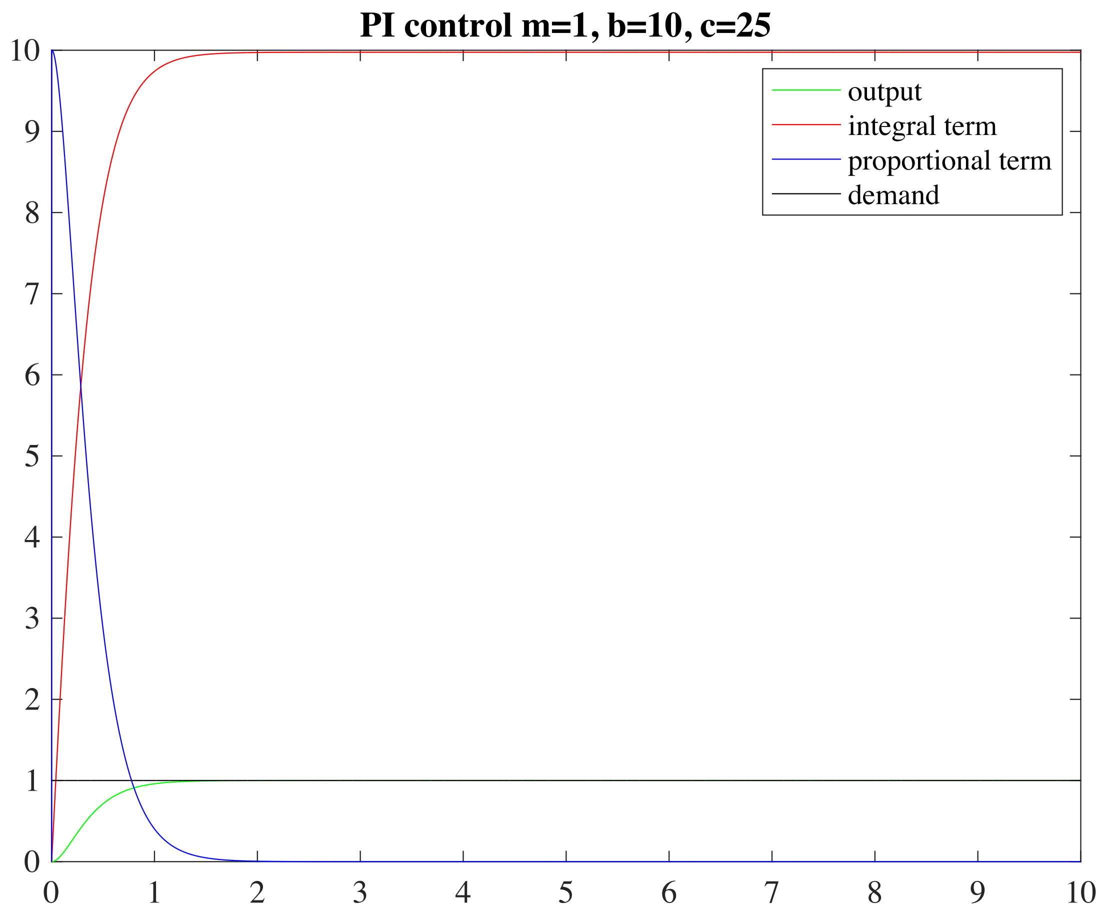


Cases:

I: two real roots; sum of exponentials

II: two copies of the same root

III: sinusoid with exponential amplitude



Cases:

b^2-4cm >0   (two real roots; sum of exponentials)

b^2-4cm=0 (two copies of the same root - this is known as critical damping)

b^2-4cm<0 (sinusoid with exponential amplitude)

Cases:

I: two real roots; sum of exponentials

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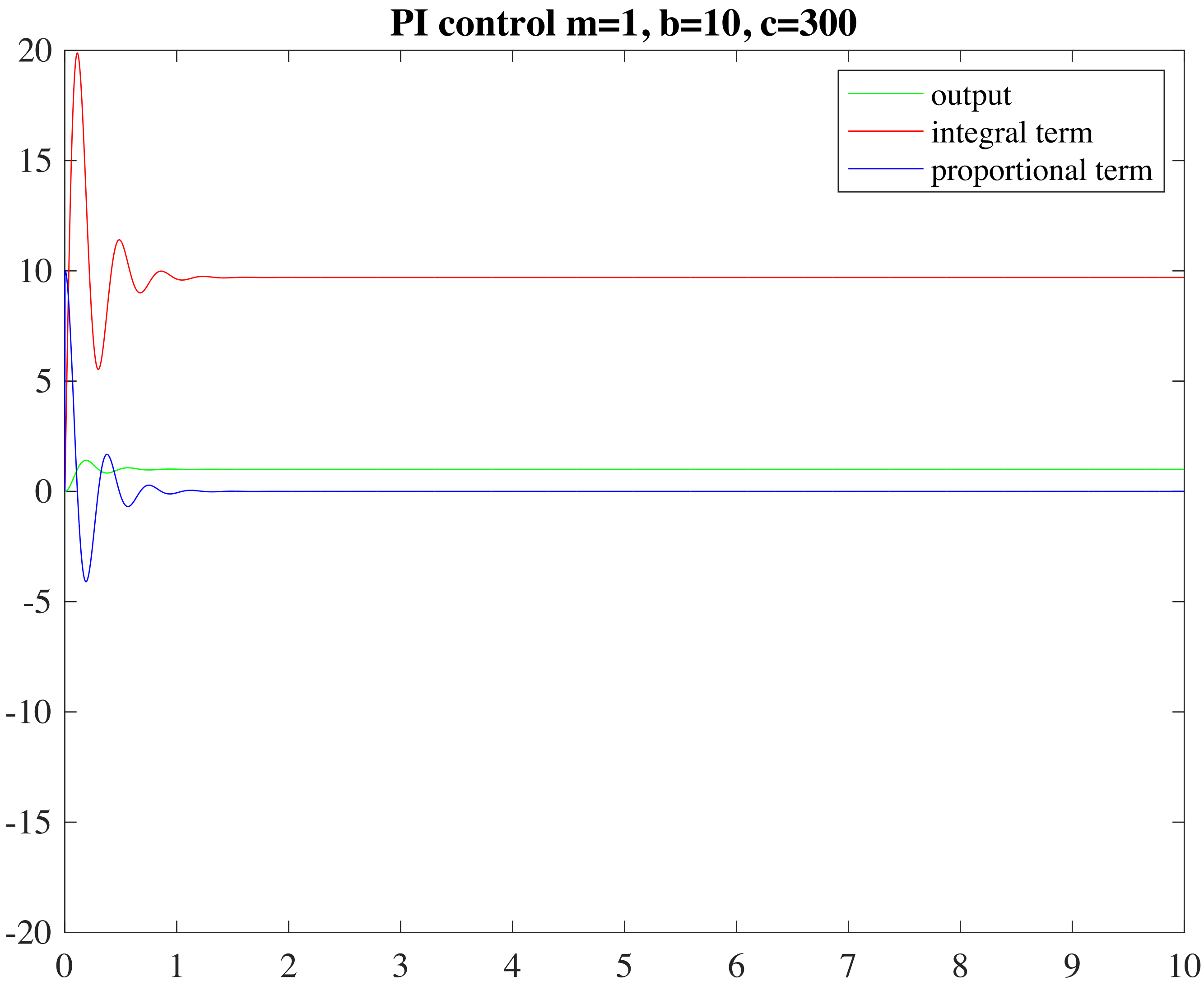
III: sinusoid with exponential amplitude



Cases:

I: two real roots; sum of exponentials

II: two copies of the same root

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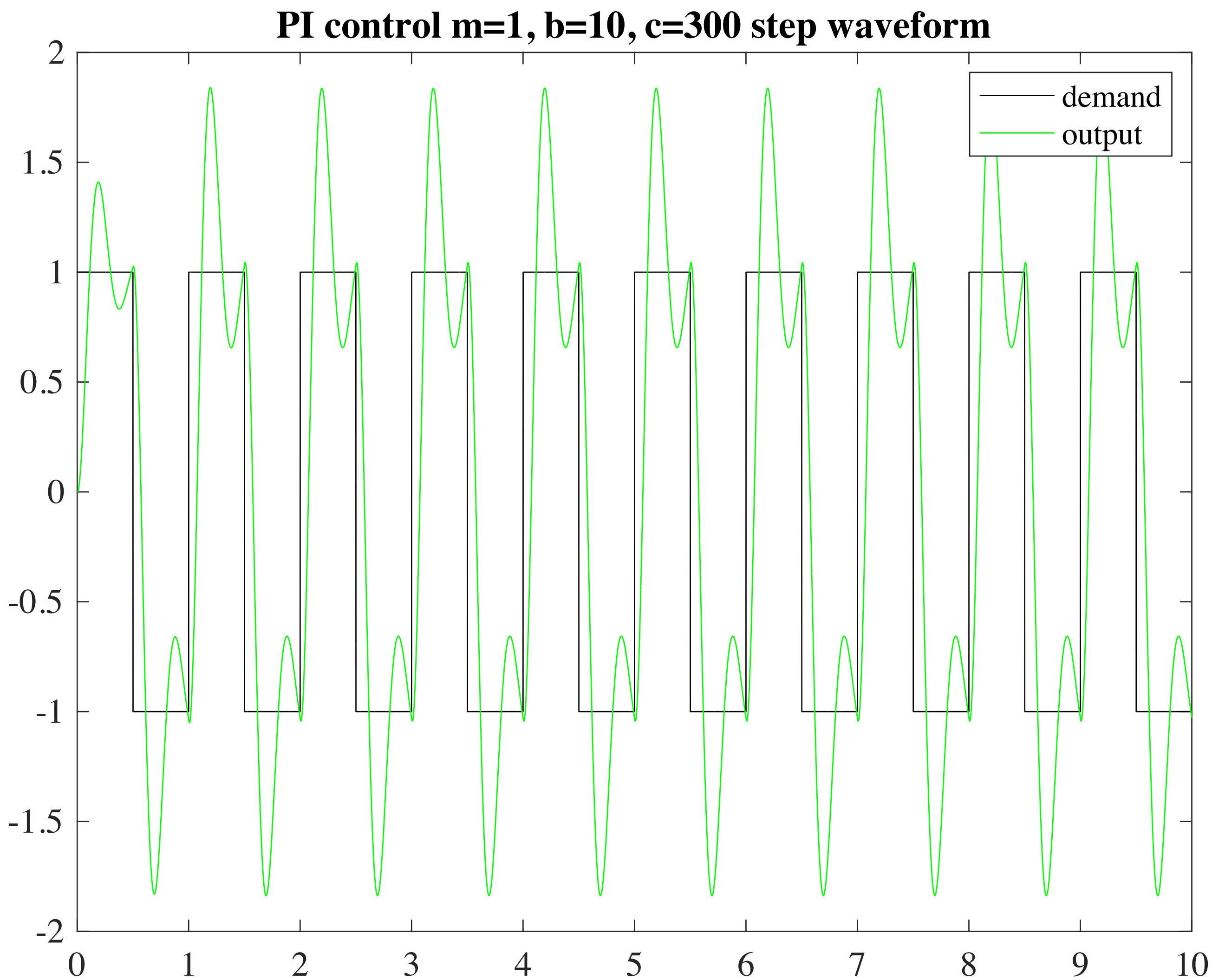
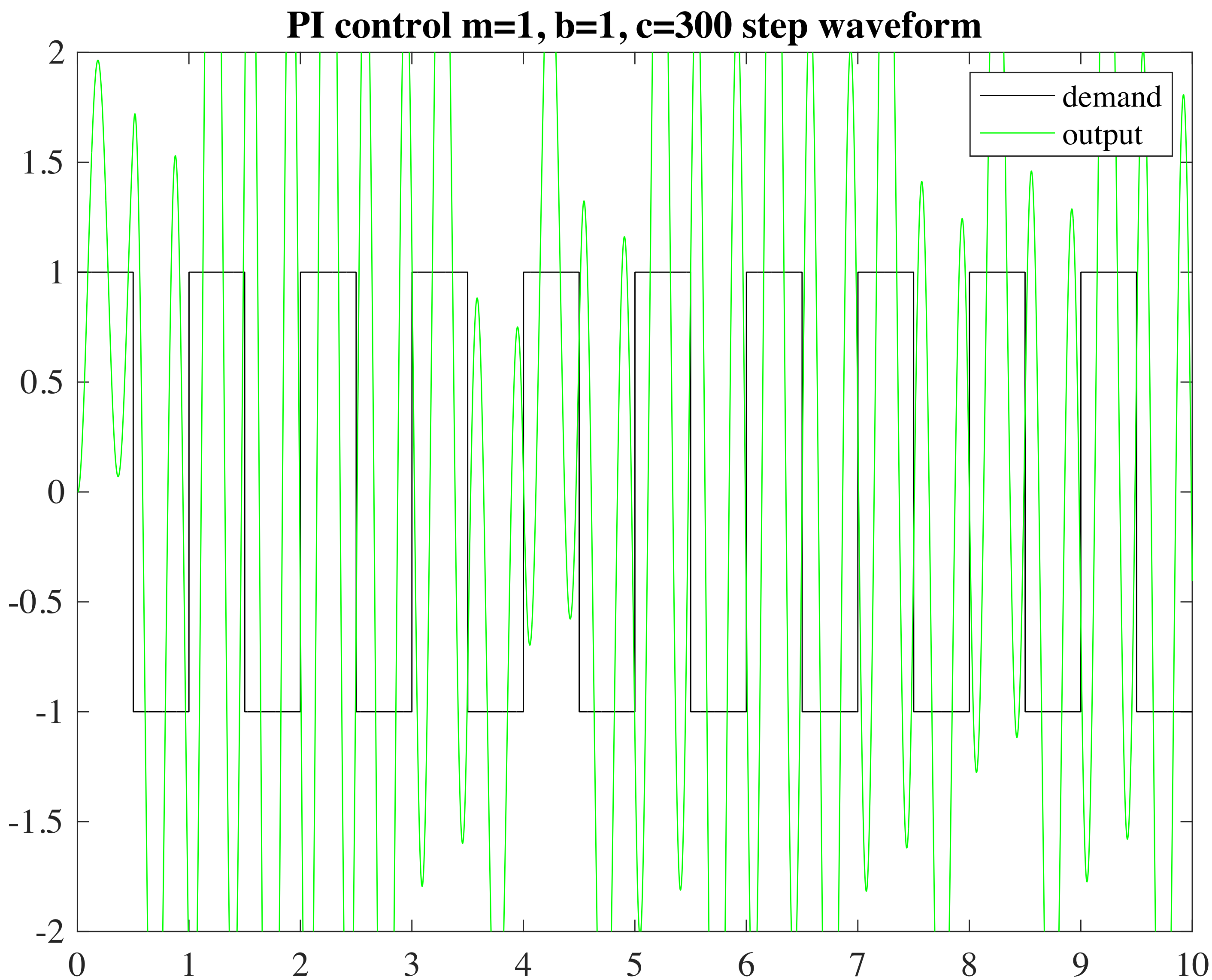

PI control m=1, b=10, c=300 step waveform
demand
output
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10


PI control m=1, b=1, c=300 step waveform
demand
output
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10

More on quadratic equations!

$$z^2 + \frac{b}{m}z + \frac{c}{m} = 0$$

$$z^2 + 2\zeta\omega z + \omega^2 = 0$$

Damping

$$z = -\omega\left(\zeta \pm i\sqrt{1-\zeta^2}\right)$$

Natural frequency

Critical damping occurs when there is a double root
equivalently when zeta=1
zeta <1 underdamped (soln. wobbles)
zeta>1 overdamped (slow rise time)

More on quadratic equations!

$$z^2 + 2\zeta\omega z + \omega^2 = 0$$

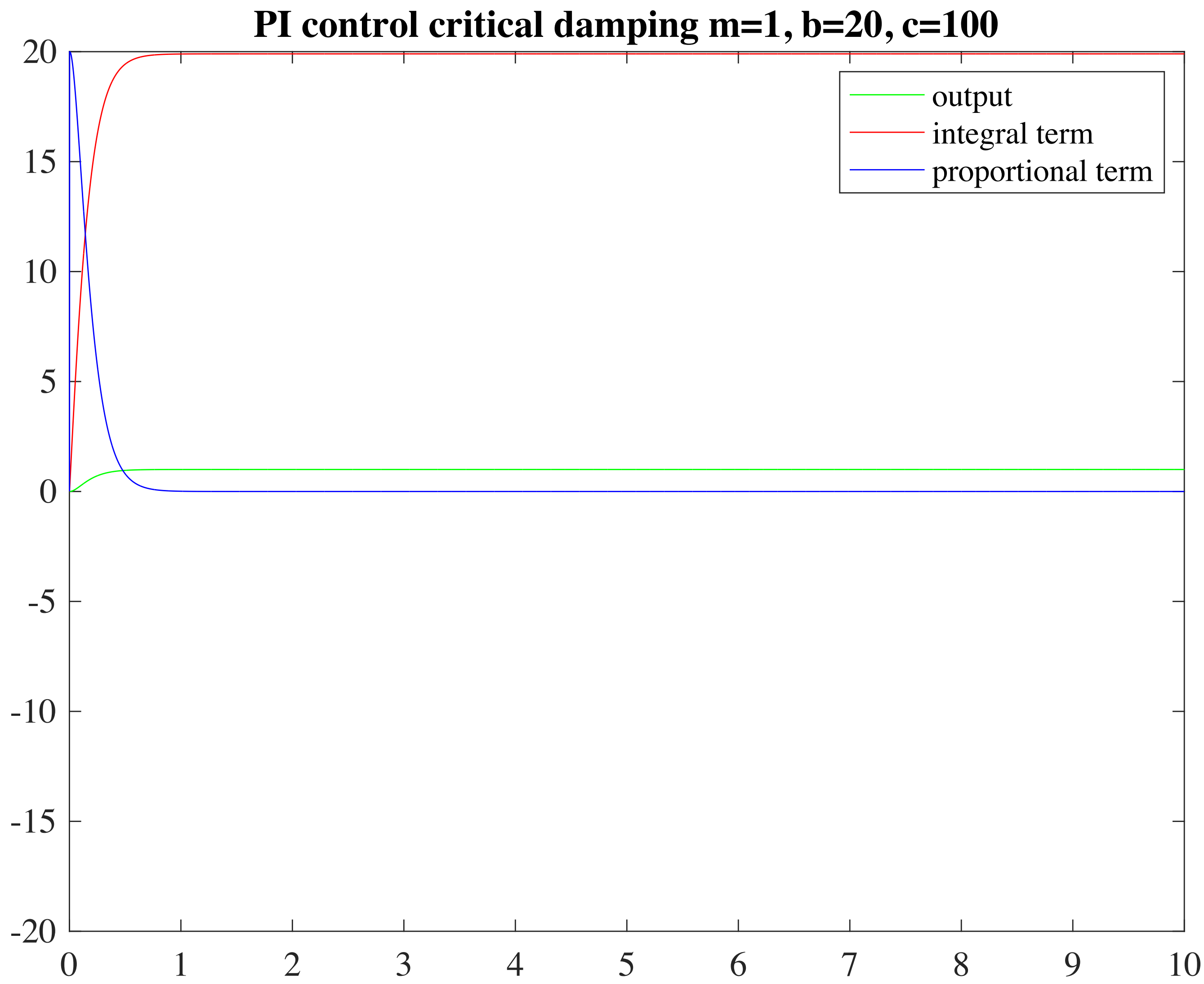
$$z = \underset{\substack{\uparrow \\ \text{Natural frequency}}}{-\omega} \left(\overset{\substack{\text{Damping} \\ \downarrow}}{\zeta} \pm i\sqrt{1-\zeta^2} \right)$$

Our equation

$$z^2 + \frac{b}{m} z + \frac{c}{m} = 0$$

$$\omega = \sqrt{\frac{c}{m}} \qquad \zeta = \frac{1}{2} \frac{b}{\sqrt{cm}}$$

Critical damping: $b = 2\sqrt{cm}$



PI control critical damping m=1, b=20, c=100
output
integral term
proportional term
20
15
10
5
0
-5
-10
-15
-20
0
1
2
3
4
5
6
7
8
9
10

A derivative term

- Issue:
  - may be hard to get fast rise time
    - big m requires big b for critical damping
  - this may be because we are feeding back the current error
- Idea:
  - predict future error
  - this is equivalent to feeding back some fraction of the derivative

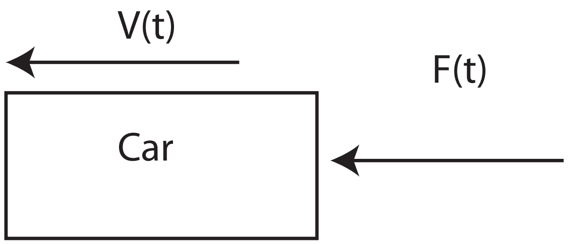
The most important slide

- A very high fraction of all controllers in the real world are:

$$Gx(t) = K_i \int_0^t x(u)du + K_p x(t) + K_d \frac{dx}{dt}$$

- PID controller

A more interesting plant



Force

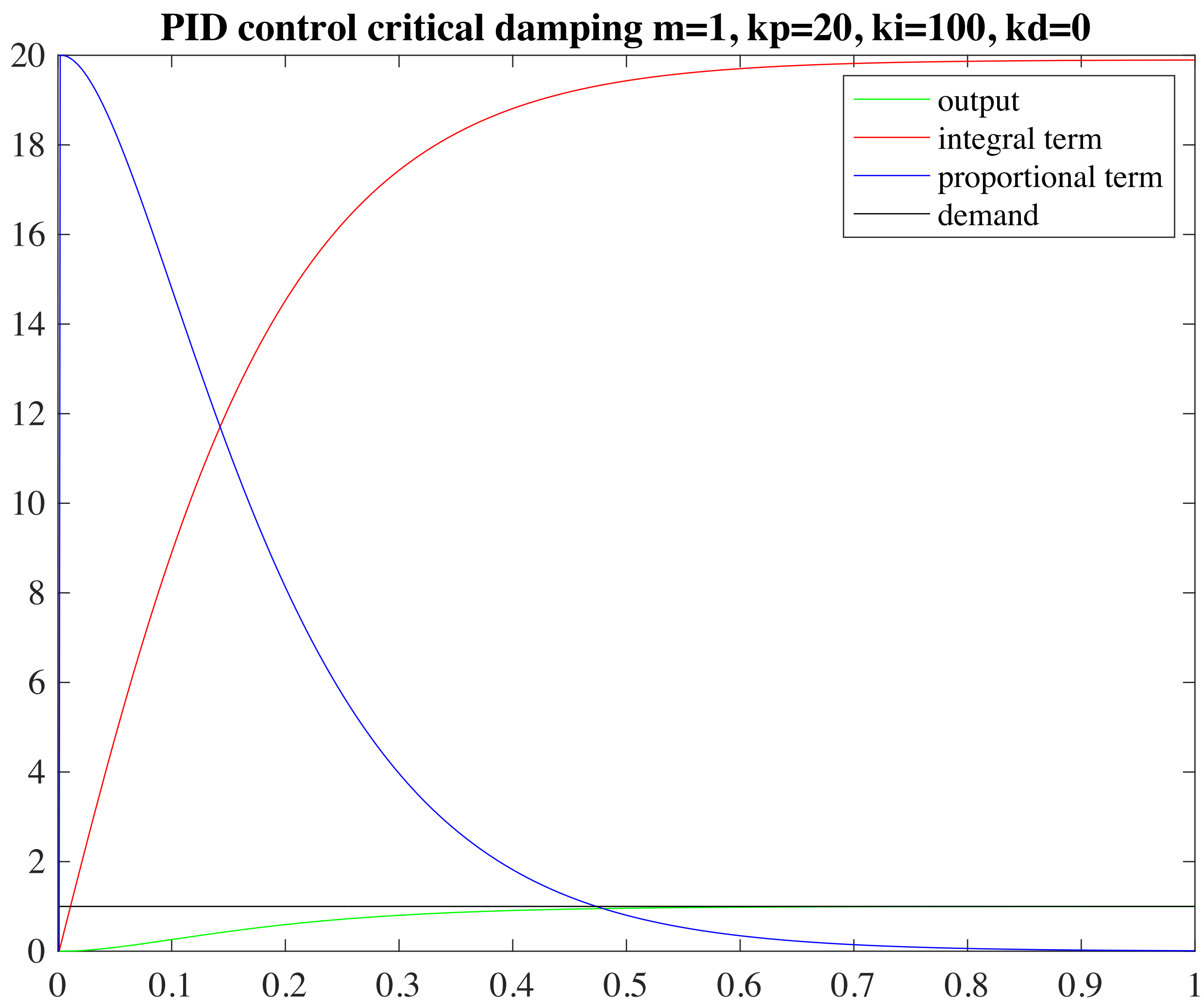
$$v(t) = v(0) + \int_0^t \frac{F(s)}{m} dt$$

- Apply a force to the car to control its velocity
  - eg braking

Input

Output

$$v(t) = \int_0^t \frac{F(s)}{m} dt$$


PID control critical damping m=1, kp=20, ki=100, kd=0
output
integral term
proportional term
demand
0
2
4
6
8
10
12
14
16
18
20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Proportional-Integral-Derivative (PID) control

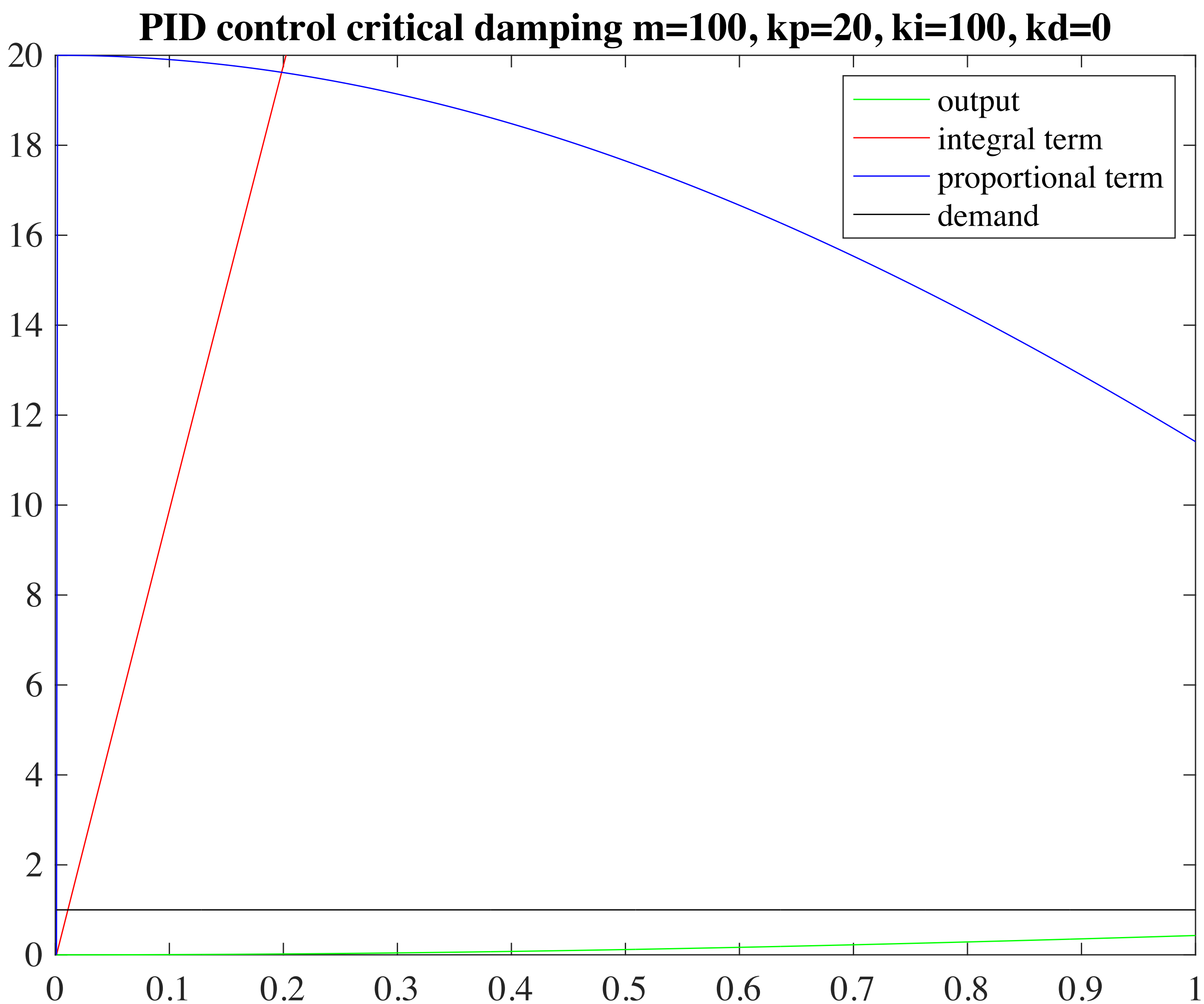
Thrash through math of PI slide, and end up with:

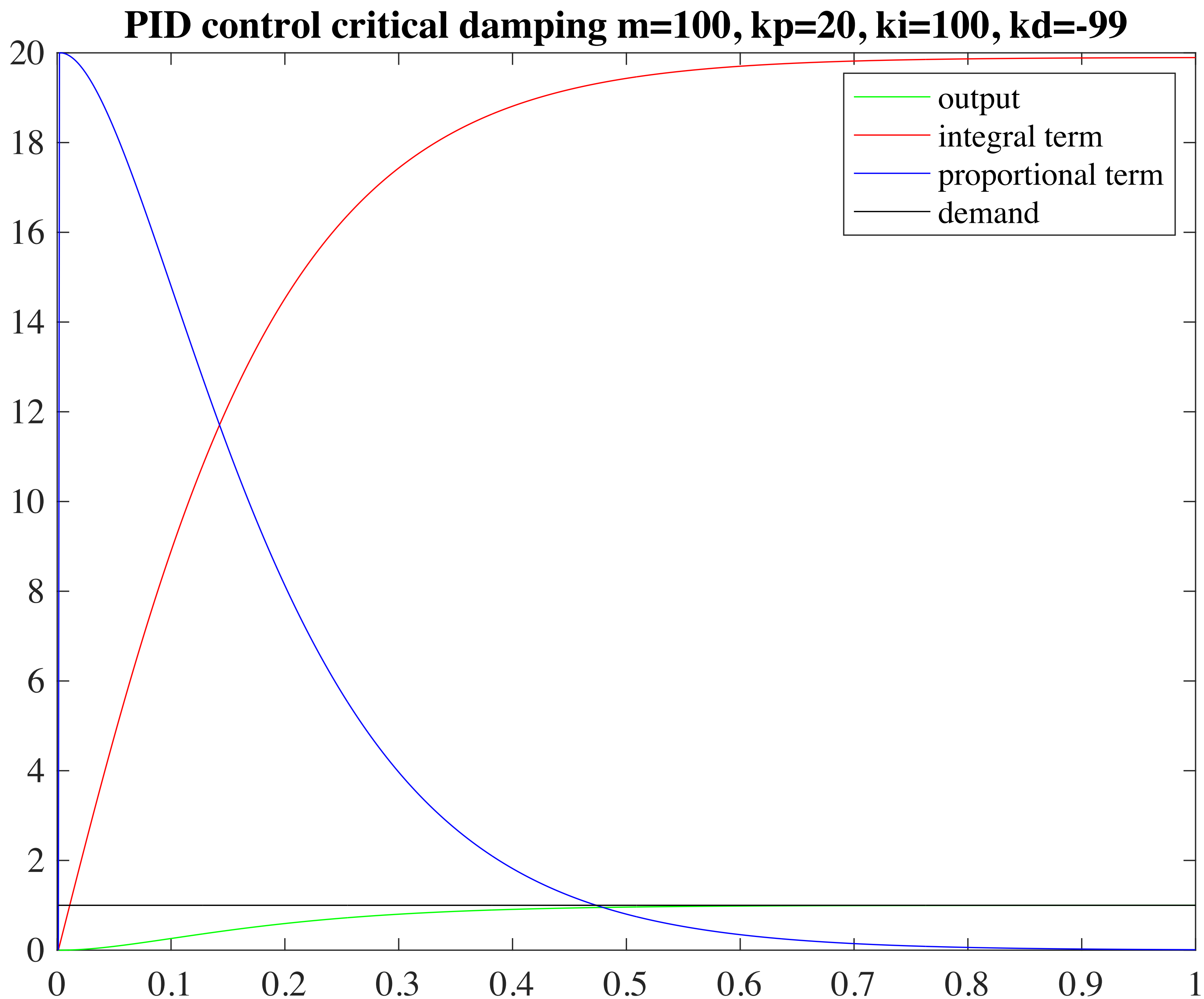
$$\frac{d^2o}{dt^2} + \frac{K_p}{m + K_d}\frac{do}{dt} + \frac{K_i}{m + K_d}o = \frac{K_i}{m + K_d}$$

Compare to:

$$\frac{d^2o}{dt^2} + \frac{b}{m}\frac{do}{dt} + \frac{c}{m}o(t) = \frac{c}{m}$$

Kd makes the mass look smaller!


PID control critical damping m=100, kp=20, ki=100, kd=0
output
integral term
proportional term
demand
0
2
4
6
8
10
12
14
16
18
20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

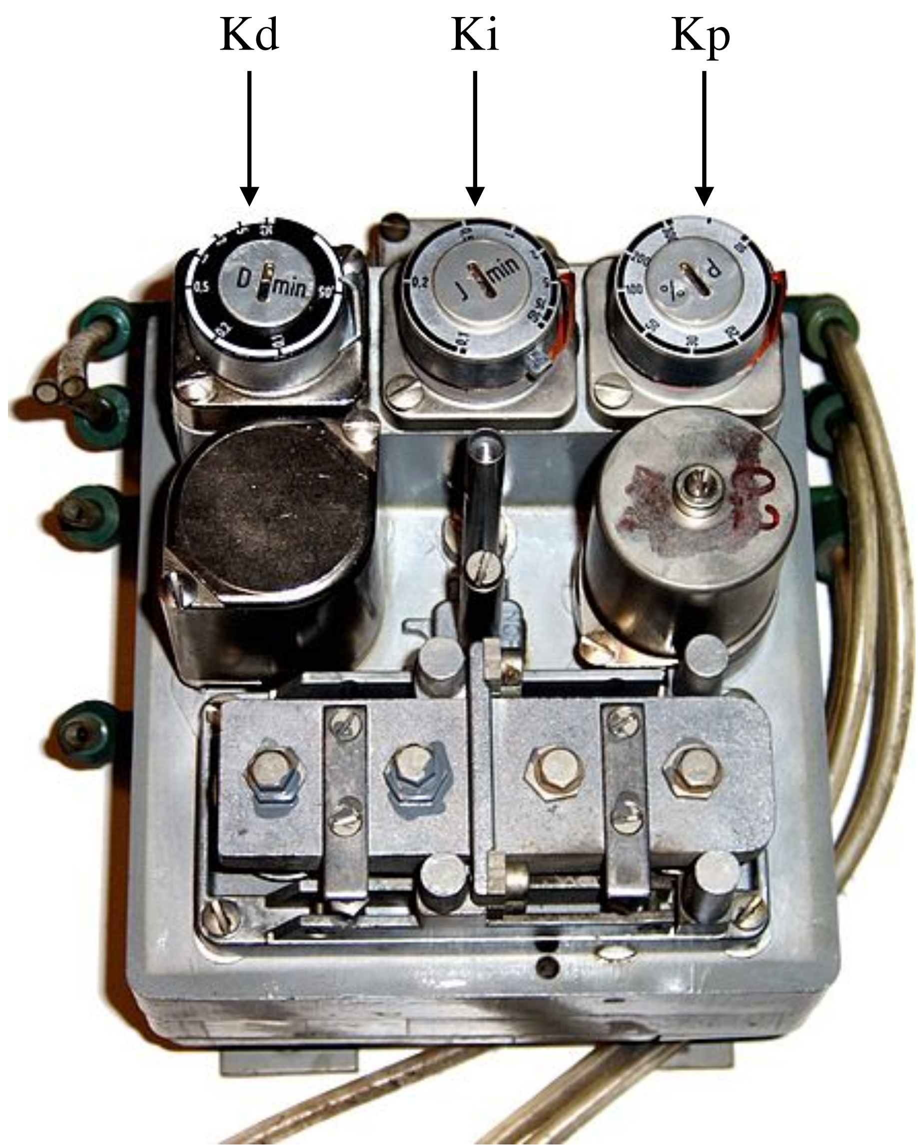

PID control critical damping m=100, kp=20, ki=100, kd=-99
output
integral term
proportional term
demand
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

In case you missed it…

- A very high fraction of all controllers in the real world are:

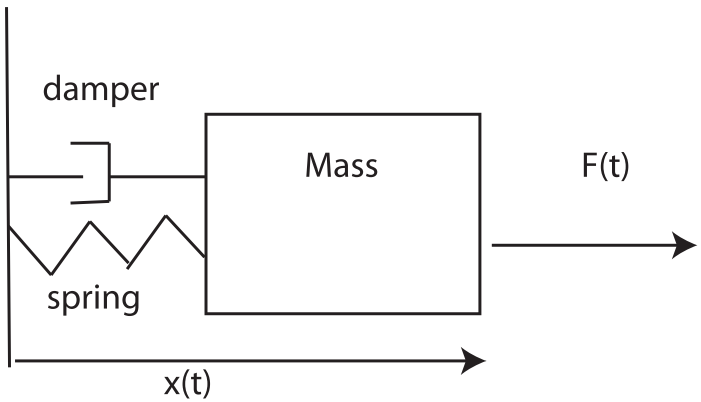
$$Gx(t) = K_i \int_0^t x(u)du + K_p x(t) + K_d \frac{dx}{dt}$$

- PID controller


Kd
Ki
Kp


radix
X96P
249.8
250.0
33
1
2
3
4
RLY
1
2
3
4
SP
Rx
Tx
MAN
A/M

Yet more interesting plant



Apply a force to the mass,
want to control its position.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F$$

Proportional-Integral-Derivative (PID) control

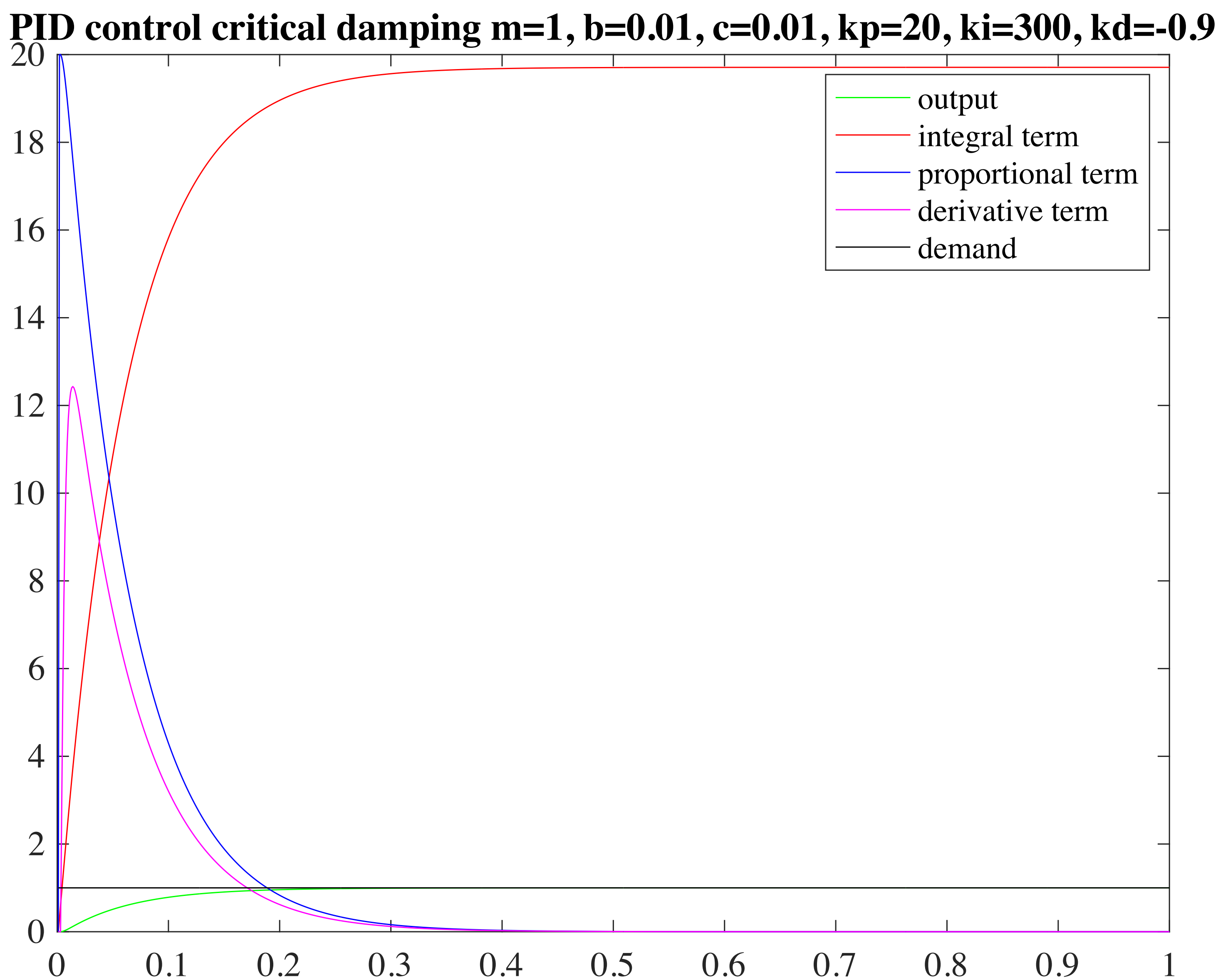
Thrash through math of past slides, and end up with:

$$\frac{d^2 o}{dt^2} + \frac{K_p + b}{m + K_d}\frac{dx}{dt} + \frac{K_i + k}{m + K_d}x = \frac{K_i + k}{m + K_d}$$

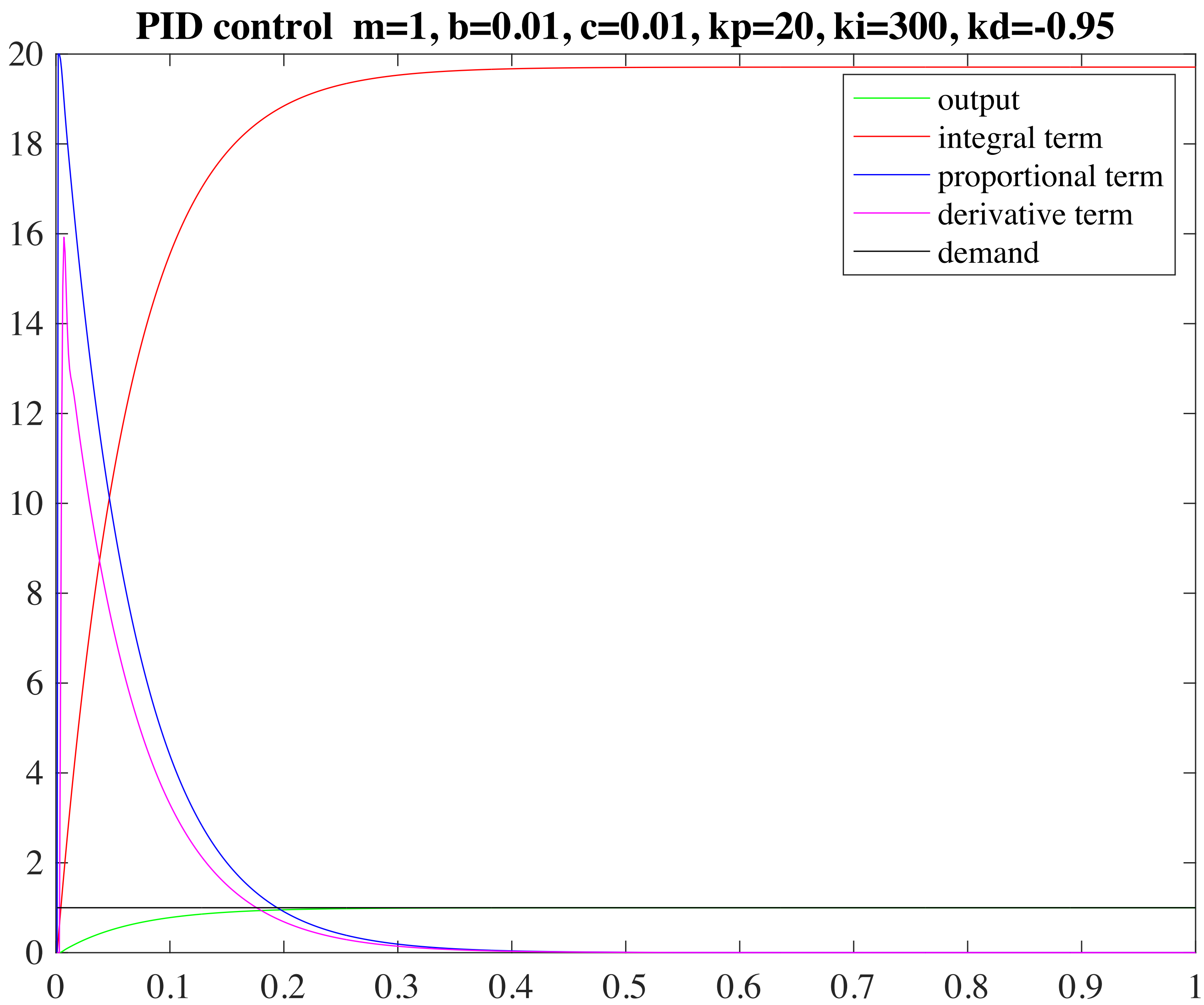
Compare to:

$$m\frac{d^2 x}{dt^2} + b\frac{dx}{dt} + kx = F$$

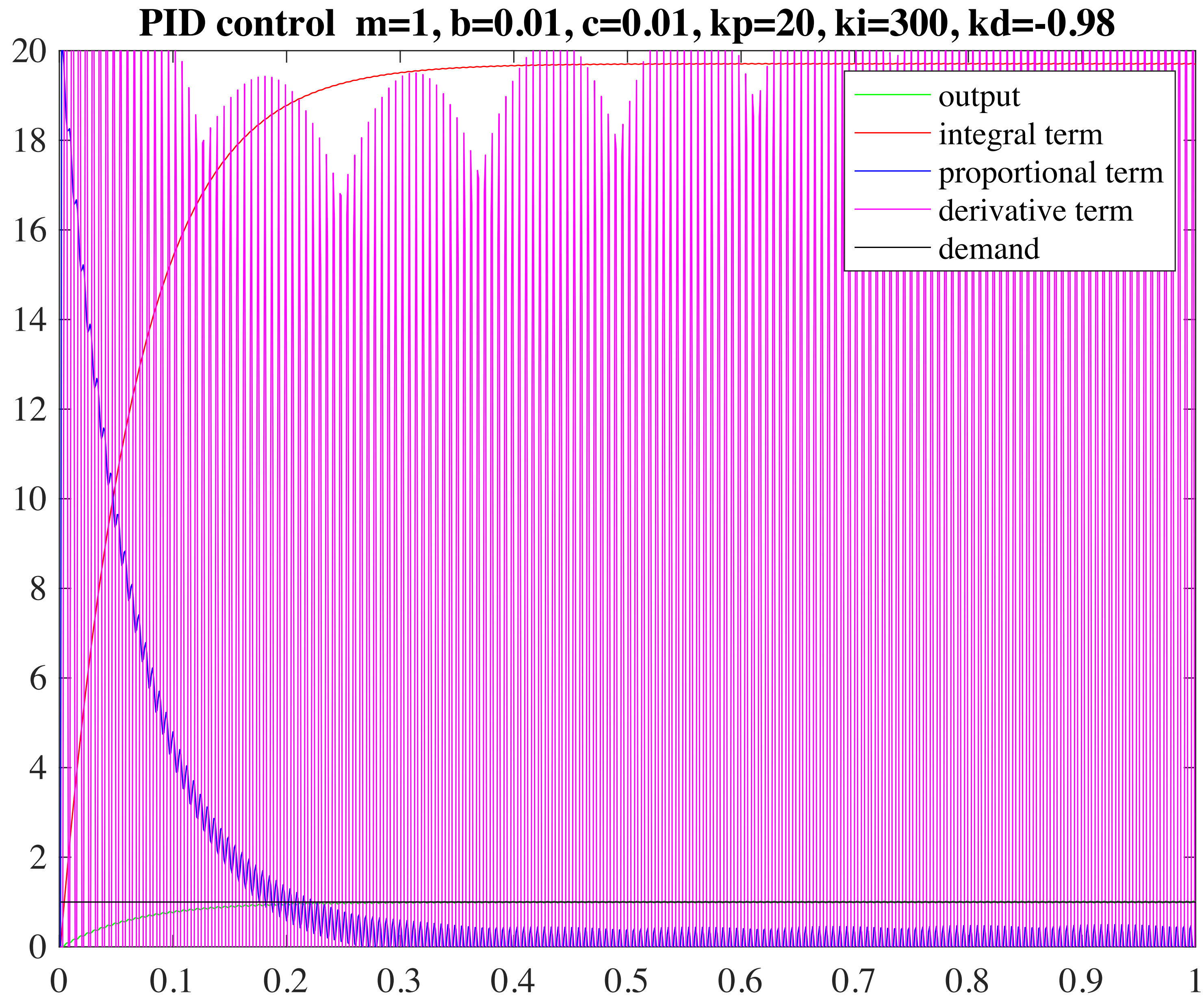
Kd makes the mass look smaller!   Kp changes the damping constant! Ki changes the spring constant!



PID control critical damping m=1, b=0.01, c=0.01, kp=20, ki=300, kd=-0.9
output
integral term
proportional term
derivative term
demand
0
2
4
6
8
10
12
14
16
18
20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



PID control m=1, b=0.01, c=0.01, kp=20, ki=300, kd=-0.95
output
integral term
proportional term
derivative term
demand
0
2
4
6
8
10
12
14
16
18
20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


PID control m=1, b=0.01, c=0.01, kp=20, ki=300, kd=-0.98
output
integral term
proportional term
derivative term
demand
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Proportional-Integral-Derivative (PID) control

Thrash through math of past slides, and end up with:

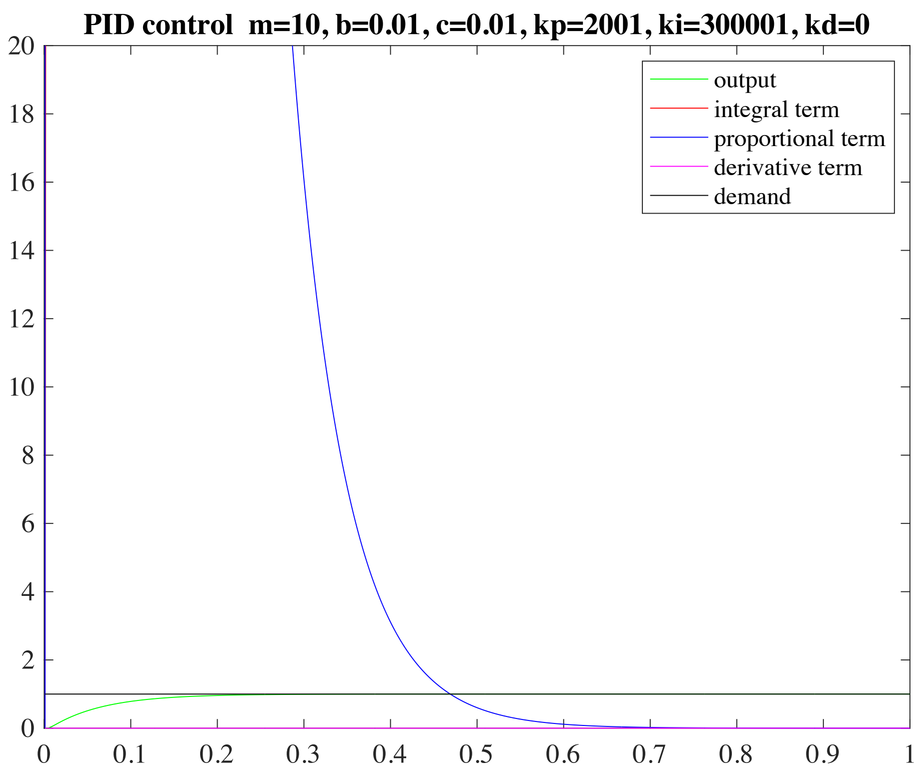
$$\frac{d^2o}{dt^2} + \frac{K_p + b}{m + K_d}\frac{dx}{dt} + \frac{K_i + k}{m + K_d}x = \frac{K_i + k}{m + K_d}$$

Compare to:

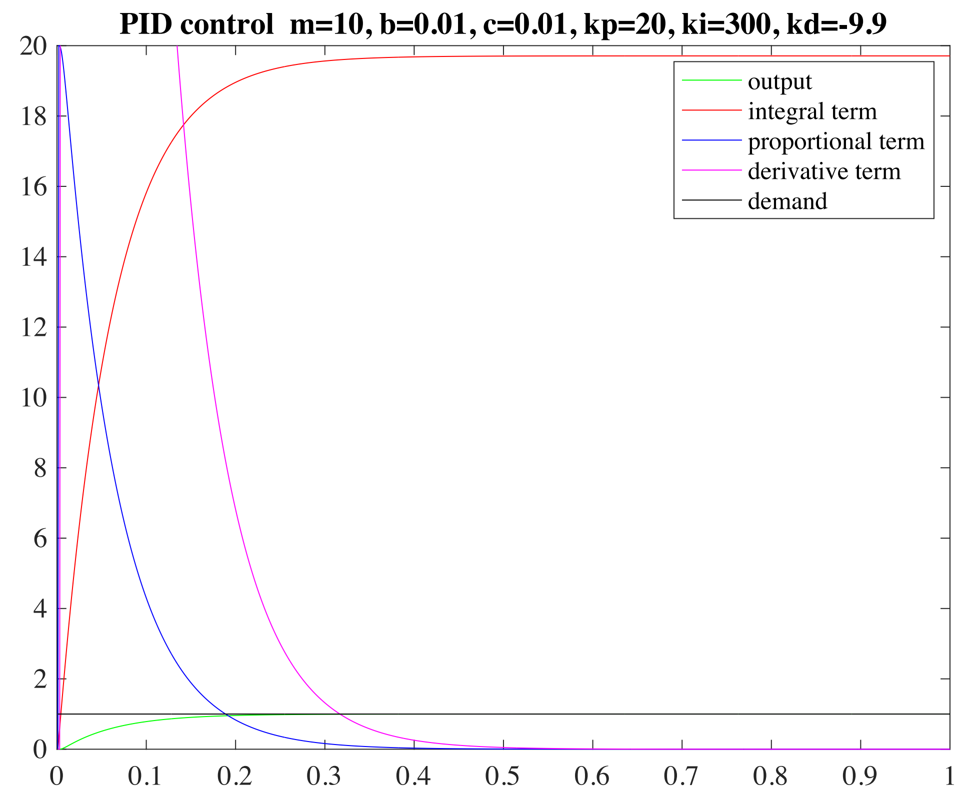
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F$$

Kd makes the mass look smaller! Kp changes the damping constant! Ki changes the spring constant!

Examples

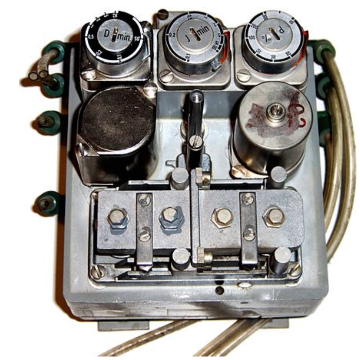


PID control m=10, b=0.01, c=0.01, kp=2001, ki=300001, kd=0
output
integral term
proportional term
derivative term
demand
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



PID control m=10, b=0.01, c=0.01, kp=20, ki=300, kd=-9.9
output
integral term
proportional term
derivative term
demand
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

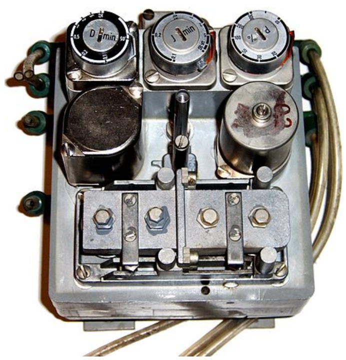
Tuning



- Usually, you don't know the plant and can't do the math
- Powerful rule of thumb (manual tuning)

If the system must remain online, one tuning method is to first set K_i and K_d values to zero. Increase the K_p until the output of the loop oscillates, then the K_p should be set to approximately half of that value for a "quarter amplitude decay" type response. Then increase K_i until any offset is corrected in sufficient time for the process. However, too much K_i will cause instability. Finally, increase K_d, if required, until the loop is acceptably quick to reach its reference after a load disturbance. However, too much K_d will cause excessive response and overshoot. A fast PID loop tuning usually overshoots slightly to reach the setpoint more quickly; however, some systems cannot accept overshoot, in which case an overdamped closed-loop system is required, which will require a K_p setting significantly less than half that of the K_p setting that was causing oscillation.

Tuning, II



Effects of *increasing* a parameter independently[22][23]

Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	Decrease	Increase	Small change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor change	Decrease	Decrease	No effect in theory	Improve if K_d small

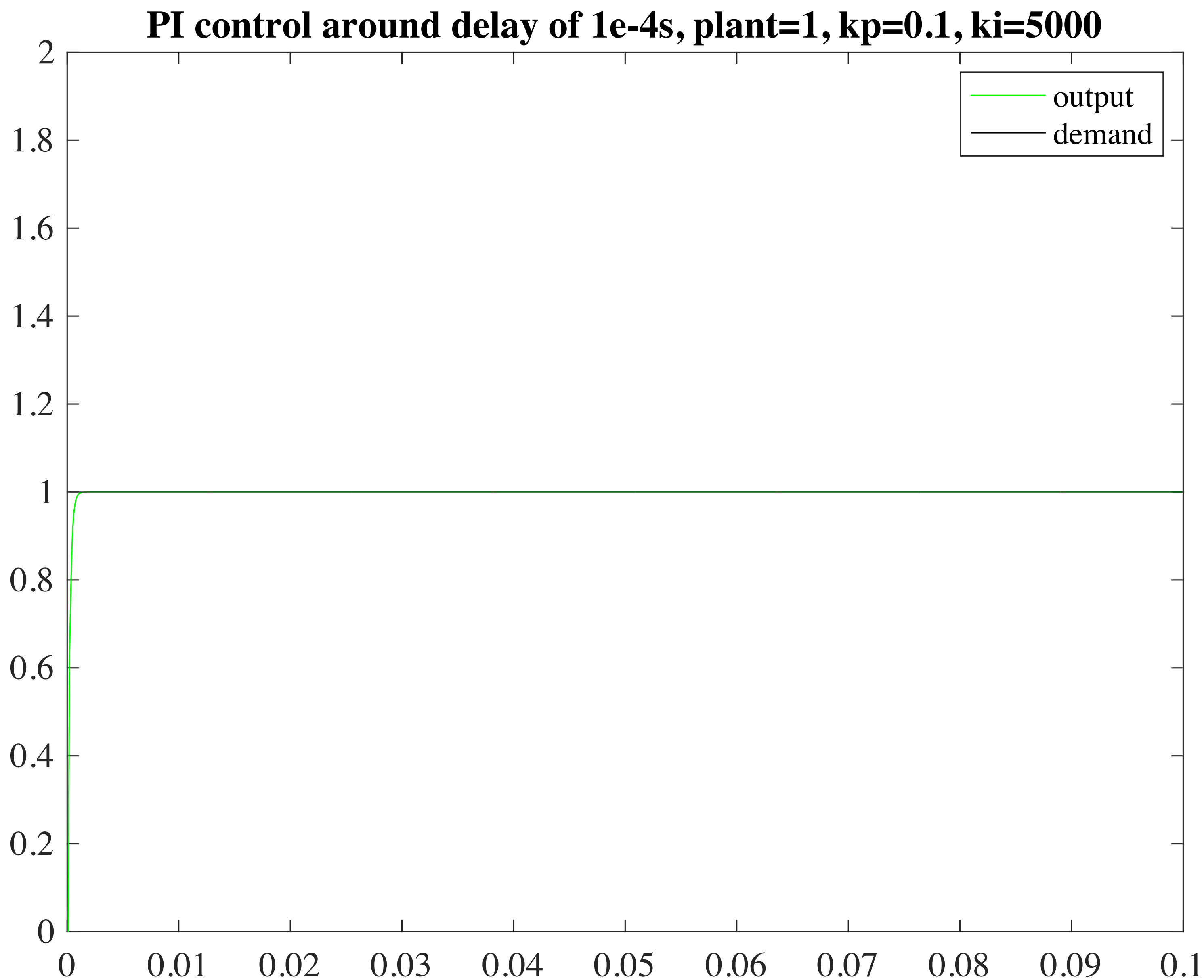
Kd = 0 for about 75% of deployed systems

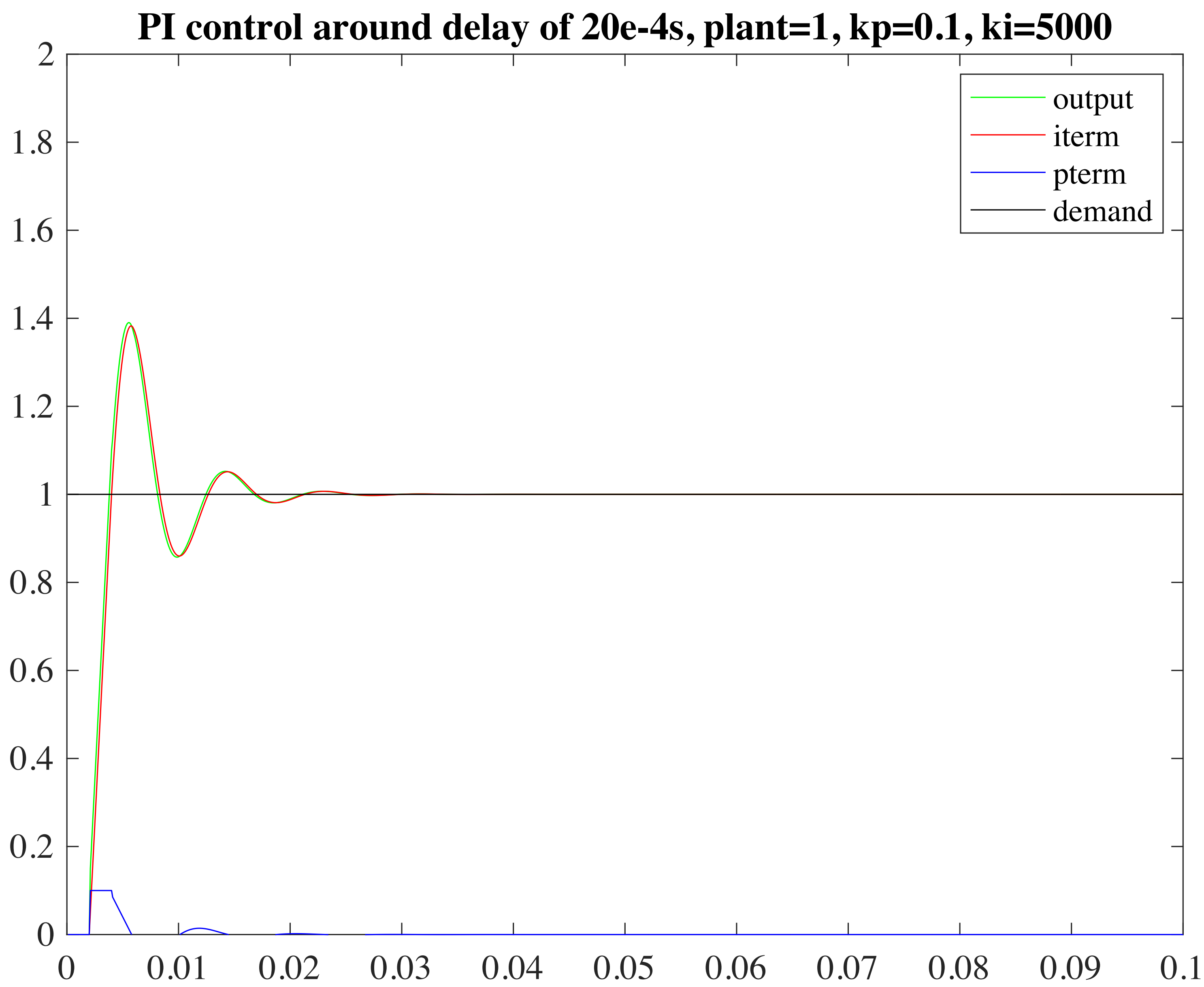
Question

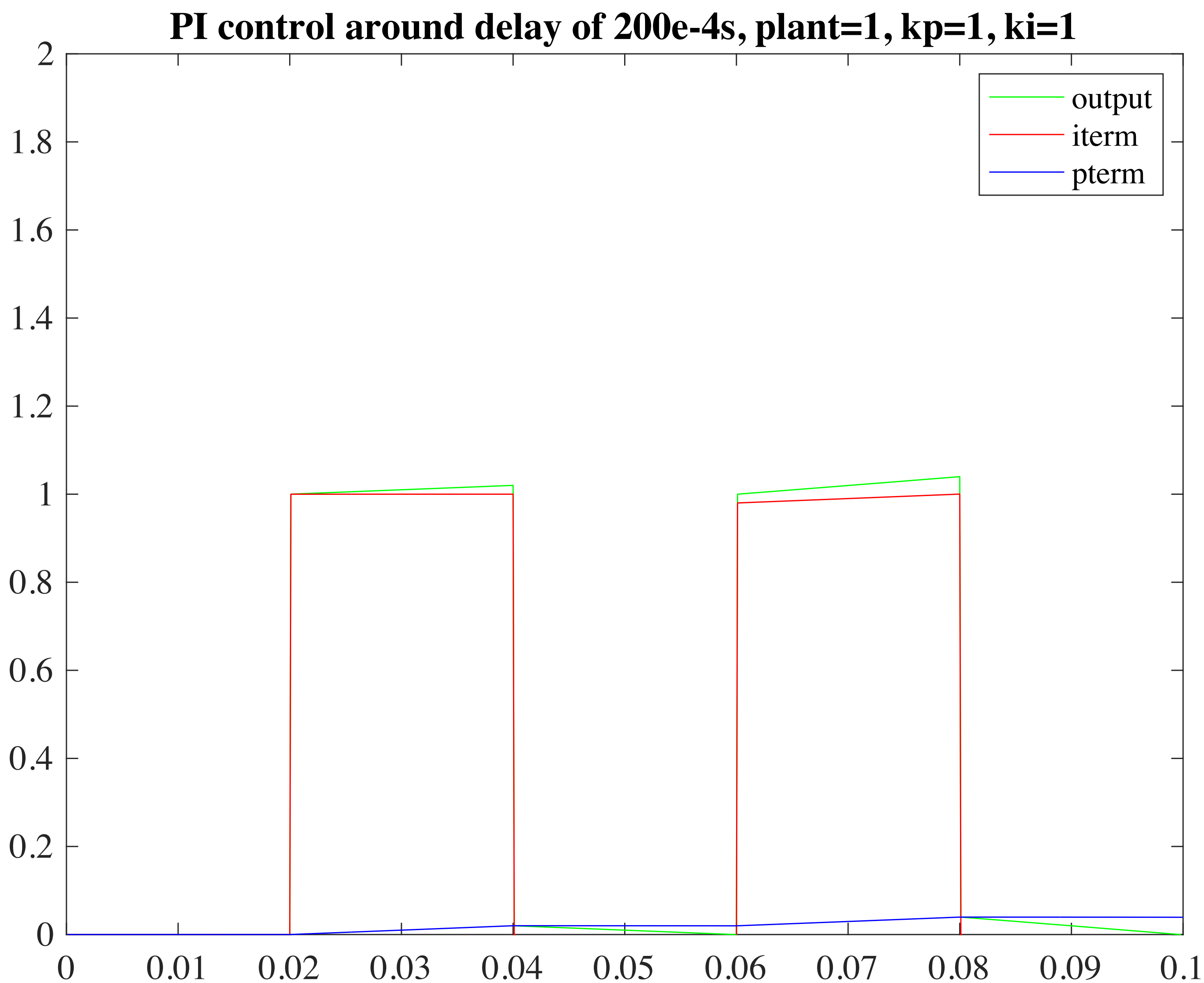
- Q: Why does the Federal Reserve Bank not just:
  - stick a PID controller on the inflation rate
    - input: desired inflation
    - output: base rate
  - and forget it?
- A: (obvious) it wouldn't work
  - otherwise they'd be doing it
- Q: Why wouldn't it work?

Stability and oscillation (rough)

- Linear systems can clearly oscillate
  - generally, too big a Kp or Kd can cause problems
- Nonlinearities can easily cause oscillations
- Delays cause oscillations


PI control around delay of 1e-4s, plant=1, kp=0.1, ki=5000
output
demand
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1


PI control around delay of 20e-4s, plant=1, kp=0.1, ki=5000
output
iterm
pterm
demand
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

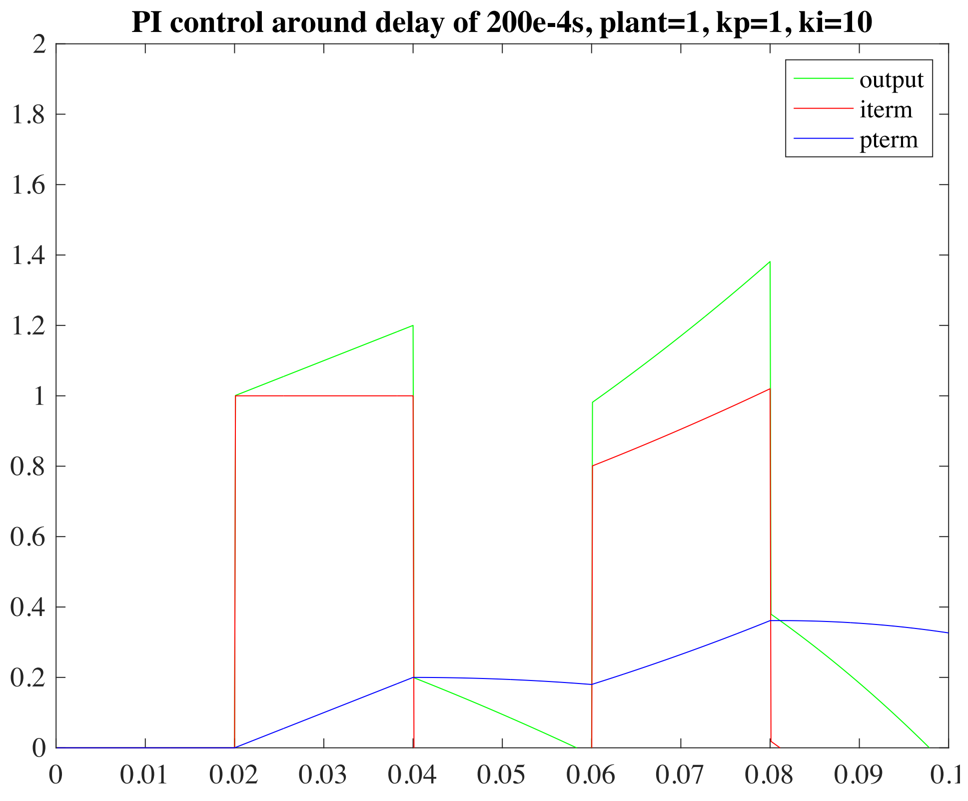


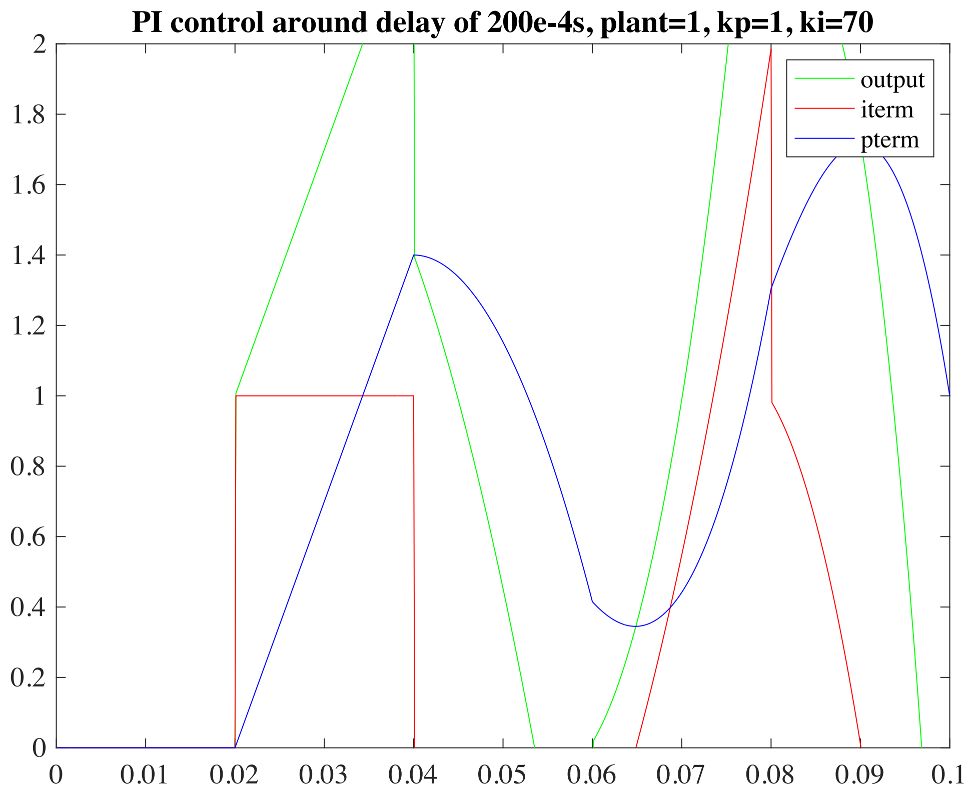
Demand is a step - this should look unpromising…

NOTICE Plant is 1 (really simple)

Unrecoverable

Pushing up Ki speculatively doesn't help





Ideas

- Plant/process
- control
- Open vs closed loop
- stability
- Linear vs non-linear
- Simplest linear feedback control
    - x constant
    - with derivative term
    - large gains can cause instability
    - steady state error is a problem
- Delay is a problem
- non-linearities can create excitement

Ideas

- PID control
  - standard procedure
    - (there are tons in the car software)
  - P controls; I reduces steady state error; D increases response speed
  - Straightforward tuning procedure
    - (see software example)

The next few lectures

- You can stick a PID controller on
  - speed
  - distance
  - steering angle
  - etc.
- So you can cause the vehicle to follow a given path
  - at a given speed, etc.

- Where does the path come from?
  - simple planning (next)
- How do we know where obstacles are?
  - simple localization from point clouds (after that)