Back to MRF's

recall that

$$\max_x \quad \frac{x^T A x}{2} + b^T x$$
$$\text{st} \quad x \in \{0,1\}$$

is easy l.p. if $a_{ij} > 0$

for a 2 label mrf, write Energy.

$$E = \sum_{i \in \text{sites}} E_i(l_i) + \sum_{ij \in \text{sites}} E_{ij}(l_i, l_j)$$

But 2 labels a, b

write $x_i = \begin{cases} 0 & l_i = a \\ 1 & l_i = b \end{cases}$

then

$$E = \sum_{i \in \text{Sites}} \left[E_i(a).(1-x_i) + E_i(b)\, x_i \right]$$
$$+ \sum_{ij \in \text{Sites}} \left[\begin{array}{l} E_{ij}(a,a)\,(1-x_i)(1-x_j) \\ + E_{ij}(a,b)\,(1-x_i)\,x_j \\ + E_{ij}(b,a)\,x_i\,(1-x_j) \\ + E_{ij}(b,b)\,x_i\,x_j \end{array} \right]$$

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But 2 labels a, b $\quad l_i = a$

write $x_i = \begin{cases} 0 & l_i = a \\ 1 & l_i = b \end{cases}$

then

$$E = \sum_{i \in \text{Sites}} \left[E_i(a).(1-x_i) + E_i(b)\, x_i \right]$$

$$+ \sum_{ij \in \text{Sites}} \begin{bmatrix} E_{ij}(a,a)\,(1-x_i)(1-x_j) \\ + E_{ij}(a,b)\,(1-x_i)\,x_j \\ + E_{ij}(b,a)\,x_i(1-x_j) \\ + E_{ij}(b,b)\,x_i\,x_j \end{bmatrix}$$

we want to <u>min</u> this

$$\max_x \quad x^T A x + b^T x$$
$$\text{st} \quad x \in \{0, 1\}$$

$$|||$$

$$\min_x \quad - x^T A x - b^T x$$
$$\text{st} \quad x \in \{0, 1\}.$$

So easy energy $\Rightarrow$

$$E_{ij}(b, b) + E_{ij}(a, a) < E_{ij}(a, b) + E_{ij}(b, a)$$

(better to <u>agree</u> than disagree).

And this is a cut problem

$\Rightarrow$ dual to max-flow

$\Rightarrow$ flow cut alg.

What if there are more than 2 labels?

$$\min_x \; -x^T A x - b^T x.$$

st $\quad x \in \{0,1\}^n$

and $\quad Lx = b$

(use the x to pick the label).

Two important strategies:

α - β swap.

- take points w/ label α, β ONLY
- now either
  - leave
  - swap.

α - expansion

- take points w/ label NOT α
- now either
  - leave
  - make α.

α - β swap:

~~ignore~~

- ~~the~~ x_i $\leftarrow$ one entry for each node labelled either α or β

$$x_i = \begin{cases} 1 & \cancel{\text{leave}} \ \alpha \\ 0 & \cancel{\text{swap}} \ \beta \end{cases}$$ — notice $\equiv$ leave, swap.

energy:

- unary terms for non (α, β) nodes are fixed
- binary " " fixed
- binary terms for 1 non(α, β), 1 $(\alpha - \beta)$ become unary terms
- binary terms for (α, β) terms are what matter.

Energy:

$$E = \begin{bmatrix} \text{unary, binary terms} \\ \text{for non-}\alpha,\beta \end{bmatrix}$$

$$+ \sum_{i \in \alpha,\beta \text{ sites}} \left\{ E_i(\alpha)\, x_i + E_i(\beta)(1-x_i) \right\}$$

$$+ \sum_{j \in \alpha,\beta \text{ sites}} \left\{ \left[\sum_k E_{kj}(l_k,\alpha)\right] x_i + \left[\sum_k E_{kj}(l_k,\beta)\right](1-x_i) \right\}$$

$$+ \sum_{\substack{ij \in \alpha,\beta \\ \text{sites}}} \left[E_{ij}(\alpha,\alpha)\, x_i x_j + E_{ij}(\alpha,\beta)\, x_i (1-x_j) + E_{ij}(\beta,\alpha)(1-x_i)\, x_j + E_{ij}(\beta,\beta)\, \cancel{x_i}\, (1-x_i)(1-x_j) \right]$$

$\therefore$ <u>condition</u>

want $E_{ij}(\alpha,\beta) + E_{ij}(\beta,\alpha) > E_{ij}(\alpha,\alpha) + E_{ij}(\beta,\beta)$

α - expansion:

$$x_i = \begin{cases} 0 & \text{leave} \\ 1 & \text{make label } \alpha. \end{cases}$$

energy:

- unary terms for non- α- nodes are unary in x_i
- α nodes are fixed
- binary non α, α terms become unary
- binary α, α terms become const
- binary non-α, non-α become binary.

Energy =

$$\left[\sum \text{unary, binary terms in } \alpha \text{ nodes} \right]$$

$$+ \sum_{i \in \text{non } \alpha} \left[E_i(l_i)(1-x_i) + E_i(\alpha)\, x_i \right]$$

$$+ \sum_{i \in \text{non-}\alpha} \left[\left(\sum_{j \in \alpha} E_{ij}(l_i, \alpha) \right)(1-x_i) + \left(\sum_{j \in \alpha} E_{ij}(\alpha, \alpha) \right) x_i \right]$$

$$+ \sum_{i,j \in \text{non-}\alpha} \left[\begin{array}{l} \quad E_{ij}(l_i, l_j)\,(1-x_i)(1-x_j) \\ + \; E_{ij}(l_i, \alpha)\,(1-x_i)\, x_j \\ + \; E_{ij}(\alpha, l_j)\, x_i\,(1-x_j) \\ + \; E_{ij}(\alpha, \alpha)\, x_i\, x_j \end{array} \right]$$

Condition: want

$$E_{ij}(l_i, \alpha) + E_{ij}(\alpha, l_j) > E_{ij}(l_i, l_j) + E_{ij}(\alpha, \alpha).$$

Ways to meet conditions:

E is metric if

a $\quad E(\alpha,\beta) = 0 \iff \alpha = \beta$

b $\quad E(\alpha,\beta) = E(\beta,\alpha) \geq 0$

c $\quad E(\alpha,\beta) \leq E(\alpha,\gamma) + E(\gamma,\beta)$ ($\triangle$ inequality)

E is semimetric if a, b ~~but not c~~ true

Notice

E semimetric $\Rightarrow$

$$E(\alpha,\beta) + E(\beta,\alpha) > E(\alpha,\alpha) + E(\beta,\beta)$$

$\therefore$ α - β swap OK

E metric

$$\Rightarrow \quad E(l_i,\alpha) + E_{ij}(\alpha, l_j) \geq E_{ij}(l_i, l_j) + \cancel{E_{ij}(\alpha,\alpha)}_{=0}$$

equality no big worry.

$\therefore$ α - exp OK

What graph should I cut?

I have

$$\sum_{ij} \left[\begin{array}{l} E_{ij}(0,0)\,(1-x_i)(1-x_j) \;+ \\ E_{ij}(1,0)\;x_i\;(1-x_j) \;+ \\ E_{ij}(0,1)\,(1-x_i)\,x_j \;+ \\ E_{ij}(1,1)\;\;x_i\,x_j \end{array} \right] + \sum_i \left[\begin{array}{l} E_i(0)(1-x_i) \\ \;+\;E_i(1)\,x_i \end{array} \right]$$

$x \in \{0,1\}$

→ I need to get it onto a graph, and cut that — what graph do I cut? (earlier notes rather vague)

Strategy:
1) describe a graph rep'n for component fn's
2) Show how to add.

Graph reph:

(a) ~~Notice~~ graph has $n+2$ nodes

s, t, x_i

(b) Set up so:

$x_i \in S$ side of cut $\Rightarrow x_i = 0$

$x_i \in t$ side $\Rightarrow x_i = 1$

(c) Val(Cut) = Energy(vars) + const

Reph linear functions:

$$\sum_i \left[E_i(0)(1-x_i) + E_i(1)\, x_i \right]$$

- do one term (then sum).

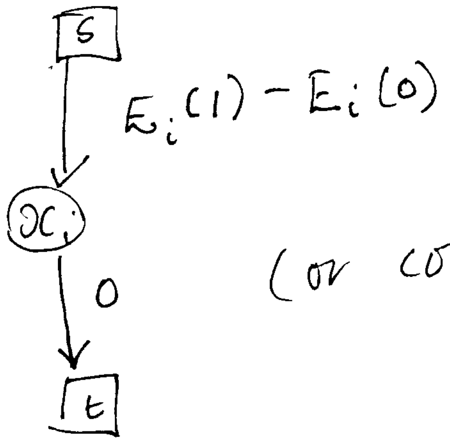
Recall capacities are ≥ 0 for max flow - mincut.

So we want a graph where
(a) ~~cap.~~ cuts rep'n cost
(b) capacities ≥ 0

<u>linear</u>

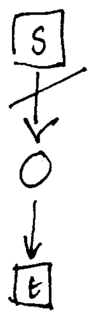
we have $E_i(1)\, x_i + (1-x_i)\, E_i(0)$

<u>Case 1</u>: $E_i(1) > E_i(0)$.



(or could omit, which is usual)

<u>Notice 2 cuts</u>:



← value is $E_i(1) - E_i(0)$

- Value is 0

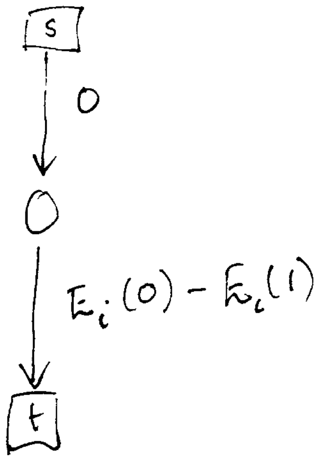
So:

Value of cut = Energy represented by cut $- E_i(0)$

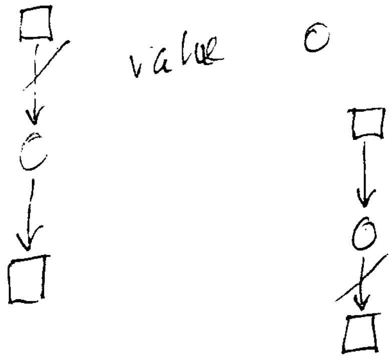
Case 2:

$E_i(1) \not> E_i(0)$

not greater than.



Again 2 cuts:



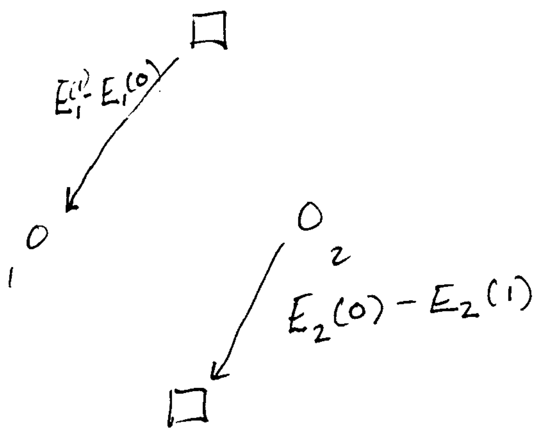
So:

value of cut = Energy repn by cut $- E_i(1)$

So now we can do <u>any</u> linear function. (13)

$$E_1(0)(1-x_i) + E_1(1)\,x_i + E_2(0)(1-x_i) + E_2(1)\,x_i$$

$$|||$$



(say)

<u>etc.</u>

<u>Rep'n Binary (= Quadratic) terms</u>

Notice we can decompose

$$\sum_{ij}\Big[E_{ij}(0,0)(1-x_i)(1-x_j) + E_{ij}(1,0)\,x_i(1-x_j) + E_{ij}(0,1)(1-x_i)\,x_j + E_{ij}(1,1)\,x_i x_j\Big]$$

as a sum of terms. — we need only show how to deal w/ 1 pair

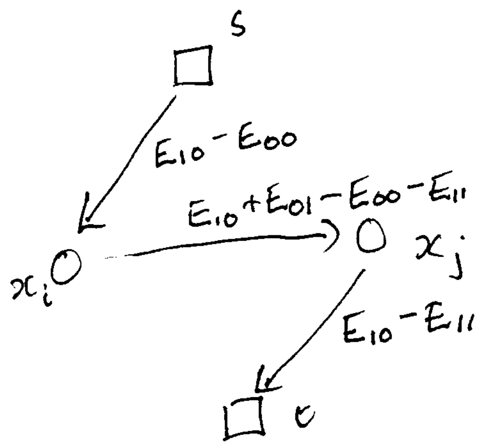
Decompose as:

$$
\begin{aligned}
& E_{ij}(0,0). && \text{constant} \\
+ & (1-x_i)\,0 + x_i(E_{10}-E_{00}) && \\
+ & (1-x_j)\,0 + x_j(E_{11}-E_{10}) && \text{linear} \\
+ & (1-x_i)\,x_j\,(E_{10}+E_{01}-E_{00}-E_{11}) && \text{interesting}
\end{aligned}
$$

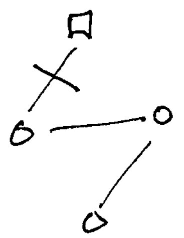
Cases depend on signs of linear coeffs:

se 1) $E_{10} - E_{00} > 0$, $E_{10} - E_{11} > 0$

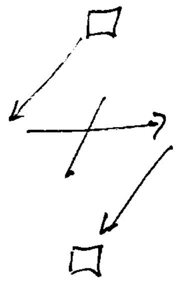
(Notice 1,0 <u>can't</u> be soln)



<u>cuts</u>

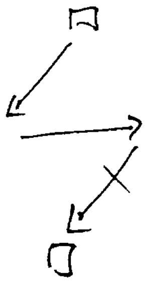


$$
\begin{aligned}
Val(cut) &= E_{10} - E_{00} \\
Energy &= E_{11} \\
&= Val(cut) - E_{10} + E_{00} + E_{11}
\end{aligned}
$$



$$Val(Cut) = E_{10} + E_{01} - E_{00} - E_{11}$$

$$\begin{aligned} Energy &= E_{01} \\ &= Val(cut) - E_{10} + E_{00} + E_{11} \end{aligned}$$



$$Val(Cut) = E_{10} - E_{11}$$

$$\begin{aligned} Energy &= E_{00} \\ &= Val(cut) - E_{10} + E_{00} + E_{11} \end{aligned}$$

So: ~~Val(Cut)~~ $Energy = Val(cut) + Const$

Case 2)

$$E_{10} - E_{00} > 0, \quad E_{10} - E_{11} < 0$$

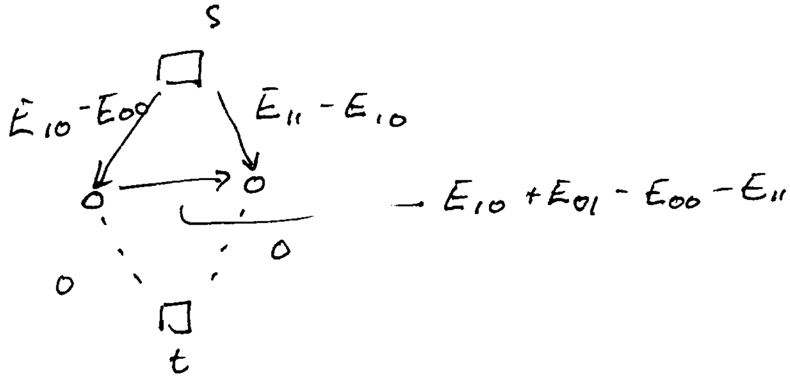
Notice this implies E_{00} is smallest

$$E_{11} > E_{10} > E_{00}$$

and

$$E_{10} + E_{01} > E_{00} + E_{11}$$

$$\Rightarrow E_{01} > E_{00}$$



$\mathrm{Val(Cut)} = \cancel{E_{00}}\ 0$ for all cuts

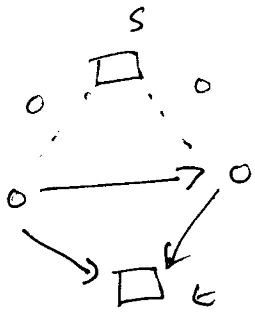
$\mathrm{Energy} = E_{00}$

$= \mathrm{Val(Cut)} + E_{00}$

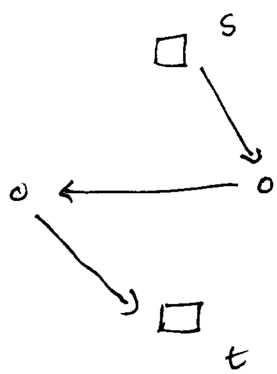
Case 3)

$E_{00} > E_{10}$

$E_{10} > E_{11}$



Case 4)



We can now do *any* 01 Q.F. such that

$$E_{ij}(10) + E_{ij}(01) > E_{ij}(00) + E_{ij}(11)$$

- one node for each var
- for each linear term, insert edges, summing weights as needed
- " quad "

Val(Cut) = Energy + Const

⇒ Min (cut) gives min(Energy)

This works both ways

(i.e. represent as cut ⇒ energy cond).