Semi Definite programming,

or how to solve R.

SDP:

$$\min \quad c^T x$$

s.t. $\quad \underbrace{F(x) \succeq 0}$

↳ is positive Semi Definite

and $\quad F(x) = F_0 + \sum_i x_i F_i$

: symmetric matrices

- this problem is convex

(obvious that objective is convex)

Domain is convex because

$$F_\alpha \succeq 0, \quad F_\beta \succeq 0$$

$$\Rightarrow \quad t F_\alpha + (1-t) F_\beta \succeq 0$$

$$t \in [0,1]$$

Cases :

· Linear program :

$$\min \quad c^T x$$

$$\text{st} \quad Ax + b \geq 0$$

↑ component wise inequality

<u>now</u> $\quad \operatorname{diag}(v) \succeq 0 \iff v \geq 0$

$\therefore \quad Ax + b \geq 0 \iff \operatorname{diag}[Ax + b] \succeq 0$

$$A = [a_1 \; \cdots \; a_n]$$

↑ cols

$$\operatorname{diag}[Ax + b] = \operatorname{diag}(b) + \sum_i x_i \operatorname{diag}(a_i)$$

So $\quad \min \quad c^T x$

$$\text{st} \quad \operatorname{diag}(b) + \sum_i x_i \operatorname{diag}(a_i) \succeq 0$$

- A problem that is an SDP but not an LP

(I) $$\min \frac{(c^T x)^2}{d^T x}$$

$$\text{st} \quad Ax + b \geq 0$$

(where $d^T x > 0$ for x s.t. $Ax + b \geq 0$)

- this is convex, <u>Not</u> LP.

(II) $$\min \quad t$$

$$\text{st} \quad Ax + b \geq 0$$

$$\frac{(c^T x)^2}{(d^T x)} \leq t$$

$I \equiv II$ <u>Trick</u>:

$$\begin{bmatrix} t & (c^T x) \\ (c^T x) & (d^T x) \end{bmatrix} \succeq 0$$

<u>is</u>: $t(d^T x) - (c^T x)^2 \geq 0$

<u>but</u> $d^T x > 0$ by assumption, so $t - \frac{(c^T x)^2}{d^T x} \geq 0$

So $\quad \min \quad t$

III $\quad$ st $\begin{bmatrix} \operatorname{diag}(Ax+b) & 0 & 0 \\ 0 & t & (c^Tx) \\ 0 & (c^Tx) & (d^Tx) \end{bmatrix} \succcurlyeq 0$

is equivalent to I ; but III is an SDP.

Fact: SDP's can be solved fairly efficiently with interior point methods (estimates later).

~~Further examples:~~

~~Quadratically constrained Quadratic program~~

Schur complements

consider

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \qquad A, C, \text{Symmetr}$$

and assume A positive definite

now

$$X \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$Ax + By = u$$

$$B^T x + Cy = v$$

so

$$x = A^{-1}u - A^{-1}By$$

and

$$B^T A^{-1} u - B^T A^{-1} B y + Cy = v$$

write $S = C - B^T A^{-1} B$

$$B^T A^{-1} u + Sy = v$$

$$y = S^{-1}(v - B^T A^{-1} u)$$

S is *very* interesting, w/ applications to positive definiteness.

Consider

$$\min_u \left[u^T A u + 2v^T B^T u + v^T C v \right]$$

(for fixed v)

then $u = -A^{-1} B v$ so the *value* is

$$v^T B^T A^{-1} A A^{-1} B v - 2 v^T B^T A^{-1} B v + v^T C v$$

$$= v^T S v$$

this gives

$$\boxed{\begin{array}{l} X \succeq 0 \iff A \succ 0,\ S \succ 0 \\ A \succ 0 \implies \left[X \succeq 0 \iff S \succeq 0 \right] \end{array}}$$

worth remembering!

SDP example: Quadratically constrained Q.P.

assume convex quadratic constraints

$$(Ax+b)^T(Ax+b) - c^Tx - d \preceq 0$$

By Schur complement equiv to

$$\begin{bmatrix} I & Ax+b \\ (Ax+b)^T & c^Tx+d \end{bmatrix} \succeq 0$$

this has right form!

so if we have

minimize $\quad (A_0x+b_0)^T(A_0x+b_0) - c_0^Tx - d_0$

s.t.

$$(A_ix+b_i)^T(A_ix+b_i) - c_i^Tx - d_i \preceq 0$$

we can rearrange to get

min $\quad t$

subject to

$$\begin{bmatrix} \text{OF block} & 0 & 0 & 0 & \cdots \\ 0 & C_1 \text{ block} & 0 & & \cdots \\ 0 & 0 & C_2 \text{ block} & & \\ \vdots & \vdots & & \ddots & \end{bmatrix} \succeq 0$$

<u>Where</u> $\quad C_i \text{ block} = \begin{bmatrix} I & A_i x + b_i \\ (A_i x + b_i)^T & c_i x + d_i \end{bmatrix}$

$$\cancel{C_0} \text{ OF block} = \begin{bmatrix} I & A_0 x + b_0 \\ (A_0 x + b_0)^T & c_0 x + d_0 + t \end{bmatrix}$$