Matchings:

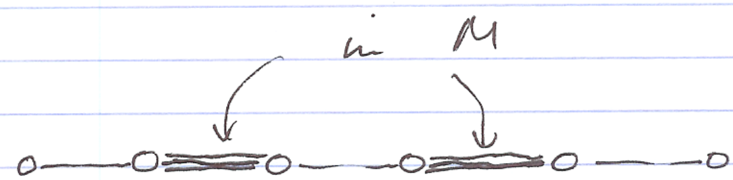
$G = (V, E)$ undirected

A **matching** is a set of disjoint edges

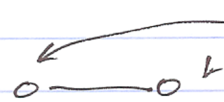
An **augmenting path** for a matching M is a path with

- odd length
- first, last vert not covered by M
- alternate edges in M

Examples:



not covered by M.



Define symmetric difference Δ by

$$A \Delta B = (A-B) \cup (B-A)$$

$$= \{\text{everything in one, but not both}\}$$

if P is an M-augmenting path

then $M' := M \Delta \underbrace{EP}_{\text{edges of } P}$

is a matching with $|M'| = |M| + 1$

<u>Thm</u>: $G=(V,E)$ a graph, M a matching. Then either M is a matching of max size, or there exists an M-augmenting path.

Proof:

if M is max size matching, cannot have M-augmenting P because $M \triangle EP$ would be bigger than M

if M' is a matching, larger than M, consider

$$G' = (V, M' \cup M)$$

This has max degree 2

So each component is a path (perhaps length 0) or a circuit.

$|M'| > |M|$ so one component must have more M' edges than M edges

this is an M augmenting path.

$\square$

Maximum size bipartite matching

(we did this w/ flow already)

$G = (V, E)$, bipartite] input
M matching]

Output: M', such that $|M'| > |M|$

Alg: – G has color classes U, W.

– D_M is directed.
  - verts are verts of G
  - edges are edges of G

    $e \in M$, e goes $W \to U$

    $e \notin M$, e goes $U \to W$

– U_M is elements in U not covered by M
  W_M " W "

- find a <u>directed</u> path from U_M to W_M in D_M
- this is M augmenting, so gives a matching larger than M.

<u>Weighted</u> bipartite matching. (Hungarian alg)

- ~~obt~~ each edge has a weight $w(e)$
- we have $G = (V, E)$, color classes U, W, $w : E \to \mathbb{Q}$

<u>Method</u>

- Start with $M = \emptyset$
- construct D_M, directed by
  - orienting each edge in M to go $W \to U$, with length $= w_e$
  - all others go $U \to W$, length $= -w_e$

- Write U_M for U verts not covered by M

  W_M " W "

- find shortest $U_M - W_M$ path (if it exists), say P

- ~~form~~ $M \longrightarrow M \triangle EP$

  and iterate.

- Stop when no P can be found.

Thm: call a matching <u>extreme</u> if it has max weight among all size $|M|$ matchings. Each M found by this method is extreme.

Proof: (Induction)

- True for $M = \emptyset$
- Suppose M is extreme. P and M' from matching alg. Show M' is extreme
- Consider any N, a matching, extreme,
  $|N| = |M| + 1$.
- $|N| > |M|$, so $M \cup N$ has a component Q that is M-augmenting
- P is shortest such, so $\ell(P) \leq \ell(Q)$
- $N \triangle Q$ is a matching, $|N \triangle Q| = |M|$
  →|Why|←
- but M is extreme, so $w(M) \geq w(N \triangle Q)$

  →|Why|←
- $w(N) = w(N \triangle Q) - \ell(Q) \leq w(M) - \ell(P) = w(M')$
- So M' is extreme

$\square$

$|C| \geq |U|$. This indeed is the case, since $N(U \setminus C) \subseteq C \cap W$, and hence

(16.8) $$|C| = |C \cap U| + |C \cap W| \geq |C \cap U| + |N(U \setminus C)| \geq |C \cap U| + |U \setminus C| = |U|.$$ ∎

This can be extended to general subsets of V. First, Hoffman and Kuhn [1956b] and Mendelsohn and Dulmage [1958a] showed:

Theorem 16.8. *Let $G = (V, E)$ be a bipartite graph with colour classes U and W and let $R \subseteq V$. Then there exists a matching covering R if and only if there exist a matching M covering $R \cap U$ and a matching N covering $R \cap W$.*

Proof. Necessity being trivial, we show sufficiency. We may assume that G is connected, that $E = M \cup N$, and that neither M nor N covers R. This implies that there is a $u \in R \cap U$ missed by N and a $w \in R \cap W$ missed by M. So G is an even-length $u - w$ path, a contradiction, since $u \in U$ and $w \in W$. ∎

(This theorem goes back to theorems of F. Bernstein (cf. Borel [1898] p. 103), Banach [1924], and Knaster [1927] on injective mappings between two sets.)

Theorem 16.8 implies a characterization of sets that are covered by some matching:

Corollary 16.8a. *Let $G = (V, E)$ be a bipartite graph with colour classes U and W and let $R \subseteq V$. Then there is a matching covering R if and only if $|N(S)| \geq |S|$ for each $S \subseteq R \cap U$ and for each $S \subseteq R \cap W$.*

Proof. Directly from Theorems 16.7 and 16.8. ∎

It also gives the following exchange property:

Corollary 16.8b. *Let $G = (V, E)$ be a bipartite graph, with colour classes U and W, let M and N be maximum-size matchings, let U' be the set of vertices in U covered by M, and let W' be the set of vertices in W covered by N. Then there exists a maximum-size matching covering $U' \cup W'$.*

Proof. Directly from Theorem 16.8: the matching found is maximum-size since $|U'| = |W'| = \nu(G)$. ∎

Notes. These results also are special cases of the exchange results on paths discussed in Section 9.6c. Perfect [1966] gave the following linear-algebraic argument for Corollary 16.8b. Make a $U \times W$ matrix A with $a_{u,w} = x_{u,w}$ if $uw \in E$ and $a_{u,w} := 0$ otherwise, where the $x_{u,w}$ are independent variables. Let U' be any maximum-size subset of U covered by some matching and let W' be any maximum-size subset of W covered by some matching. Then U' gives a maximum-size set of linearly independent rows of A and W' gives a maximum-size set of linearly independent columns of A. Then the $U' \times W'$ submatrix of A is nonsingular, hence of nonzero determinant. It implies (by the definition of determinant) that G has a matching covering $U' \cup W'$.
(Related work includes Perfect and Pym [1966], Pym [1967], Brualdi [1969b,1971b], and Mirsky [1969].)

16.7. Further results and notes

16.7a. Complexity survey for cardinality bipartite matching

Complexity survey for cardinality bipartite matching ($*$ indicates an asymptotically best bound in the table):

	$O(nm)$	König [1931], Kuhn [1955b]
	$O(\sqrt{n}\,m)$	Hopcroft and Karp [1971,1973], Karzanov [1973a]
$*$	$\tilde{O}(n^\omega)$	Ibarra and Moran [1981]
	$O(n^{3/2}\sqrt{\frac{m}{\log n}})$	Alt, Blum, Mehlhorn, and Paul [1991]
$*$	$O(\sqrt{n}\,m \log_n(n^2/m))$	Feder and Motwani [1991,1995]

Here ω is any real such that any two $n \times n$ matrices can be multiplied by $O(n^\omega)$ arithmetic operations (e.g. $\omega = 2.376$).

Goldberg and Kennedy [1997] described a bipartite matching algorithm based on the push-relabel method, of complexity $O(\sqrt{n}\,m \log_n(n^2/m))$. Balinski and Gonzalez [1991] gave an alternative $O(nm)$ bipartite matching algorithm (not using augmenting paths).

16.7b. Finding perfect matchings in regular bipartite graphs

By König's matching theorem, each k-regular bipartite graph has a perfect matching (if $k \geq 1$). One can use the regularity also to find quickly a perfect matching. This will be used in Chapter 20 on bipartite edge-colouring.

First we show the following result of Cole and Hopcroft [1982] (which will not be used any further in this book):

Theorem 16.9. *A perfect matching in a regular bipartite graph can be found in $O(m \log n)$ time.*

Proof. We first describe an $O(m \log n)$-time algorithm for the following problem:

(16.9) given: a k-regular bipartite graph $G = (V, E)$ with $k \geq 2$,
find: a nonempty proper subset F of E with (V, F) regular.

Proof. See above.

17.5. Further results and notes

17.5a. Complexity survey for maximum-weight bipartite matching

Complexity survey for the maximum-weight bipartite matching (* indicates an asymptotically best bound in the table):

	$O(nW \cdot \mathrm{VC}(n,m))$	Egerváry [1931] (implicitly)
	$O(2^n n^2)$	Easterfield [1946]
	$O(nW \cdot \mathrm{DC}(n,m,W))$	Robinson [1949]
	$O(n^4)$	Kuhn [1955b], Munkres [1957][28] Hungarian method
	$O(n^2 m)$	Iri [1960]
	$O(n^3)$	Dinits and Kronrod [1969]
*	$O(n \cdot \mathrm{SP}_+(n,m,W))$	Edmonds and Karp [1970], Tomizawa [1971]
	$O(n^{3/4} m \log W)$	Gabow [1983b,1985a,1985b]
*	$O(\sqrt{n}\, m \log(nW))$	Gabow and Tarjan [1988b,1989] (cf. Orlin and Ahuja [1992])
	$O(\sqrt{n}\, mW)$	Kao, Lam, Sung, and Ting [1999]
*	$O(\sqrt{n}\, mW \log_n(n^2/m))$	Kao, Lam, Sung, and Ting [2001]

Here $W := \|w\|_\infty$ (assuming w to be integer-valued). Moreover, $\mathrm{SP}_+(n,m,W)$ is the time needed to find a shortest path in a directed graph with n vertices and m arcs, with nonnegative integer lengths on the arcs, each at most W. Similarly, $\mathrm{DC}(n,m,W)$ is the time required to find a negative-length directed circuit in a directed graph with n vertices and m arcs, with integer lengths on the arcs, each at most W in absolute value. Moreover, $\mathrm{VC}(n,m)$ is the time required to find a minimum-size vertex cover in a bipartite graph with n vertices and m edges.

Dinits [1976] gave an algorithm for finding a minimum-weight matching in $K_{p,q}$ of size p, with time bound $O(|p|^3 + pq)$ (taking $p \leq q$).

17.5b. Further notes

Simplex method. Finding a maximum-weight matching in a bipartite graph is a special case of a linear programming problem (see Chapter 18), and hence linear programming methods like the simplex method apply.

[28] Munkres showed that Kuhn's 'Hungarian method' takes $O(n^4)$ time.

signment problem. Using the 'strongly feasible' trees of Cunningham [1976], Roohy-Laleh [1980] showed that a version of the simplex method solves the assignment problem in less than n^3 pivots (cf. Hung [1983], Orlin [1985], and Akgül [1993]; the last paper gives a method with $O(n^2)$ pivots, yielding an $O(n(m + n \log n))$ algorithm).

Balinski [1985] (cf. Goldfarb [1985]) showed that a version of the dual simplex method (the *signature method*) solves the assignment problem in strongly polynomial time ($O(n^2)$ pivots, yielding an $O(n^3)$ algorithm). More can be found in Dantzig [1963], Barr, Glover, and Klingman [1977], Balinski [1986], Ahuja and Orlin [1988,1992], Akgül [1988], Paparrizos [1988], and Akgül and Ekin [1991].

For further algorithmic studies of the assignment problem, consult Flood [1960], Kurtzberg [1962], Hoffman and Markowitz [1963], Balinski and Gomory [1964], Tabourier [1972], Carpaneto and Toth [1980a,1983,1987], Hung and Rom [1980], Karp [1980], Bertsekas [1981,1987,1992] ('auction method'), Engquist [1982], Avis [1983], Avis and Devroye [1985], Derigs [1985b,1988a], Carraresi and Sodini [1986], Derigs and Metz [1986a], Glover, Glover, and Klingman [1986], Jonker and Volgenant [1986], Kleinschmidt, Lee, and Schannath [1987], Avis and Lai [1988], Bertsekas and Eckstein [1988], Motwani [1989,1994], Kalyanasundaram and Pruhs [1991, 1993], Khuller, Mitchell, and Vazirani [1991,1994], Goldberg and Kennedy [1997] (push-relabel), and Arora, Frieze, and Kaplan [1996,2002].

For computational studies, see Silver [1960], Florian and Klein [1970], Barr, Glover, and Klingman [1977] (simplex method), Gavish, Schweitzer, and Shlifer [1977] (simplex method), Bertsekas [1981], Engquist [1982], McGinnis [1983], Lindberg and Ólafsson [1984] (simplex method), Glover, Glover, and Klingman [1986], Jonker and Volgenant [1987], Bertsekas and Eckstein [1988], and Goldberg and Kennedy [1995] (push-relabel). Consult also Johnson and McGeoch [1993].

Linear-time algorithm for weighted bipartite matching problems satisfying a quadrangle or other inequality were given by Karp and Li [1975], Buss and Yianilos [1994,1998], and Aggarwal, Bar-Noy, Khuller, Kravets, and Schieber [1995].

For generating *all* minimum-weight perfect matchings, see Fukuda and Matsui [1992]. For studies of the 'most vital' edges in a weighted bipartite graph, see Hung, Hsu, and Sung [1993].

Aráoz and Edmonds [1985] gave an example showing that iterative dual improvements in the linear programming problem dual to the assignment problem, need not converge for irrational data.

For the 'bottleneck' assignment problem, see Gross [1959] and Garfinkel [1971]. An algebraic approach to assignment problems was described by Burkard, Hahn, and Zimmermann [1977].

For surveys on matching algorithms, see Galil [1983,1986a,1986b]. Books covering the weighted bipartite matching and assignment problems include Ford and Fulkerson [1962], Dantzig [1963], Christofides [1975], Lawler [1976b], Bazaraa and Jarvis [1977], Burkard and Derigs [1980], Papadimitriou and Steiglitz [1982], Gondran and Minoux [1984], Rockafellar [1984], Derigs [1988a], Bazaraa, Jarvis, and Sherali [1990], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000].