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AN ANALYSIS OF APPROXIMATIONS FOR MAXIMIZING SUBMODULAR SET FUNCTIONS – I

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Let N be a finite set and z be a real-valued function defined on the set of subsets of N that satisfies $z(S)+z(T)\geq z(S\cup T)+z(S\cap T)$ for all S, T in N. Such a function is called submodular. We consider the problem $\max_{S\subseteq N}\{z(S)\colon |S|\leq K, z(S) \text{ submodular}\}$.

Several hard combinatorial optimization problems can be posed in this framework. For example, the problem of finding a maximum weight independent set in a matroid, when the elements of the matroid are colored and the elements of the independent set can have no more than K colors, is in this class. The uncapacitated location problem is a special case of this matroid optimization problem.

We analyze greedy and local improvement heuristics and a linear programming relaxation for this problem. Our results are worst case bounds on the quality of the approximations. For example, when $z(S)$ is nondecreasing and $z(\emptyset)=0$, we show that a "greedy" heuristic always produces a solution whose value is at least $1-[(K-1)/K]^K$ times the optimal value. This bound can be achieved for each K and has a limiting value of $(e-1)/e$, where e is the base of the natural logarithm.

Key words: Heuristics, Greedy Algorithm, Interchange Algorithm, Linear Programming, Matroid Optimization, Submodular Set Functions.

1. Introduction

In a recent paper Cornuejols, Fisher and Nemhauser [2] give bounds on approximations (heuristics and relaxations) for the uncapacitated location problem. Here we extend these results and generalize them to a larger class of problems. We can view the location problem as the following combinatorial one. Given a non-negative $m\times n$ matrix $C=(c_{ij})$ with column index set N and row index set I, for each non-empty $S\subseteq N$ define

$$z(S)=\sum_{i\in I}\max_{j\in S} c_{ij} \tag{1.1}$$

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and $z(\emptyset) = 0$. The problem is to find an S of cardinality less than or equal to a specified integer K ($K < n$) such that $z(S)$ is maximum; i.e.,

$$\max_{S \subseteq N} \{z(S): |S| \leq K, z(S) \text{ given by (1.1)}\}. \tag{1.2}$$

As noted in [2], problem (1.2) is a member of the class of NP-hard problems.

A fundamental property of the function given by (1.1) is that, for $R \subset S$ and $k \in N - S$, adding k to S will increase z by no more than by adding k to R. Note that

$$\begin{aligned} \max_{j \in S \cup \{k\}} c_{ij} - \max_{j \in S} c_{ij} &= \max(0, c_{ik} - \max_{j \in S} c_{ij}) \\ &\leq \max(0, c_{ik} - \max_{j \in R} c_{ij}) \\ &= \max_{j \in R \cup \{k\}} c_{ij} - \max_{j \in R} c_{ij}, \quad i \in I, \end{aligned} \tag{1.3}$$

where the inequality follows from $\max_{j \in S} c_{ij} \geq \max_{j \in R} c_{ij}$. By summing the inequalities of (1.3) over i we obtain that $z(S)$ defined by (1.1) satisfies

$$z(S \cup \{k\}) - z(S) \leq z(R \cup \{k\}) - z(R), \quad R \subset S \subset N, \quad k \in N - S. \tag{1.4}$$ [1]

Furthermore,

$$z(S \cup \{k\}) - z(S) \geq 0, \qquad S \subset N, \quad k \in N - S. \tag{1.5}$$

A real-valued function $z(S)$ defined on the set of subsets of N that satisfies (1.4) [and (1.5)] is called a *submodular* [*nondecreasing*] set function. Thus a natural generalization of (1.2), which we study in this paper, is

$$\max_{S \subseteq N} \{z(S): |S| \leq K, z(S) \text{ submodular}\}. \tag{1.6}$$

Note that when z is nondecreasing the cardinality constraint is necessary in (1.6) to obtain a nontrivial problem. However when z does not satisfy (1.5), the problem is interesting even without the cardinality constraint (i.e., $K = |N| = n$). Although our results apply to arbitrary submodular functions, they are much sharper for nondecreasing submodular functions.

In Section 2 we give several equivalent definitions of submodular functions and we will see that submodularity is in some sense a combinatorial analogue of concavity. Most of these results are not original; we collect them together and prove them here to facilitate availability and use throughout this paper.

A rich variety of combinatorial optimization problems can be modeled as the maximization of submodular functions as in (1.6). In Section 3 we present three classes of these problems. One class, which contains the location problem, arises from matroids, another from the assignment problem and a third from boolean polynomials. To motivate the representation of combinatorial optimization

[1] This property for the uncapacitated location problem has been observed by Babayev [1], Frieze [6] and Spielberg [9].

problems by (1.6), we now present the location problem in its matroid context.

Let E be a finite set and $\mathscr{F}$ a nonempty collection of subsets of E. The system $\mathcal{M} = (E, \mathscr{F})$ is called a *matroid* if

(a) $F_1 \in \mathscr{F}$ and $F_2 \subset F_1 \Rightarrow F_2 \in \mathscr{F}$.

(b) For all $E' \subseteq E$ every maximal member of $\mathscr{F}(E') = \{F: F \in \mathscr{F}, F \subseteq E'\}$ has the same cardinality.

The members of $\mathscr{F}$ are called *independent* sets.

Suppose that the elements $\{e\}$ of E are assigned weights $\{c_e\}$ and consider the matroid optimization problem of determining

$$v(E') = \max_{F \in \mathscr{F}(E')} \sum_{e \in F} c_e. \tag{1.7}$$

The function $v(E')$ is submodular and nondecreasing (see Proposition 3.1) but it is completely unnecessary to consider approximations for (1.7) since it is well-solved by a simple greedy algorithm [5].

Consider, however, a generalization of (1.7) in which the elements of E are assigned colors as well as weights; in other words E is partitioned into subsets $\{Q_j: j \in N\}$ and the elements of Q_j are colored j. The problem is to find a maximum weight independent set that contains no more than K colors. Let

$$z(S) = v\Big(\bigcup_{j \in S} Q_j\Big) = \max\Big\{\sum_{e \in F} c_e: F \in \mathscr{F}\Big(\bigcup_{j \in S} Q_j\Big)\Big\}, \quad S \subseteq N. \tag{1.8}$$

The function $z(S)$ in (1.8) gives the value of a maximum weight independent set that is restricted to the colors contained in S. The submodularity of $v(E')$ in (1.7) implies the submodularity of $z(S)$ in (1.8), see Proposition 2.5. Thus the matroid optimization problem of finding a maximum weight independent set that contains no more than K colors is a case of (1.6) where $z(S)$ is given by (1.8).

To show that the location problem can be placed in the framework of (1.7) and (1.8) let $E = \{(i, j): i \in I, j \in N\}$ and partition E into the subsets $E_i = \{(i, j): j \in N\}$, $i \in I$. Define an $F \subseteq E$ to be independent if and only if $|F \cap E_i| \leq 1$, $i \in I$. This system clearly satisfies matroid property (a) and property (b) follows from the fact that every maximal independent set of E' has cardinality $|\{i \in I: E' \cap E_i \neq \emptyset\}|$. The matroid defined by this system is the well-known *partition matroid.*

Let $c_{ij} \geq 0$ be the weight of $(i, j) \in E$, $i \in I$, $j \in N$ and $Q_j = \{(i, j): i \in I\}$, $j \in N$. When $E' = \bigcup_{j \in S} Q_j$, $S \subseteq N$, we obtain

$$z(S) = v\Big(\bigcup_{j \in S} Q_j\Big) = \sum_{i \in I} \max_{j \in S} c_{ij}. \tag{1.9}$$

Note that (1.9) is identical to (1.1) so that we have described the location problem as an optimization problem with respect to a partition matroid and have obtained another proof that the location problem belongs to (1.6). The result obtained in this indirect and tedious way provided impetus for our work.

Sections 4–7 contain the results on approximations. Here we briefly summarize the main results for $z(\emptyset)=0$, z nondecreasing and $z\not\equiv 0$.

In Section 4 we study a greedy heuristic that first solves (1.6) for $K=1$ and then iteratively approximates for larger K. If S, $|S|=p$, is the approximate solution for $K=p$, then the approximate solution for $K=p+1$ is determined by adding to S (if possible) a j^* such that $z(S\cup\{j^*\})=\max_{j\notin S} z(S\cup\{j\})$ and $z(S\cup\{j^*\})>z(S)$. We obtain the bound

$$\frac{\text{value of greedy approximation}}{\text{value of optimal solution}}\geq 1-\left(\frac{K-1}{K}\right)^K\geq\frac{\mathrm{e}-1}{\mathrm{e}}$$

where $\mathrm{e}=2.718\cdots$ is the base of the natural logarithm. This result is generalized to the case in which the greedy algorithm selects a subset of cardinality R at each iteration.

In Section 5 we analyze an interchange heuristic for (1.6) that begins with an arbitrary set S of cardinality K and attempts to improve on the value of this set by replacing r members of S by r members of $N-S$, $1\leq r\leq R$. The procedure stops when no such improvement is possible. When R divides K we obtain the bound

$$\frac{\text{value of interchange approximation}}{\text{value of optimal solution}}\geq\frac{K}{2K-R}.$$

For the matroid case, in which $z(S)$ is given by (1.8), we can state (1.6) as an integer program. In Section 6 we study a linear programming relaxation of this integer program. The quality of the linear programming approximation depends on the matroid structure relative to the sets $\{Q_i\}$. For a particular case that includes the location problem we obtain the result

$$\frac{\text{value of greedy approximation}}{\text{value of linear programming solution}}\geq 1-\left(\frac{K-1}{K}\right)^K.$$

All of the bounds mentioned above are achieved by worst-case examples.

The performance of a heuristic sometimes can be improved by combining it with partial enumeration. Suppose for some heuristic we have (value of heuristic)/(value of optimal solution) $\geq 1-\beta(K)$. In Section 7 we analyze the approximation that first enumerates all $\binom{|N|}{R}$ subsets of N for a specified $R<K$ and then applies the heuristic to each of the $\binom{|N|}{R}$ problems (1.6) with a fixed subset of cardinality R, and K replaced by $K-R$. Choosing the best of these solutions yields

$$\frac{\text{value of } R\text{-enumeration plus heuristic approximation}}{\text{value of optimal solution}}\geq$$

$$\geq 1-\frac{(K-R)}{K}\beta(K-R).$$

2. Submodular set functions[2]

Definition 2.1. Given a finite set E, a real-valued function z on the set of subsets of E is called *submodular* if

(i) $$z(A)+z(B)\geq z(A\cup B)+z(A\cap B),\quad \forall A,B\subseteq E.$$

We shall often make use of the incremental value of adding element j to the set S; let $\rho_j(S)=z(S\cup\{j\})-z(S)$.

Proposition 2.1. *Each of the following statements is equivalent and defines a submodular set function.*

(i) $$z(A)+z(B)\geq z(A\cup B)+z(A\cap B),\quad \forall A,B\subseteq E.$$

(ii) $$\rho_j(S)\geq\rho_j(T),\quad \forall S\subseteq T\subseteq E\quad\text{and}\quad j\in E-T.$$

(iii) $$\rho_j(S)\geq\rho_j(S\cup\{k\}),\quad \forall S\subseteq E\quad\text{and}\quad j\in E-(S\cup\{k\}).$$

(iv) $$z(T)\leq z(S)+\sum_{j\in T-S}\rho_j(S)-\sum_{j\in S-T}\rho_j(S\cup T-\{j\}),\quad \forall S,T\subseteq E.$$

(v) $$z(T)\leq z(S)+\sum_{j\in T-S}\rho_j(S),\quad \forall S\subseteq T\subseteq E.$$

(vi) $$z(T)\leq z(S)-\sum_{j\in S-T}\rho_j(S-\{j\}),\quad \forall T\subseteq S\subseteq E.$$

(vii) $$z(T)\leq z(S)-\sum_{j\in S-T}\rho_j(S-\{j\})+\sum_{j\in T-S}\rho_j(S\cap T),\quad \forall S,T\subseteq E.$$

Proof. We will prove the equivalence of (i) and (ii), and then (iii)$\Rightarrow$(ii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(iii). Statements (vi) and (vii) can be shown to be equivalent in a similar manner.

(i)$\Rightarrow$(ii). Take $S\subseteq T$, $j\notin T$, $A=S\cup\{j\}$ and $B=T$ in (i). This yields

$$z(S\cup\{j\})+z(T)\geq z(T\cup\{j\})+z(S),$$

or

$$\rho_j(S)=z(S\cup\{j\})-z(S)\geq z(T\cup\{j\})-z(T)=\rho_j(T).$$

(ii)$\Rightarrow$(i). Let $\{j_1,\dots,j_r\}=A-B$. From (ii) we obtain

$$\rho_{j_i}(A\cap B\cup\{j_1,\dots,j_{i-1}\})\geq\rho_{j_i}(B\cup\{j_1,\dots,j_{i-1}\}),\quad i=1,\dots,r.$$

Summing these r inequalities yields

$$z(A)-z(A\cap B)\geq z(A\cup B)-z(B).$$

[2] Edmonds [4] and Shapley [8], among others, give various properties of submodular set functions. See also [1] and [6].

(iii)$\Rightarrow$(ii). Take $S\subseteq T$, $j\notin T$, and $T-S=\{j_1,\dots,j_r\}$. Then from (iii) we have

$$\rho_j(S)\geq\rho_j(S\cup\{j_1\}),\quad \rho_j(S\cup\{j_1\})\geq\rho_j(S\cup\{j_1,j_2\}),\dots,$$
$$\rho_j(S\cup\{j_1,\dots,j_{r-1}\})\geq\rho_j(T).$$

Summing these r inequalities yields (ii).

(ii)$\Rightarrow$(iv). For arbitrary S and T with $T-S=\{j_1,\dots,j_r\}$ and $S-T=\{k_1,\dots,k_q\}$ we have

$$\begin{aligned}z(S\cup T)-z(S)&=\sum_{t=1}^{r}[z(S\cup\{j_1,\dots,j_t\})-z(S\cup\{j_1,\dots,j_{t-1}\})]\\&=\sum_{t=1}^{r}\rho_{j_t}(S\cup\{j_1,\dots,j_{t-1}\})\leq\sum_{t=1}^{r}\rho_{j_t}(S)=\sum_{j\in T-S}\rho_j(S)\end{aligned}\tag{2.1}$$

where the inequality follows from (ii). Similarly

$$\begin{aligned}z(S\cup T)-z(T)&=\sum_{t=1}^{q}[z(T\cup\{k_1,\dots,k_t\})-z(T\cup\{k_1,\dots,k_{t-1}\})]\\&=\sum_{t=1}^{q}\rho_{k_t}(T\cup\{k_1,\dots,k_t\}-\{k_t\})\geq\sum_{t=1}^{q}\rho_{k_t}(T\cup S-\{k_t\})\\&=\sum_{j\in S-T}\rho_j(S\cup T-\{j\}).\end{aligned}\tag{2.2}$$

We obtain (iv) by subtracting (2.2) from (2.1).

(iv)$\Rightarrow$(v). If $S\subseteq T$, $S-T=\emptyset$ and the last term of (iv) vanishes.

(v)$\Rightarrow$(iii). Substitute $T=S\cup\{j,k\}$, $j\notin S\cup\{k\}$ in (v) to obtain

$$z(S\cup\{j,k\})\leq z(S)+\rho_j(S)+\rho_k(S),$$

or

$$\begin{aligned}\rho_j(S\cup\{k\})&=z(S\cup\{j,k\})-z(S\cup\{k\})\\&=z(S\cup\{j,k\})-\rho_k(S)-z(S)\leq\rho_j(S).\end{aligned}$$

In many cases we consider nondecreasing submodular functions, which, in addition to (i), satisfy $z(S)\leq z(T)$, $\forall S\subseteq T\subseteq E$.

Proposition 2.2. *Each of the following statements is equivalent and defines a nondecreasing submodular set function.*

(i′) $$z(A)+z(B)\geq z(A\cup B)+z(A\cap B),\quad \forall A,B\subseteq E,$$
$$z(A)\leq z(B),\quad \forall A\subseteq B\subseteq E.$$

(ii′) $$\rho_j(S)\geq\rho_j(T)\geq 0,\quad \forall S\subseteq T\subseteq E,\quad \forall j\in E.$$

(iv′) $$z(T)\leq z(S)+\sum_{j\in T-S}\rho_j(S),\quad \forall S,T\subseteq E.$$

Proof. (i′)$\Leftrightarrow$(ii′) is a trivial consequence of (i)$\Leftrightarrow$(ii). Now (ii′)$\Rightarrow$(ii)$\Rightarrow$(iv). But $\rho_j(T)\geq 0$ implies that the last term of (iv) is nonpositive, and this fact gives (iv′).

Finally, (iv′)⇒(ii′) since choosing $S = T \cup \{j\}$ in (iv′) yields $z(T) \le z(T \cup \{j\})$ or $\rho_j(T) \ge 0$.

We remark that definitions (ii) and (ii′) suggest decreasing returns to scale and the notion of concave set functions. Furthermore (i) and (i′) when rewritten as

$$[z(A) - z(A \cap B)] + [z(B) - z(A \cap B)] \ge$$
$$\ge [z(A \cup B) - z(A \cap B)], \quad \forall A, B \subseteq E$$

can be thought of as generalized subadditivity. However, subadditivity by itself is not sufficient to obtain any of our results on approximations given in Sections 4, 5 and 6.

Now we look at other properties of submodular set functions that are useful in the following sections. Since E is a finite set, upper and lower bounds on $\rho_j(S)$ exist.

Proposition 2.3. *If z is a submodular set function on E with $-\theta \le \rho_j(S) \le \psi$, $\forall S \subseteq E$, $j \in E - S$, then*

(a) $$z(T) \le z(S) + \sum_{j \in T-S} \rho_j(S) + |S - T|\theta, \quad \forall S, T \subseteq E.$$

(b) $$z(T) \le z(S) - \sum_{j \in S-T} \rho_j(S - \{j\}) + |T - S|\psi, \quad \forall S, T \subseteq E.$$

Proof. Substitute the bounds on ρ_j into definitions (iv) and (vii).

We shall also make use of the incremental value of adding subset J to subset S. Let $\rho_J(S) = z(S \cup J) - z(S)$.

Proposition 2.4. *If z is a nondecreasing submodular set function on E and $\{J_1, \ldots, J_r\}$ is a partition of $T - S$, Then*

$$z(T) \le z(S) + \sum_{i=1}^{r} \rho_{J_i}(S).$$

The next four propositions show how to generate new submodular functions from given ones. Of particular interest to us is the creation of submodular functions by letting subsets $\{Q_j\}$ of E be the elements in a set function.

Proposition 2.5 (See (70) of [4]). *Given a submodular set function v on the set of subsets of E and a collection of subsets $\{Q_j\}$, $j \in N$, of E; if* (a) *v is nondecreasing, or* (b) *the $\{Q_j\}$ are disjoint, then $z(S) = v(\bigcup_{j \in S} Q_j)$ is a submodular set function on the set of subsets of N.*

Proof. (Throughout the proof we abbreviate $\bigcup_{j \in T} Q_j$ by $\bigcup_T Q_j$.) Submodularity of v implies

$$\begin{aligned} z(A)+z(B) &= v(\bigcup_A Q_j)+v(\bigcup_B Q_j) \\ &\geq v[(\bigcup_A Q_j)\cup(\bigcup_B Q_j)]+v[(\bigcup_A Q_j)\cap(\bigcup_B Q_j)]. \end{aligned}$$

Also $(\bigcup_A Q_j)\cap(\bigcup_B Q_j)\supseteq \bigcup_{A\cap B} Q_j$ with equality if (b) holds. Therefore either (a) or (b) implies $v[(\bigcup_A Q_j)\cap(\bigcup_B Q_j)]\geq v(\bigcup_{A\cap B} Q_j)$. Since $(\bigcup_A Q_j)\cup(\bigcup_B Q_j)=\bigcup_{A\cup B} Q_j$ we obtain $z(A)+z(B)\geq z(A\cup B)+z(A\cap B)$.

Proposition 2.6. *Let d_j be the weight of $j\in E$. The linear set function $z(S)=\sum_{j\in S} d_j$, $S\subseteq E$, is submodular.*

Proposition 2.7. *A positive linear combination of submodular functions is submodular.*

Proposition 2.8. *Given a submodular set function $v(S)$, $S\subseteq E$, then the set function $z(S)=v(E-S)$ is submodular on $S\subseteq E$.*

Proof.

$$\begin{aligned} z(A)+z(B) &= v(E-A)+v(E-B) \\ &\geq v[(E-A)\cup(E-B)]+v[(E-A)\cap(E-B)] \\ &= v(E-A\cap B)+v(E-A\cup B)=z(A\cap B)+z(A\cup B). \end{aligned}$$

3. Some classes of submodular functions

A. Matroid optimization

Let $\mathcal{M}=(E,\mathcal{F})$ be a matroid, c_e the weight of $e\in E$, and $\mathcal{F}(E')=\{F\colon F\in\mathcal{F}, F\subseteq E'\}$, $E'\subseteq E$. Members of the complement of $\mathcal{F}$ are called *dependent* sets, minimal dependent sets are called *circuits*. If F is independent and $F\cup\{j\}$ is dependent then $F\cup\{j\}$ contains exactly one circuit. If C^1 and C^2 are distinct circuits of F and $e\in C^1\cap C^2$, then $C^1\cup C^2-\{e\}$ contains a circuit. (Refer to Section 1 for other terminology.)

Edmonds [5] has given a greedy algorithm that finds a maximum weight independent set in a matroid. (Note that, since subsets of independent sets are independent, every maximum weight independent set contains a maximum weight set in which all elements have positive weight. Thus we assume, without loss of generality, that all elements of E have positive weight.) The greedy algorithm proceeds as follows. Arrange the elements of E in a list such that e' above e implies $c_{e'}\geq c_e$. Examine the topmost element of the list not yet considered and select it if it does not form a circuit with some of the elements already selected. The subset selected after the last element has been considered is of maximum weight.

Lemma 3.1. *Assume that the elements of E have been put in a list as described above. Let F be the maximum weight independent set in $\mathcal{F}(E')$ given by the greedy algorithm, G the maximum weight independent set in $\mathcal{F}(E' \cup \{e\})$, $e \notin E'$, given by the greedy algorithm and $f \in E-(E' \cup \{e\})$. Exactly one of the following four statements is true.*

(1) *$F \cup \{f\}$ and $G \cup \{f\}$ are both independent.*

(2) *$F \cup \{f\}$ and $G \cup \{f\}$ are both dependent and contain the same unique circuit.*

(3) *$F \cup \{f\}$ is independent and $G \cup \{f\}$ is dependent.*

(4) *$F \cup \{f\}$ and $G \cup \{f\}$ are both dependent. $G = F \cup \{e\} - \{e_r\}$, $e_r \neq e$, and e_r is in the circuit of $F \cup \{f\}$.*

Proof. These four statements are mutually exclusive. Furthermore, the greedy algorithm implies that G must be F, $F \cup \{e\}$ or $F \cup \{e\} - \{e_r\}$, $e_r \neq e$. More precisely $G = F \cup \{e\}$ if $F \cup \{e\}$ is independent and otherwise $G = F \cup \{e\} - \{e_r\}$, where e_r (e_r can be e) is a minimum weight element in the unique circuit of $F \cup \{e\}$.

We now prove, by considering separately the three possibilities for G, that one of the four statements must be obtained.

(a) $G = F$. Either (1) or (2) must be true.

(b) $G = F \cup \{e\}$. If $G \cup \{f\} \in \mathcal{F}$ then $F \cup \{f\} \in \mathcal{F}$ as $F \subset G$ and we obtain (1). If $G \cup \{f\} \notin \mathcal{F}$, we obtain (2) if $F \cup \{f\} \notin \mathcal{F}$ and we obtain (3) if $F \cup \{f\} \in \mathcal{F}$.

(c) $G = F \cup \{e\} - \{e_r\}$, $e_r \neq e$. Let C be the unique circuit in $F \cup \{e\}$. Suppose first that $F \cup \{f\} \in \mathcal{F}$. Then C is also the unique circuit in $F \cup \{e, f\}$. Thus $e_r \in C$ implies that $F \cup \{e, f\} - \{e_r\} = G \cup \{f\}$ is independent and we obtain (1). Now suppose that $F \cup \{f\} \notin \mathcal{F}$ and let $\hat{C}$ be the unique circuit in $F \cup \{f\}$. If $e_r \in \hat{C}$, then $C \cup \hat{C} - \{e_r\}$ contains a circuit in $G \cup \{f\}$ and we obtain (4). If $e_r \notin \hat{C}$, then $\hat{C} \subseteq F \cup \{f\} - \{e_r\} \subset G \cup \{f\}$. Hence $\hat{C}$ is the unique circuit of $G \cup \{f\}$ and we obtain (2).

Proposition 3.1.[3] *The set function*

$$v(E') = \max_{F \in \mathcal{F}(E')} \sum_{e \in F} c_e, \quad E' \subseteq E,$$

is submodular and nondecreasing.

Proof. Since $\hat{E} \subset E'$ implies $\mathcal{F}(\hat{E}) \subseteq \mathcal{F}(E')$, v is nondecreasing.

To prove submodularity let F, G, e and f be defined as in Lemma 3.1. By Proposition 2.1 it suffices to show that $\rho_f(E') \geq \rho_f(E' \cup \{e\})$. We will establish this result for the four mutually exclusive and collectively exhaustive statements of Lemma 3.1. Note that for all $\hat{E} \subset E$, $\rho_f(\hat{E}) \leq c_f$.

[3] J. Edmonds has pointed out to us that Proposition 3.1 also can be proved using Proposition 2.5 and the fact that the rank function of a matroid is submodular [4]. We prove Proposition 3.1 here because it does not seem to be in the literature, except in an unpublished paper by Woodall [10], which proves a more general result for polymatroids.

(1) and (3). Since $F \cup \{f\} \in \mathcal{F}$, $\rho_f(E') = c_f$.

(2) $\rho_f(E') = \rho_f(E' \cup \{e\}) = c_f - \min_{g \in C} c_g$, where C is the unique circuit in $F \cup \{f\}$ and $G \cup \{f\}$.

(4) In this case $G = F \cup \{e\} - \{e_r\}$, $e_r \neq e$, where $e_r \in C$, the unique circuit of $F \cup \{e\}$, and $e_r \in \hat{C}$, the unique circuit of $F \cup \{f\}$. Thus the unique circuit C' of $G \cup \{f\}$ is contained in $C \cup \hat{C} - \{e_r\}$ and

$$\rho_f(E' \cup \{e\}) = c_f - \min_{g \in C'} c_g \leq c_f - \min_{g \in C \cup \hat{C} - \{e_r\}} c_g$$
$$\leq c_f - \min_{g \in \hat{C}} c_g = \rho_f(E'),$$

where the last inequality follows from $G = F \cup \{e\} - \{e_r\}$, which implies $c_{e_r} = \min_{g \in C} c_g$.

Combining Proposition 2.5 with Proposition 3.1 we obtain.

Proposition 3.2. *Let* $\{Q_j\}$, $j \in N$, *be a collection of subsets of* E. *The set function* $z(S) = \max\{\sum_{e \in F} c_e : F \in \mathcal{F}(\bigcup_{j \in S} Q_j)\}$, $S \subseteq N$, *is submodular and nondecreasing.*

To interpret the use of Proposition 3.2 in the context of problem (1.6) we consider a graphic matroid. Given a graph G, a subset of its edges is an independent set if the subgraph induced by these edges is a forest (contains no cycles). Each edge is assigned a weight and one or more colors from the set N; Q_j, $j \in N$, is the subset of edges that are colored j. $z(S)$, $S \subseteq N$, is the value of a maximum weight forest that contains colors only in the set S. Problem (1.6) is to find a maximum weight forest that contains no more than K colors.

B. Generalized transportation problems

Let I be a set of sources, J a set of sinks and c_{ik} the value of assigning source i to sink k. Consider the family of transportation (or assignment) problems parametrized by $T \subseteq I$:

$$\begin{aligned} v(T) = \max & \sum_{i \in T} \sum_{k \in J} c_{ik} x_{ik}, \\ & \sum_{i \in T} x_{ik} \leq b_k, \quad k \in J, \\ & \sum_{k \in J} x_{ik} \leq 1, \quad i \in T, \\ & x_{ik} \geq 0, \quad i \in T, \quad k \in J. \end{aligned} \tag{3.1}$$

Proposition 3.3 (Shapley [7]). $v(T)$, $T \subseteq I$, *given by* (3.1) *is submodular.*

We now consider a generalization of (3.1) in which there is a set N of suppliers. The jth supplier, $j \in N$, can provide a_{ij} units from source i and has a fixed cost of d_j. Let $z(S)$ be the value of an optimal solution to the transportation problem when the set of suppliers $S \subseteq N$ is employed, i.e.,

$$\begin{aligned} z(S) = \max & \sum_{i\in I}\sum_{k\in J} c_{ik}x_{ik} - \sum_{j\in S} d_j, \\ & \sum_{i\in I} x_{ik} \le b_k, \quad k \in J, \\ & \sum_{k\in J} x_{ik} \le \sum_{j\in S} a_{ij}, \quad i \in I, \\ & x_{ik} \ge 0, \qquad i \in I, \quad k \in J. \end{aligned} \tag{3.2}$$

Proposition 3.4. *$z(S)$, $S \subseteq N$, given by* (3.2) *is submodular.*

Proof. For the moment ignore the fixed costs $\{d_j\}$ and consider a problem of the form (3.1) with $\sum_{i\in I}\sum_{j\in N} a_{ij}$ sources each with an availability of one unit. Let M be the set of these sources and $\bar{v}(T)$, $T \subseteq M$, the maximum value solution of (3.1) that can be obtained from the subset T. By Proposition 3.3, $\bar{v}(T)$ is submodular. Now let Q_j be the subset of M corresponding to the jth supplier. By Proposition 2.5, $\bar{z}(S) = \bar{v}(\bigcup_{j\in S} Q_j)$, $S \subseteq N$, is submodular. Finally, the submodularity of $z(S) = \bar{z}(S) - \sum_{j\in S} d_j$ is a consequence of Propositions 2.6 and 2.7.

We note that with $z(S)$ given by (3.2), problem (1.6), without the cardinality constraint, gives an optimal set of suppliers. In the particular case in which each supplier is associated with only one source, we obtain the so-called "capacitated location problem".

C. Boolean polynomials

Let g be a real-valued function on the set of subsets of N and for all $S \subseteq N$ define

$$\begin{aligned} & z(S) = \sum_{T\subseteq S} g(T) = \sum_{T\subseteq N} g(T) \prod_{j\in T} x_j, \\ & \text{where } x_j = \begin{cases} 1, & j \in S, \\ 0, & j \notin S. \end{cases} \end{aligned} \tag{3.3}$$

We note that any set function $z(S)$ can be represented by (3.3) by defining $g(\emptyset) = z(\emptyset)$ and $g(S) = z(S) - \sum_{T\subset S} g(T)$. The problem of maximizing $z(S)$ is a nonlinear boolean program.

Proposition 3.5. *$z(S)$ given by* (3.3) *is submodular if and only if $\sum_{T\subseteq S} g(T \cup \{j, k\}) \le 0$ for all $S \subset N$ and $j, k \notin S$, $j \ne k$, and nondecreasing if and only if $\sum_{T\subseteq S} g(T \cup \{j\}) \ge 0$ for all $S \subset N$ and $j \notin S$.*

Proof. We have

$$\rho_j(S) = \sum_{T\subseteq S\cup\{j\}} g(T) - \sum_{T\subseteq S} g(T) = \sum_{T\subseteq S} g(T \cup \{j\})$$

and

$$\rho_j(S) - \rho_j(S \cup \{k\}) = \sum_{T \subseteq S} g(T \cup \{j\}) - \sum_{T \subseteq S \cup \{k\}} g(T \cup \{j\})$$

$$= - \sum_{T \subseteq S} g(T \cup \{j, k\}).$$

An interesting case of (3.3) is the quadratic boolean polynomial that is obtained by putting $g(T) = 0$ for $|T| > 2$. Here a simple necessary and sufficient condition for submodularity is $g(\{i, j\}) \leq 0$ for all $\{i, j\}$ with $i \neq j$, and z is nondecreasing if and only if $g(\{j\}) \geq -\sum_{i \neq j} g(\{i, j\})$ for all $j \in N$. In this context, problem (1.6) takes the form:

$$\max\{xHx: ex \leq K, x_j \in \{0, 1\}, j = 1, \dots, n\},$$

where H is a symmetric matrix with non-positive off-diagonal elements and $e = (1, 1, \dots, 1)$.

Submodular quadratic polynomials can be used to describe the cut function of a graph. Given an undirected graph $G = (V, E)$, let $w'_{ij} \geq 0$, $i < j$ be the weight of edge $(i, j) \in E$. A cut in G is a partition of the vertices into sets S and $V - S$. The value of the cut is given by the function $v(S) = \sum_{i \in S} \sum_{j \in V-S} w_{ij}$, where $w_{ij} = w_{ji} = w'_{ij}$ if $(i, j) \in E$ and $w_{ij} = 0$ otherwise.

Let $z(S) = -\frac{1}{2} \sum_{i \in S} \sum_{j \in S} w_{ij}$; z is submodular since

$$\rho_r(S) - \rho_r(S \cup \{k\}) = \sum_{j \in S \cup \{k\}} w_{rj} - \sum_{j \in S} w_{rj} = w_{rk} \geq 0.$$

The submodularity of v follows by noting that $v(S) = z(S) + z(V - S) - z(V)$ and then using Propositions 2.7 and 2.8. Problem (1.6) in this context, without the cardinality constraint, is to find a maximum cut in the graph. Using the set function z we see that it is equivalent to

$$\max\{\tfrac{1}{2}[eWe - xWx - (e - x)W(e - x)]: x_j \in \{0, 1\}, j = 1, \dots, n\},$$

where $W = \{w_{ij}\}$.

4. The greedy heuristic for submodular set functions

A natural way to find solutions to problem (1.6) quickly is to start from the null set and add elements one at a time, taking at each step that element which increases z the most. The resulting solution is called a "greedy" solution, and the procedure, which we now define formally, the "greedy" heuristic. As before, we let $\rho_j(S) = z(S \cup \{j\}) - z(S)$.

The greedy heuristic for set functions

Initialization. Let $S^0 = \emptyset$, $N^0 = N$ and set $t = 1$.

Iteration t. Select $i(t) \in N^{t-1}$ for which $\rho_{i(t)}(S^{t-1}) = \max_{i \in N^{t-1}}(S^{t-1})$ with ties settled arbitrarily. Set $\rho_{t-1} = \rho_{i(t)}(S^{t-1})$.

Step 1. If $\rho_{t-1} \leq 0$, stop with the set S^{K^*}, $K^* = t-1 < K$. If $\rho_{t-1} > 0$, set $S^t = S^{t-1} \cup \{i(t)\}$ and $N^t = N^{t-1} - \{i(t)\}$. Continue.

Step 2. If $t = K$, stop with the set S^{K^*}, $K^* = K$. Otherwise set $t \to t+1$.

Let Z be the value of an optimal solution to problem (1.6) and let Z^G be the value of a particular solution to (1.6) constructed by the greedy heuristic. Note that

$$Z^G = z(\emptyset) + \rho_0 + \cdots + \rho_{K^*-1}, \quad K^* \leq K,$$

and, since the $\{\rho_t\}$ may depend on how ties are settled, Z^G may as well. We will assume throughout this section that $K^* \geq 1$, thus excluding trivial problems with $Z = Z^G = z(\emptyset)$.

Let $C(\theta)$ be the class of submodular set functions satisfying $\rho_j(S) \geq -\theta$, $\forall S \subset N$, $j \in N - S$.

Proposition 4.1. *Suppose $z \in C(\theta)$, $\theta \geq 0$, and the greedy heuristic stops after K^* steps, then the corresponding $\{\rho_i\}_{i=0}^{K^*-1}$ satisfy*

$$Z \leq z(\emptyset) + \sum_{i=0}^{t-1} \rho_i + K\rho_t + t\theta, \quad t = 0, \ldots, K^*-1$$

and also (4.1)

$$Z \leq z(\emptyset) + \sum_{i=0}^{K^*-1} \rho_i + K^*\theta, \quad \textit{if } K^* < K.$$

Proof. By Proposition 2.3, $z(T) \leq z(S) + \sum_{j \in T-S} \rho_j(S) + |S - T|\theta$.

Taking T to be an optimal solution of problem (1.6), S to be the set S^t generated after t iterations of the greedy heuristic, and using

$$Z = z(T), \quad \rho_j(S^t) \leq \rho_t, \quad \rho_t \geq 0,$$
$$|S^t - T| \leq t, \quad \theta \geq 0, \quad |T - S^t| \leq K,$$

and

$$z(S^t) = z(\emptyset) + \sum_{i=0}^{t-1} \rho_i,$$

we obtain

$$Z \leq z(\emptyset) + \sum_{i=0}^{t-1} \rho_i + K\rho_t + t\theta, \quad t = 0, \ldots, K^*-1.$$

In addition, if $K^* < K$, taking $S = S^{K^*}$ yields

$$Z \leq z(\emptyset) + \sum_{i=0}^{K^*-1} \rho_i + K^*\theta$$

as $\rho_{K^*} \leq 0$.

From Proposition 4.1 we immediately obtain some simple results. For example, putting $\theta = 0$ in the last inequality of (4.1) yields

Proposition 4.2. *If the greedy heuristic is applied to problem* (1.6) *with z non-decreasing and stops after $K^* < K$ steps, the greedy solution is optimal.*

From the first inequality of (4.1) with $t = 0$ we obtain

Proposition 4.3. *If the greedy heuristic is applied to problem* (1.6) *then*

$$\frac{Z - Z^G}{Z - z(\emptyset)} \leq \frac{K-1}{K}.$$

Proof. The inequality for $t = 0$ of (4.1) yields $Z - z(\emptyset) \leq K\rho_0 \leq K(Z^G - z(\emptyset))$ or, equivalently,

$$\frac{Z - Z^G}{Z - z(\emptyset)} \leq \frac{K-1}{K}.$$

The bound of Proposition 4.3 can be tight only for very large values of θ. For example, when $\theta = 0$ we will obtain the much sharper bound

$$\frac{Z - Z^G}{Z - z(\emptyset)} \leq \left(\frac{K-1}{K}\right)^K.$$

In fact, we will obtain a family of K bounds, each one of which is tight for a different interval containing θ. These intervals cover the whole nonnegative domain of θ that is of interest. The bounds are obtained by applying linear programming to the problem of minimizing Z^G subject to the inequalities (4.1). Lemma 4.1 states the linear program and its solution. After proving the lemma, we will use it to establish the bounds. Let $\alpha = (K-1)/K$.

Lemma 4.1. *Given positive integers j and K, $j < K$, and a non-negative real number b, let*

$$\begin{aligned} P(b) &= Kb + \min \sum_{i=0}^{j} x_i, \\ &\sum_{i=0}^{t-1} x_i + Kx_t \geq 1 - (K+t)b, \quad t = 0, \dots, j, \\ &\sum_{i=0}^{j} x_i \geq 1 - (K+j+1)b, \end{aligned} \tag{4.2}$$

then

$$P(b) = \begin{cases} 1 - (j+1)b & \text{if } b \leq \alpha^{j+1}/K, \\ 1 + (K-j-1)b - \alpha^{j+1} & \text{if } b \geq \alpha^{j+1}/K, \end{cases}$$

and

$$\min_{b \geq 0} P(b) = 1 - \left(\frac{j+1}{K}\right)\alpha^{j+1}.$$

If the last constraint is omitted from (4.2), *then* $P(b) = 1 + (K - j - 1)b - \alpha^{j+1}$ *for all* $b \geq 0$.

Proof. The dual of problem (4.2) is

$$\begin{aligned} &W(b) = Kb + \max \sum_{t=0}^{j+1} \{1 - (K+t)b\}u_t, \\ &Ku_i + \sum_{t=i+1}^{j+1} u_t = 1, \quad i = 0, \dots, j, \\ &u_t \geq 0, \quad t = 0, \dots, j+1. \end{aligned} \tag{4.3}$$

We now proceed to calculate $W(b)$, and hence by LP duality $P(b)$. Let $\lambda = 1 - u_{j+1}$ in (4.3). Then we observe that feasible values of the remaining variables u_t, $t = 0, \dots, j$, are uniquely determined with $u_t = (\lambda/K)\alpha^{j-t}$ and

$$\sum_{t=0}^{j} \{1 - (K+t)b\}u_t = \lambda[1 - \alpha^{j+1} - (j+1)b].$$

Therefore

$$\begin{aligned} W(b) &= \max_{0 \leq \lambda \leq 1} \{Kb + \lambda[1 - \alpha^{j+1} - (j+1)b] + (1-\lambda)[1 - (K+j+1)b]\} \\ &= \max_{0 \leq \lambda \leq 1} \{1 - (j+1)b + \lambda(Kb - \alpha^{j+1})\}. \end{aligned}$$

It follows immediately that $\lambda = 0$ ($u_{j+1} = 1$), if $Kb < \alpha^{j+1}$, and $\lambda = 1$ ($u_{j+1} = 0$) if $Kb > \alpha^{j+1}$. Therefore

$$W(b) = \max\{1 - (j+1)b, 1 + (K - j - 1)b - \alpha^{j+1}\}$$

and

$$P(b) = W(b) = \begin{cases} 1 - (j+1)b, & \text{if } b \leq \alpha^{j+1}/K, \\ 1 + (K - j - 1)b - \alpha^{j+1}, & \text{if } b \geq \alpha^{j+1}/K. \end{cases} \tag{4.4}$$

Now we observe that as $j + 1 > 0$ and $K - j - 1 \geq 0$,

$$\min_{b \geq 0} P(b) = P\left(\frac{\alpha^{j+1}}{K}\right) = 1 - \left(\frac{j+1}{K}\right)\alpha^{j+1}.$$

Consider now the case where the last constraint of (4.2) is omitted. Dropping this constraint is equivalent to finding an optimal dual solution with $u_{j+1} = 0$. But then from (4.4) we obtain $P(b) = 1 + (K - j - 1)b - \alpha^{j+1}$.

Theorem 4.1. *If the greedy heuristic is applied to problem* (1.6) *with* $z \in C(\theta)$, *then*

(a) *if it terminates after* K^* *steps, then*

$$\frac{Z - Z^G}{Z - z(\emptyset) + K\theta} \leq \left(\frac{K^*}{K}\right)\alpha^{K^*},$$

(b) *if*

$$0 \leq \frac{\theta}{Z - z(\emptyset) + K\theta} \leq \frac{\alpha^k}{K},$$

then

$$\frac{Z - Z^G}{Z - z(\emptyset) + K\theta} \le \alpha^{k+1} - \frac{\theta(K-k-1)}{Z - z(\emptyset) + K\theta} \le \alpha^{k+1}, \quad k = 0, \dots, K-1,$$

(c) *there is a family of problems of the form* (1.6) *such that for* $k = 1, \dots, K-2$ *and*

$$\frac{\alpha^{k+1}}{K} \le \frac{\theta}{Z - z(\emptyset) + K\theta} \le \frac{\alpha^k}{K},$$

and for $k = K - 1$ *and*

$$0 \le \frac{\theta}{Z - z(\emptyset) + K\theta} \le \frac{\alpha^k}{K},$$

the first inequality of (b) *is an equality and* $K^* = k + 1$.

Before proving Theorem 4.1 we state a much simpler version of it. Although far less general, the simplified version presents the most useful part of Theorem 4.1. The right-hand side of (a) increases with K^*, hence setting $K^* = K$ we obtain

Theorem 4.2.[4] *If the greedy heuristic is applied to problem* (1.6) *with* $z \in C(\theta)$, *then*

$$\frac{Z - Z^G}{Z - z(\emptyset) + K\theta} \le \alpha^K.$$

Proof of Theorem 4.1. We first dispose of the case $\theta < 0$ (z strictly increasing) in (a). Let $z'(S) = z(S) + |S|\theta$. We observe that $Z - Z^G = Z' - Z'^G$, $Z - z(\emptyset) + K\theta = Z' - z'(\theta)$ and z' is nondecreasing. Therefore applying the result for $\theta = 0$ to z' yields the desired conclusion for z. For the remainder of the proof we assume that $\theta \ge 0$.

The rest of the proof uses Lemma 4.1 with

$$b = \frac{\theta}{Z - z(\emptyset) + K\theta} \quad \text{and} \quad x_i = \frac{\rho_i}{Z - z(\emptyset) + K\theta}.$$

With this transformation of variables the inequalities of the linear program are identical to the inequalities (4.1) of Proposition 4.1 and we obtain

$$P(b)(Z - z(\emptyset) + K\theta) \le K\theta + \sum_{i=0}^{j} \rho_i. \tag{4.5}$$

Now for $j < K^*$, $\sum_{i=0}^{j} \rho_i \le Z^G - z(\emptyset)$ and (4.5) yields

$$P(b)(Z - z(\emptyset) + K\theta) \le K\theta + Z^G - z(\emptyset). \tag{4.6}$$

[4] For uncapacitated location problems this result appears in [2].

The proof of (a) is now separated into two parts.

($K^* < K$). Here with $j+1 = K^*$ all of the inequalities of (4.2) are valid and from Lemma 4.1,

$$P(b) \geq 1 - \left(\frac{K^*}{K}\right)\alpha^{K^*}. \tag{4.7}$$

By substituting (4.7) into (4.6) and doing some algebraic manipulation we obtain the result (a).

($K^* = K$). Here with $j+1 = K$ only the first K inequalities of (4.2) are valid and from Lemma 4.1

$$P(b) \geq 1 + (K-K)b - \alpha^K = 1 - \alpha^K. \tag{4.8}$$

Now substituting (4.8) into (4.6) and doing some algebraic manipulation yields (a).

To prove (b) for a given non-negative integer k, suppose first that $k < K^*$. Thus the first $k+1$ inequalities of (4.2) are valid and by Lemma 4.1

$$P(b) \geq 1 + \frac{\theta(K-k-1)}{Z - z(\emptyset) + K\theta} - \alpha^{k+1}. \tag{4.9}$$

Substituting (4.9) into (4.6) and doing some algebraic manipulation yields

$$\frac{Z - Z^G}{Z - z(\emptyset) + K\theta} \leq \alpha^{k+1} - \frac{\theta(K-k-1)}{Z - z(\emptyset) + K\theta}, \quad \text{if } k < K^*. \tag{4.10}$$

The right-hand side of (4.10) decreases with $\theta/(Z - z(\emptyset) + K\theta)$ and $\alpha^{k+1} - (\alpha^k/K)(K-k-1) = (k/K)\alpha^k$. Therefore

$$\left(\frac{k}{K}\right)\alpha^k \leq \alpha^{k+1} - \frac{\theta(K-k-1)}{Z - z(\emptyset) + K\theta}, \quad \text{if } 0 \leq \frac{\theta}{Z - z(\emptyset) + K\theta} \leq \frac{\alpha^k}{K}. \tag{4.11}$$

Now from (a) we have

$$\frac{Z - Z^G}{Z - z(\emptyset) + K\theta} \leq \left(\frac{K^*}{K}\right)\alpha^{K^*} \leq \left(\frac{k}{K}\right)\alpha^k, \quad \text{if } k \geq K^* \tag{4.12}$$

since $(k/K)\alpha^k$ increases with k. Combining (4.11) and (4.12) yields

$$\frac{Z - Z^G}{Z - z(\emptyset) + K\theta} \leq \alpha^{k+1} - \frac{\theta(K-k-1)}{Z - z(\emptyset) + K\theta},$$
$$\text{if } k \geq K^* \quad \text{and} \quad 0 \leq \frac{\theta}{Z - z(\emptyset) + K\theta} \leq \frac{\alpha^k}{K}. \tag{4.13}$$

Finally, combining (4.10) and (4.13) yields (b).

(c) For $K = 2, 3, \ldots$, let C^K be a $K(K-1)$ by $2K-1$ matrix with entries as follows:

$$\text{for } j = 1, \ldots, K-1, \quad c_{ij}^K = \begin{cases} (K-1)K^{K-2}\alpha^{j-1}, & \text{if } i = (j-1)K+1, \ldots, jK, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\text{for } j = K, \dots, 2K-1, \quad c_{ij}^K = \begin{cases} K^{K-1}, & \text{if } i = 1 + j + (l-2)K, \quad l = 1, \dots, K-1, \\ 0, & \text{otherwise,} \end{cases}$$

and let θ be a nonnegative scalar. We claim that $z(S) = \sum_{i \in I} \max_{j \in S} c_{ij}^K - \theta|S|$ is the required function.

For this function, problem (1.6) is an uncapacitated location problem with a fixed cost of θ at each location. For the case $\theta = 0$ it can be shown (the details are given in [2]) that the greedy heuristic can select the first K columns so that $Z^G = (K-1)K^K(1-\alpha^K)$, while the last K columns are an optimal set so that $Z = (K-1)K^K$. More generally when

$$\alpha^{k+1} \le \frac{\theta}{(K-1)K^{K-1}} \le \alpha^k, \quad k = 0, \dots, K-2$$

$$\text{or} \quad 0 \le \theta \le (K-1)^K \quad \text{for} \quad k = K-1,$$

the greedy heuristic can select the first $k+1$ columns while the last K columns are an optimal set. We then obtain $Z^G = (K-1)K^K(1-\alpha^{k+1}) - (k+1)\theta$ and $Z = (K-1)K^K - K\theta$. Using these values of θ, Z^G and Z, (c) is easily verified.

Having analyzed the worst-case behavior of the greedy heuristic, we consider whether the results can be substantially improved by finding the R best possible elements to add to the given solution at each iteration. The R-step greedy heuristic requires $O(n^{R+1})$ evaluations of z so that increasing R by one increases the number of computations by a factor of n. However we will show that, in the worst case, increasing R does not yield a substantial improvement in the quality of solutions. For simplicity, we consider only nondecreasing functions.

The R-step greedy heuristic for set functions

Suppose $K = qR - p$, where q is a positive integer and $0 \le p < R$. Let $S^t = \bigcup_{i=1}^t I^i$ and $S^0 = \emptyset$. For $t = 1, \dots, q-1$ choose $I^t \subseteq N - S^{t-1}$ with $|I^t| = R$ so as to maximize $\zeta_{t-1} = z(S^t) - z(S^{t-1})$. Finally choose $I^* \subseteq N - S^{q-1}$ with $|I^*| = R - p$ so as to maximize $\eta = z(S^{q-1} \cup I^*) - z(S^{q-1})$.

Let

$$Z^{G(R)} = z(\emptyset) + \sum_{i=0}^{q-2} \zeta_i + \eta = z(S^{q-1} \cup I^*)$$

denote the value of an R-step greedy solution.

Theorem 4.3. *Suppose z is nondecreasing and the R-step greedy heuristic is applied to problem* (1.6):

(a) *If $K = qR - p$, with q a positive integer, p integer $0 \le p \le R-1$,*

$$\frac{Z - Z^{G(R)}}{Z - z(\emptyset)} \le \left(\frac{q-\lambda}{q}\right)\left(\frac{q-1}{q}\right)^{q-1} < \left(\frac{q-1}{q}\right)^{q-1}, \quad \textit{where } \lambda = \frac{R-p}{R}.$$

(b) *If $p = 0$, i.e., K is a multiple of R, the bound is tight.*

Proof. (a) Let $\zeta_{q-1} = \max_{|I|=R} z(S^{q-1} \cup I) - z(S^{q-1})$ and T be an optimal solution to (1.6). By submodularity and Proposition 2.4, $z(T) \le z(S^t) + \sum_{i=1}^{q} \rho_{J_i}(S^t)$, where $\{J_i\}_{i=1}^{q}$ is any partition of $T - S^t$ with $|J_i| \le R$, $i = 1, \ldots, q$. Since $z(S^t) = z(\emptyset) + \sum_{i=0}^{t-1} \zeta_i$ and $\rho_{J_i}(S^t) \le \zeta_t$, we have

$$Z \le z(\emptyset) + \zeta_0 + \zeta_1 + \cdots + \zeta_{t-1} + q\zeta_t, \quad t = 0, \ldots, q-1. \tag{4.14}$$

Also

$$Z^{G(R)} = z(\emptyset) + \sum_{i=0}^{q-2} \zeta_i + \eta \ge z(\emptyset) + \sum_{i=0}^{q-2} \zeta_i + \lambda\zeta_{q-1} = \tilde{Z}^{G(R)},$$

since submodularity implies $\eta \ge \lambda\zeta_{q-1}$. Therefore

$$\frac{Z - \tilde{Z}^{G(R)}}{Z - z(\emptyset)} \ge \frac{Z - Z^{G(R)}}{Z - z(\emptyset)}.$$

It now suffices to prove that

$$\min_{\{\zeta_i\}_{i=0}^{q-1}} \frac{\tilde{Z}^{G(R)} - z(\emptyset)}{Z - z(\emptyset)} \quad \text{subject to (4.14)} \tag{4.15}$$

equals $1 - [(q-\lambda)/q][(q-1)/q]^{q-1}$. Reasoning as in the analysis of the 1-step greedy heuristic (see Lemma 4.1 with $b = 0$), we can formulate problem (4.15) as the linear program

$$\min \sum_{i=0}^{q-2} \zeta_i + \lambda\zeta_{q-1},$$

$$\sum_{i=0}^{t-1} \zeta_i + q\zeta_t \ge 1, \quad t = 0, \ldots, q-1.$$

Its dual is

$$\max \sum_{t=0}^{q-1} u_t,$$

$$qu_i + \sum_{t=i+1}^{q-1} u_t = 1, \quad i = 0, \ldots, q-2$$

$$qu_{q-1} = \lambda, \quad u_t \ge 0, \quad t = 0, \ldots, q-1.$$

The solutions

$$\zeta_i = \frac{1}{q}\left(\frac{q-1}{q}\right)^i, \quad i = 0, \ldots, q-1,$$

$$u_t = \frac{1}{q}\left(\frac{q-\lambda}{q-1}\right)\left(\frac{q-1}{q}\right)^{q-t-1}, \quad t = 0, \ldots, q-2, \quad u_{q-1} = \frac{\lambda}{q},$$

are primal and dual feasible, and give the required result.

(b) To show that the bound $(Z - Z^{G(R)})/(Z - z(\emptyset)) = ((q-1)/q)^q$ is attained when $K = qR$, take the uncapacitated location problem with

$$C = \begin{bmatrix} C^q & & & \\ & C^q & & \\ & & \ddots & \\ & & & C^q \end{bmatrix},$$

where the matrix C^q defined in the proof of part (c) of Theorem 4.1 is repeated R times along the diagonal. At iteration t the R-step greedy heuristic can choose the tth column of each matrix C^q, and therefore the behavior is the same same as if the greedy heuristic is applied to C^q.

Example. $K = 6$, $R = 3$, $q = 2$,

$$C = \begin{bmatrix} C^2 & & \\ & C^2 & \\ & & C^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

We obtain

$$Z = 12, \qquad Z^{G(3)} = 9, \qquad \frac{Z - Z^{G(R)}}{Z - z(\emptyset)} = \frac{12-9}{12} = \left(\frac{2-1}{2}\right)^2.$$

Consider the problem

$$\max_{S \subseteq N} \{z(S) \colon |S| \geq |N| - K,\ z(S) \text{ submodular}\}. \tag{4.16}$$

For problem (4.16) we can apply a "stingy heuristic", where one starts with the set N at each step removes an element so as to maximize the value of the remaining ones. An alternative but equivalent view of this heuristic is to apply the greedy heuristic with the submodular set function $v(S) = z(N - S)$, see Proposition 2.8. This observation allows us to use Theorem 4.1 to obtain a worst case analysis for the stingy heuristic applied to (4.16). Let Z be the optimal value and Z^S the value of a stingy heuristic solution to (4.16).

Theorem 4.4. *If the stingy heuristic is applied to the problem* (4.16) *with* $\rho_j(S) \leq \psi$ *and terminates after* K^* *steps, then*

$$\frac{Z - Z^S}{Z - z(N) + K\psi} \leq \left(\frac{K^*}{K}\right)\alpha^{K^*}.$$

When $K = |N|$ problems (1.6) and (4.16) are identical and we can apply both the greedy and stingy heuristics and consequently obtain the better of the two bounds from Theorems 4.1 and 4.4.

5. The interchange heuristic for submodular set functions

Here we consider another familiar way of trying to generate a good solution for problem (1.6). If z is nondecreasing, there is an optimal solution containing K elements. Thus it makes sense to start from an arbitrary set of K elements and look for a subset of R or fewer elements that can be profitably replaced by an equal number of elements not in the set. The procedure is then repeated until no further improvements of this type can be made. The resulting solution is called an R-interchange solution, and the procedure the R-interchange heuristic.

The R-interchange heuristic for nondecreasing set functions

Initialization. Pick an arbitrary set $S^0 \subseteq N$ with $|S^0| = K$. Set $t = 1$.
Iteration t. Given a set S^{t-1} with $|S^{t-1}| = K$, try to find a set $P \subseteq N$ with $|P| = K$, and $|P - S^{t-1}| = |S^{t-1} - P| \le R$ such that $z(P) > z(S^{t-1})$. Set $S^t = P$, $t \to t+1$, and continue. If no such set P exists, stop. $S^* = S^{t-1}$ is an R-interchange solution.

Let $Z^{I(R)} = z(S^*)$ denote the value of an R-interchange solution.

We note that the above heuristic is far from being completely specified. In particular both the choice of starting set S^0, and the method of searching for P are arbitrary. The bound given in Theorem 5.1 applies, however, regardless of how we make these choices.

Theorem 5.1.[5] *Suppose z is nondecreasing, and the R-interchange heuristic is applied to problem* (1.6).

(a) *If* $K = qR - p$ *with q a positive integer, and p integer* $0 \le p \le R - 1$,

$$\frac{Z - Z^{I(R)}}{Z - z(\emptyset)} \le \frac{K - R + p}{2K - R + p}.$$

(b) *If* $p = 0$, *i.e., K is a multiple of R, the bound is tight.*

Proof. (a) For simplicity we show the result only for $p = 0$. Apply the R-step greedy heuristic of the previous section to problem (1.6) with N replaced by S^*, the elements of the R-interchange solution. We obtain a partition $\{I^i\}_{i=1}^q$ of S^* where

$$|I^i| = R, \qquad \zeta_{t-1} = z\Big(\bigcup_{i=1}^{t} I^i\Big) - z\Big(\bigcup_{i=1}^{t-1} I^i\Big), \qquad \zeta_0 \ge \cdots \ge \zeta_{q-1},$$

and $Z^{I(R)} = z(\emptyset) + \sum_{i=0}^{q-1} \zeta_i$.

Let $I^* = \bigcup_{i=1}^{q-1} I^i$, so that $S^* = I^* \cup I^q$. Let $T = \bigcup_{k=1}^{q} T^k$ be any partition of an optimal set T of cardinality K into q disjoint sets of size R.

By submodularity, as in Proposition 2.4,

$$Z = z(T) \le z(I^*) + \sum_{k=1}^{q} \{z(I^* \cup T^k) - z(I^*)\}.$$

However $S^* = I^* \cup I^q$ is an R-interchange solution. Therefore

$$\zeta_{q-1} = z(I^* \cup I^q) - z(I^*) \ge z(I^* \cup T^k) - z(I^*), \quad k = 1, \dots, q,$$

and hence $Z \le z(I^*) + q\zeta_{q-1} = Z^{I(R)} + (q-1)\zeta_{q-1}$. As $\zeta_i \ge \zeta_{i+1}$, $(q-1)\zeta_{q-1} \le ((q-1)/q)(Z^{I(R)} - z(\emptyset))$. So finally we obtain $Z \le Z^{I(R)} + ((q-1)/q)(Z^{I(R)} - z(\emptyset))$, which, after rewriting as $(2q-1)(Z - Z^{I(R)}) \le (q-1)(Z - z(\emptyset))$, gives the required result.

[5] For uncapacitated location problems and $R = 1$ this result appears in [2].

When $p > 0$ the proof is almost identical to the proof given above, except that in applying the R-step greedy heuristic to S^* a set of size $(R-p)$ is chosen first, followed by $(q-1)$ sets of size R.

(b) The R-interchange algorithm applied to the following class of uncapacitated location problems in the form of problem (1.6) shows that the bound is tight for $p = 0$.

The matrix $C = (c_{ij})$ has K^2 rows and $2K$ columns. The first K columns consist of K $(K \times K)$ identity matrices, and column $(K+s)$, $s = 1, \ldots, K$ has K successive entries equal to $(2K-R)/K$ in rows $K(s-1)+t$, $t = 1, \ldots, K$, and zeroes elsewhere. (An example with $K = 4$ and $R = 2$ is shown below.)

We claim that the first K columns form an R-interchange solution. Suppose without loss of generality that columns $K-R+1, \ldots, K$ are removed, and columns $K+1, \ldots, K+R$ are added. The decrease in value from dropping the columns is KR. The increase when adding the new columns is

$$R(K-R)\left(\frac{2K-R}{K}-1\right)+R^2\left(\frac{2K-R}{K}\right) = KR.$$

On the other hand, if fewer than R columns are interchanged, the objective value decreases.

Therefore the first K columns form an R-interchange solution and $Z^{I(R)} = K^2$. The last K columns clearly form the optimal solution with $Z = K^2((2K-R)/K) = K(2K-R)$. Hence

$$\frac{Z - Z^{I(R)}}{Z - z(\emptyset)} = \frac{K-R}{2K-R}.$$

Example. $K = 4$, $R = 2$,

$$C = \begin{bmatrix}
1 & & & & \frac{3}{2} & & & \\
& 1 & & & \frac{3}{2} & & & \\
& & 1 & & \frac{3}{2} & & & \\
& & & 1 & \frac{3}{2} & & & \\
1 & & & & & \frac{3}{2} & & \\
& 1 & & & & \frac{3}{2} & & \\
& & 1 & & & \frac{3}{2} & & \\
& & & 1 & & \frac{3}{2} & & \\
1 & & & & & & \frac{3}{2} & \\
& 1 & & & & & \frac{3}{2} & \\
& & 1 & & & & \frac{3}{2} & \\
& & & 1 & & & \frac{3}{2} & \\
1 & & & & & & & \frac{3}{2} \\
& 1 & & & & & & \frac{3}{2} \\
& & 1 & & & & & \frac{3}{2} \\
& & & 1 & & & & \frac{3}{2}
\end{bmatrix}.$$

When K is not a multiple of R the bound of Theorem 5.1 is weaker than the corresponding tight bound for $K = qR$. It is possible that for nondecreasing submodular functions the bound $(Z - Z^{I(R)})/(Z - z(\emptyset)) \leq (K - R)/(2K - R)$ holds for all $R \leq K$. We have established this result for nondecreasing functions that arise from uncapacitated location problems.

When $\theta > 0$ an optimal solution to problem (1.6) may contain fewer than K elements. Therefore, the R-interchange heuristic given above must be modified to allow sets of differing cardinality. In particular, at iteration t, given the set S^{t-1} we attempt to find a set $P \subseteq N$ with $|P| \leq K$, and $|P - S^{t-1}| \leq R$ and $|S^{t-1} - P| \leq R$ such that $z(P) > z(S^{t-1})$.

With this generalized version of the R-interchange heuristic we have for $R = 1$

Theorem 5.2. *Suppose $z \in C(\theta)$, $\theta \geq 0$, and the generalized* 1*-interchange heuristic is applied to problem* (1.6). *Then*

$$\frac{Z - Z^{I(1)}}{Z - z(\emptyset) + K\theta} \leq \frac{K - 1}{2K - 1}.$$

Proof. Let S be the interchange solution. If $|S| = K$, the proof parallels the proof of (a) in Theorem 5.1. If $|S| \leq K - 1$, then by Proposition 2.3 and $\theta \geq 0$

$$z(T) \leq z(S) + \sum_{j \in T-S} \rho_j(S) + (K - 1)\theta.$$

As S is an interchange solution $\rho_j(S) \leq 0$ for $j \in N - S$ and $\rho_j(S - \{j\}) \geq 0$ for $j \in S$. This last inequality implies $z(S) \geq z(\emptyset)$. Hence

$$z(T) \leq z(S) + (K - 1)\theta + \frac{K - 1}{K}(z(S) - z(\emptyset)).$$

Substituting $z(T) = Z$ and $z(S) = Z^{I(1)}$ into this last inequality yields the result.

One might hope that the greedy heuristic followed by the R-interchange heuristic would yield a significant improvement on both. Unfortunately, in terms of worst case behavior this improvement is not achieved. We have constructed a family of uncapacitated location problems for which the R-interchange heuristic cannot improve on the greedy heuristic. For these problems the error approaches α^K as $K \to \infty$. Examples of this behavior are given in [2] for $R = 1$.

In the worst case the interchange heuristic does not perform as well as the greedy heuristic. Also the number of iterations required by the interchange heuristic depends on the method used to find improving solutions. A poor method can take an exponential number of iterations as shown by

Theorem 5.3. *There is a family of uncapacitated location problems with* $|N| =$

$2K$, $K = 2, 3, \ldots$, *for which the* 1-*interchange heuristic can take* $2^{K+1} - (K+2)$ *iterations.*

Proof. The Kth problem, $K = 2, 3, \ldots$, is defined by the $K \times 2K$ matrix C^K with elements c_{ij}^K where for $i = 1, \ldots, K$

$$c_{ij}^K = \begin{cases} 2^i - 1, & j = 2i - 1, \\ 2(2^i - 1), & j = 2i, \\ 0, & \text{otherwise.} \end{cases}$$

Starting from $S^0 = \{1, 3, \ldots, 2K - 3, 2K - 1\}$ with $z(S^0) = 2^{K+1} - (K+2)$ there is easily seen to be a sequence of $2^{K+1} - (K+2)$ iterations where $z(S^t) = 2^{K+1} - (K+2) + t$ for $t = 0, 1, \ldots, 2^{K+1} - (K+2)$.

The example below gives C^3 and the sequence of interchanges for this problem.

Example

$$C^3 = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}.$$

The sequence of S^t, $t = 0, \ldots, 11 = 2^{K+1} - (K+2)$ is

$$\begin{array}{lllllll} (1,3,5) & (2,3,5) & (3,4,5) & (1,4,5) & (2,4,5) & (2,5,6) & (3,5,6) \\ (1,3,6) & (2,3,6) & (3,4,6) & (1,4,6) & (2,4,6). & & \end{array}$$

Note that the first five and last five sets are identical except that the 6th element has replaced the 5th. Also these five sets without the last element are precisely the sets generated in the problem defined by $K = 2$.

6. A linear programming approximation

Here we study problem (1.6) in the case where the submodular function is generated from a matroid as in Proposition 3.2; i.e., $z(S) = \max\{\sum_{e \in F} c_e \colon F \in \mathcal{F}(\bigcup_{j \in S} Q_j)\}$, and the $\{Q_j\}$, $j \in N$ are a partition of E and satisfy an independence condition to be given below. Under these conditions problem (1.6) can be formulated as an integer program and the linear programming relaxation of this integer program provides an upper bound on Z. The result we obtain is a bound on the "duality gap" between the optimal values of the integer and real solutions to this linear program. More generally, it is a strengthening of Theorem 4.2 in the case where $z(S)$ is generated from a matroid.

Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid, c_e the weight of $e \in E$, $\mathcal{F}(E') = \{F \colon F \in \mathcal{F}, F \subseteq E'\}$ for $E' \subseteq E$ and $v(E') = \max_{F \in \mathcal{F}(E')} \sum_{e \in F} c_e$.

Proposition 6.1 (Edmonds [5]).

$$v(E') = \max \sum_{e\in E} c_e x_e,$$

$$\sum_{e\in A} x_e \le r(A), \quad A \subseteq E, \tag{6.1}$$

$$x_e \ge 0, \quad e \in E', \qquad x_e = 0 \quad \textit{otherwise},$$

where $r(A)$ is the cardinality of a largest independent set in A and $x_e = 1$ if element e is chosen.

Thus if $z(S) = \max\{\sum_{e\in F} c_e\colon F \in \mathscr{F}(\bigcup_{j\in S} Q_j)\}$, where $\{Q_j\}$, $j \in N$ is a partition of E, problem (1.6) can be written as the integer program

$$Z = \max \sum_{e\in E} c_e x_e,$$

$$\sum_{e\in A} x_e \le r(A), \quad A \subseteq E, \tag{6.2}$$

$$x_e \le y_j, \quad e \in Q_j, \quad j \in N, \tag{6.3}$$

$$\sum_{j=1}^{n} y_j \le K, \tag{6.4}$$

$$x_e \ge 0, \quad e \in E, \tag{6.5}$$

$$y_j \in \{0, 1\}, \quad j \in N.$$

Define the linear programming relaxation of this integer program by

$$Z^{\mathrm{LP}} = \max \sum_{e\in E} c_e x_e$$

subject to (6.2)–(6.5) and

$$0 \le y_j \le 1, \quad j \in N. \tag{6.6}$$

We have $Z^{\mathrm{LP}} \ge Z$ and, by Theorem 4.1, $(Z - Z^G)/Z \le \alpha^K$, where Z^G is the value of the greedy solution to (1.6).

Theorem 6.1. *Let $\mathscr{M} = (E, \mathscr{F})$ be a matroid and $\{Q_j\}$, $j \in N$, be a partition of E that satisfies the independence condition: "if e, $f \in Q_j$, there is no circuit in $\mathscr{M}$ containing both e and f." Then*

$$\frac{Z^{\mathrm{LP}} - Z^G}{Z^{\mathrm{LP}}} \le \alpha^K.$$

The "independence condition" holds for the uncapacitated location problem, for which Theorem 6.1 is given in [2]. More generally when the matroid $(E, \mathscr{F})$ is obtained by combining r distinct matroids $(E_i, \mathscr{F}_i)$ so that $E = \cup E_i$, with $E_j \cap E_k = \emptyset$, $j \ne k$, and $F \in \mathscr{F}$ only if $F \cap E_i \in \mathscr{F}_i$ for all i,[6] the independence property will hold if the $\{Q_j\}$ are chosen to satisfy $|E_i \cap Q_j| \le 1$, $i = 1, \dots, r$, $j \in N$.

[6] R. Giles has pointed out to us that these conditions are also necessary for a matroid $(E, \mathscr{F})$ to be partitionable into sets $\{Q_j\}$ that satisfy the independence condition.

The proof of Theorem 6.1 is a simple consequence of the proof of Theorem 4.2 (with $\theta = 0$) and

Lemma 6.1. *If the conditions of Theorem 6.1 hold, then*

$$Z^{\mathrm{LP}} \leq z(S) + \sum_{j \in S(K)} \rho_j(S), \quad S \subseteq N$$

where $S(K) \subseteq N - S$ is an index set of K largest $\rho_j(S)$.

The proof of Lemma 6.1 requires some properties of optimal dual variables to the linear program (6.1). The dual of (6.1) is

$$\begin{aligned} &\min \sum_{A \subseteq E} r(A) u_A, \\ &\sum_{A \ni e} u_A \geq c_e, \quad e \in E', \\ &u_A \geq 0, \quad A \subseteq E. \end{aligned} \tag{6.7}$$

Let $F = \{e_1, \dots, e_s\} \subseteq E'$ be the elements corresponding to an optimal solution of problem (6.1) with $c_{e_1} \geq \cdots \geq c_{e_s}$, and let $\mathrm{sp}(T)$, $T \subseteq E$, denote the set $\{e \in E \colon r(T \cup \{e\}) = r(T)\}$. The following proposition is known and easily verified.

Proposition 6.2. *Let $A_j = \mathrm{sp}(e_1, \dots, e_j)$, $j = 1, \dots, s$, and define $c_{e_{s+1}} = 0$. An optimal solution of (6.7) is $u^*_{A_j} = c_{e_j} - c_{e_{j+1}}$, $j = 1, \dots, s$, and $u^*_A = 0$ otherwise.*

We use Proposition 6.2 to establish other properties of the optimal dual variables.

Proposition 6.3. *Suppose $e \in E - E'$.*
(i) *If $F \cup \{e\} \in \mathcal{F}$, then $\sum_{A \ni e} u^*_A = 0$.*
(ii) *If $F \cup \{e\} \notin \mathcal{F}$, then $\sum_{A \ni e} u^*_A = \min_i \{c_{e_i} \colon e_i \in C - \{e\}\}$ where C is the unique circuit in $F \cup \{e\}$.*

Proof. (i) If $F \cup \{e\} \in \mathcal{F}$, $e \notin A_j$ for any $j = 1, \dots, s$, and hence $u^*_A = 0$, $\forall A \ni e$.
(ii) Suppose $\{e_1, \dots, e_j, e\} \in \mathcal{F}$, but $\{e_1, \dots, e_j, e_{j+1}, e\} \notin \mathcal{F}$. Then $e \notin A_k$ for $k \leq j$, and $e \in A_k$ for $k > j$. Therefore

$$\sum_{A \ni e} u^*_A = \sum_{k > j} (c_{e_k} - c_{e_{k+1}}) = c_{e_{j+1}}.$$

Now $e_{j+1} \in C \subseteq \{e_1, \dots, e_j, e_{j+1}, e\}$, and therefore

$$\sum_{A \ni e} u^*_A = c_{e_{j+1}} = \min_i \{c_{e_i} \colon e_i \in C - \{e\}\}.$$

We now consider the change in v in problem (6.1) when the constraints $x_e = 0$ and $x_f = 0$ are suppressed, for $e, f \in E - E'$.

Proposition 6.4. (i) $v(E' \cup \{e\}) - v(E') = (c_e - \sum_{A \ni e} u_A^*)^+$, *where* $x^+ = \max\{0, x\}$.

(ii) *If there is no circuit containing e and f,*

$$v(E' \cup \{e, f\}) - v(E') = \Big(c_e - \sum_{A \ni e} u_A^*\Big)^+ + \Big(c_f - \sum_{A \ni f} u_A^*\Big)^+.$$

Proof. (i) It follows from the greedy algorithm that if e enters the new solution, either $F \cup \{e\}$ is independent or it replaces the cheapest element in the unique circuit C containing it. Therefore the result follows from (i) and (ii) of the previous proposition.

(ii) Let G be the elements corresponding to an optimal solution of (6.1) over $E' \cup \{e\}$ given by the greedy algorithm. We now consider the forms of the optimal solutions over $E' \cup \{e\}$ and $E' \cup \{e, f\}$ given in Lemma 3.1. However, note that when e and f are in no common circuit cases (3) and (4) of Lemma 3.1 cannot occur. This is obvious for case (3). In case (4) $G = F \cup \{e\} - \{e_r\}$, $e_r \neq e$ and e_r is in the circuit of $F \cup \{f\}$, which implies that $F \cup \{e, f\} - \{e_r\}$ contains a circuit. Since $F \cup \{e\} - \{e_r\}$ and $F \cup \{f\} - \{e_r\}$ are independent, e and f are contained in a common circuit of $F \cup \{e, f\} - \{e_r\}$. Thus we are left with cases (1) and (2), which imply that either $F \cup \{f\}$ and $G \cup \{f\}$ are both independent or both contain the same unique circuit. Therefore $v(E' \cup \{e, f\}) - v(E' \cup \{e\}) = v(E' \cup \{f\}) - v(E')$, and the result follows.

Proof of Lemma 6.1. Let u^S be the optimal variables as given in Proposition 6.2 for problem (6.7) with $E' = \bigcup_{j \in S} Q_j$. From Propositions 6.1 and 6.2, $z(S) = v(\bigcup_{j \in S} Q_j) = \sum_A u_A^S r(A)$, the value of the objective function of (6.7).

By definition

$$\begin{aligned}
Z^{\mathrm{LP}} &= \max_{x,y} \sum_{e \in E} c_e x_e, \quad \text{subject to (6.2)–(6.6)} \\
&\leq \max_{x,y} \Big\{ \sum_{e \in E} c_e x_e + \sum_{A \subseteq E} u_A^S \Big(r(A) - \sum_{e \in A} x_e\Big) \Big\}, \\
&\qquad\qquad \text{subject to (6.2)–(6.6), (since } u_A^S \geq 0 \text{ and (6.2) holds)} \\
&\leq z(S) + \max_{x,y} \sum_{e \in E} \Big(c_e - \sum_{A \ni e} u_A^S\Big) x_e, \quad \text{subject to (6.3)–(6.6)} \\
&= z(S) + \max_{x,y} \sum_{j \notin S} \sum_{e \in Q_j} \Big(c_e - \sum_{A \ni e} u_A^S\Big) x_e, \quad \text{subject to (6.3)–(6.6),}
\end{aligned}$$

$$\Big(\text{as } \bigcup_{j \in N} Q_j = E,\ Q_j \cap Q_k = \emptyset,\ j \neq k \text{ and } c_e - \sum_{A \ni e} u_A^S \leq 0 \text{ for } e \in \bigcup_{j \in S} Q_j\Big)$$

$$= z(S) + \max_y \left\{ \sum_{j \notin S} \left(\sum_{e \in Q_j} \left(c_e - \sum_{A \ni e} u_A^S \right)^+ \right) y_j : \sum_{j \notin S} y_j \le K, 0 \le y_j \le 1 \right\}$$

$$= z(S) + \max_y \left\{ \sum_{j \notin S} \left(v\left(\left(\bigcup_{i \in S} Q_i \right) \cup Q_j \right) - v\left(\bigcup_{i \in S} Q_i \right) \right) y_j : \right.$$

$$\left. \sum_{j \notin S} y_j \le K, 0 \le y_j \le 1 \right\} \quad \text{(by Proposition 6.4)}$$

$$= z(S) + \max_y \left\{ \sum_{j \notin S} \rho_j(S) y_j : \sum_{j \notin S} y_j \le K, 0 \le y_j \le 1 \right\}$$

$$\left(\text{since } \rho_j(S) = z(S \cup \{j\}) - z(S) = v\left(\left(\bigcup_{i \in S} Q_i \right) \cup Q_j \right) - v\left(\bigcup_{i \in S} Q_i \right) \right)$$

$$= z(S) + \sum_{j \in S(K)} \rho_j(S).$$

Proof of Theorem 6.1. Using Lemma 6.1 and $S = S^t$, $t = 0, \dots, K-1$ as the sets chosen by the greedy heuristic we have that

$$Z^{\mathrm{LP}} \le z(S^t) + \sum_{j \in S^t(K)} \rho_j(S^t), \quad t = 0, \dots, K-1$$

and

$$Z^{\mathrm{LP}} \le \sum_{i=0}^{t-1} \rho_i + K\rho_t, \quad t = 0, \dots, K-1.$$

Exactly as in Theorem 4.1 (with $\theta = 0$), we obtain $(Z^{\mathrm{LP}} - Z^G)/Z^{\mathrm{LP}} \le \alpha^K$.

A surprising consequence of Theorem 6.1 is that if a particular problem is a worst case example for the greedy heuristic in the sense of Theorem 4.2, then $Z^{\mathrm{LP}} = Z$, while if the duality gap is maximum i.e., $(Z^{\mathrm{LP}} - Z)/Z^{\mathrm{LP}} = \alpha^K$, then the greedy heuristic gives the optimal solution.

Finally, we note that under the hypotheses of Theorem 6.1 we can prove an analogous theorem for the interchange heuristic. Here we obtain the bound $(Z^{\mathrm{LP}} - Z^{I(1)})/Z^{\mathrm{LP}} \le (K-1)/(2K-1)$ (see Theorem 5.1). This theorem is proved by applying the greedy heuristic to the interchange solution and then using the result of Lemma 6.1 for $t = K-1$.

7. Heuristics and partial enumeration

By combining partial enumeration with the heuristics of Sections 4 and 5 we can improve the bounds given previously. Suppose for each subset S of cardinality R we apply some heuristic (H) to the problem

$$\max_{T \subseteq N-S} \{ z(S \cup T) : |T| \le K - R \}. \tag{7.1}$$

Call the value of the best of these $\binom{|N|}{R}$ approximations Z^* and the method the "R-enumeration plus H" heuristic.

Theorem 7.1. *Suppose that $z(S)$, $S \subseteq N$ is submodular and nondecreasing and the heuristic* (*H*) *gives a bound of*

$$\frac{Z - Z^H}{Z - z(\emptyset)} \leq \beta(K)$$

when applied to problem (1.6). *Then*

$$\frac{Z - Z^*}{Z - z(\emptyset)} \leq \frac{K - R}{K} \beta(K - R).$$

Proof. Let T of cardinality K be an optimal solution to problem (1.6). Apply at most R steps of the greedy heuristic of Section 4 to T to obtain a set $S^* \subseteq T$, $|S^*| \leq R$. If the greedy heuristic stops before R steps have been executed, then Proposition 4.2 implies that $z(S^*) = z(T) = Z$. Otherwise submodularity implies that $z(S^*) - z(\emptyset) \geq (R/K)(Z - z(\emptyset))$. Now applying heuristic (H) to problem (7.1) with $S = S^*$, we obtain a set $S^* \cup T^*$ for which

$$\frac{Z - z(S^* \cup T^*)}{Z - z(S^*)} \leq \beta(K - R).$$

Noting that $Z^* \geq z(S^* \cup T^*)$ and substituting $z(\emptyset) + (R/K)(Z - z(\emptyset)) \leq z(S^*)$ we obtain the required result.

Theorem 7.1 suggests that the "$(R-1)$-enumeration plus 1-greedy" heuristic may outperform the R-step greedy heuristic, and similarly the "$(R-1)$-enumeration plus 1-interchange heuristic" may be preferable to the R-interchange heuristic.

We close this section by noting that all of the bounds on approximations can also be viewed as bounds on the value of an optimal solution. If we have a particular heuristic value and know that the heuristic value is at least a specified fraction of the optimal value, we then have an upper bound on the optimal value. Furthermore, if z is nondecreasing, from any heuristic solution $S = \{i_1, \dots, i_K\}$ we obtain

$$Z \leq \min_{t=0,\dots,K-1} z(S^t) + \sum_{j \in S^t(K)} \rho_j(S^t) \tag{7.2}$$

where $S^t = \{i_1, \dots, i_t\}$. For example, from the greedy heuristic and (7.2) we obtain a bound that is at most $Z/(1 - \alpha^K)$. But on specific problems the bound computed from (7.2) frequently will be much tighter.

Thus in branch-and-bound algorithms a heuristic solution can serve the dual purpose of providing an upper bound on the optimal value as well as the usual feasible solution and lower bound. In this regard, the bound of Theorem 7.1 could be helpful in an implicit enumeration algorithm that used a tree in which the nodes at level k represents subsets of N in which k elements are chosen.

Similarly, when viewed in this way, linear programming approximations provide lower bounds as well as the usual upper bounds.

8. Future work

Problem (1.6) is a particular case of

$$\max_{S \subseteq N} \{z(S): z(S) \text{ submodular}, \; S \text{ an independent set in a matroid } \mathcal{M}\} \tag{8.1}$$

where in (1.6) the maximal independent sets in $\mathcal{M}$ are all subsets of cardinality K. The approach and many of the results of this paper generalize to problem (8.1) and to the maximization of submodular functions over other independence systems as well. In a sequel to this paper we will analyze approximations for these problems.

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