Conjugate direction methods:

- a set of vectors, $p_0 \cdots p_n$ is conjugate for A positive definite if

$$p_i' A p_j = 0 \qquad i \neq j$$

- Assume we wish to min

$$\frac{x' A x}{2} - b' x$$

- useful because:

  a) solution to $Ax = b$ for A p.d.

  b) $\min_x \|ux - b\|^2$ is like this

- now write

$$x = \alpha_0 p_0 + \alpha_1 p_1 + \cdots$$

Conjugate direction in incremental form

Start with x_0, p_0

$$x_1 = x_0 + \alpha_0 p_0$$

now min wrt α_0

to get

$$\frac{(Ax_0 - b)' p_0}{p_0' A p_0} = \alpha_0$$

write

$$r_k = (Ax_k - b)$$

and get

$$x_{k+1} = x_k + \alpha_k p_k$$

$$\alpha_k = \frac{r_k' p_k}{p_k' A p_k}$$

$$r_{k+1} = r_k + \alpha_k A p_k$$

Again ~~This~~ gives

$$\tfrac{1}{2}\left[(x_K + \alpha_K p_K)' A (x_K + \alpha_K p_K) \right]$$
$$- b_K \ (x_K + \alpha_K p_K)$$

min is at:

$$-\frac{(A x_K - b)' p_K}{p_K' A p_K}$$

write

$$r_K = A x_K - b$$

so
$$\alpha_K = -\frac{r_K' p_K}{p_K' A p_K}$$

gate gradient (Simple form)

Start: x_0, $r_0 = Ax_0 - b$, $p_0 = -r_0$

Step:

$$x_{k+1} = x_k + \alpha_k p_k$$

$$\alpha_k = \frac{-r_k' p_k}{p_k' A p_k}$$

$$p_{k+1} = -r_{k+1} + \beta_{k+1} p_k$$

$$\beta_{k+1} = \frac{r_{k+1}' A p_k}{p_k^T A p_k}$$

$$r_{k+1} = r_k + \alpha_k A p_k$$

Properties of conj. direction

$$r_k' p_i = 0, \quad \forall_i < k$$

(Show this by induction)

$$r_k' r_i = 0 \quad \forall_i < k$$

(thm 5.3 at end).

Conjugate gradient.

<u>Cleaner form:</u>

- By properties, we have

$$\alpha_{K\cancel{+1}} = \frac{r_K' r_K}{p_K' A p_K}$$

- Now $\alpha_K A p_K = r_{K+1} - r_K$

So
$$\beta_{K+1} = r_{K+1}' \frac{(r_{K+1} - r_K)}{\alpha_K} \cdot \frac{1}{p_K' A p_K}$$

$$= \frac{r_{K+1}' (r_{K+1} - r_K)}{r_K' r_K}$$

$$= \frac{r_{K+1}' r_{K+1}}{r_K' r_K} \quad \text{(By properties)}$$