Linear programs

$$\max \quad c^T x$$
$$\text{st} \quad A\underline{x} = \underline{b}$$
$$M\underline{x} \leq \underline{n}$$

Notice : ~~cases~~ :

- domain is always convex
  (i.e. if x_1, x_2 are in, then $t x_1 + (1-t) x_2$ is in for $0 \leq t \leq 1$)
- called the Feasible Set
- could be
  - empty
  - compact (then its a polytope)
  - a cone
- solution could be
  - non existent (empty Feasible set)
  - a k-face of the feasible set (k usually zero)
  - infinite

Linear programs

we can always convert to

<u>Standard form</u> (equational form).

$$\max \; c^T x$$

s.t. $\quad Ax = b \quad$ ← Not the same A, b – sorry!

$\qquad x \geq 0$

by:

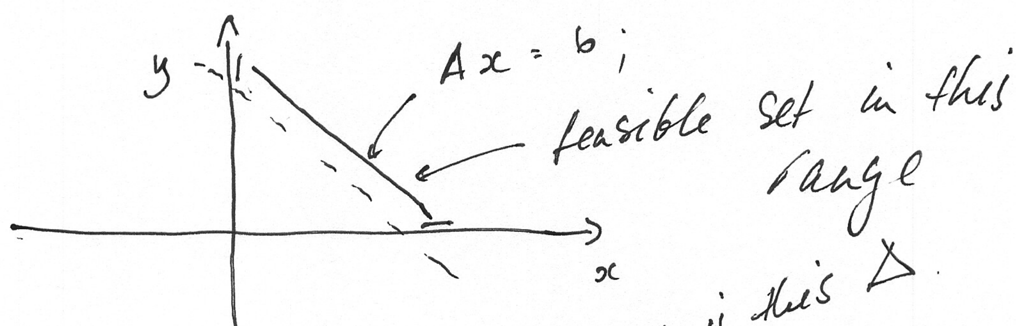
- introduce a slack for each inequality, mult by -1 if necessary.
- if we have x_i which is unbounded, then replace w/ $x_u - x_v$, $x_u \geq 0$, $x_v \geq 0$

The simplex method:

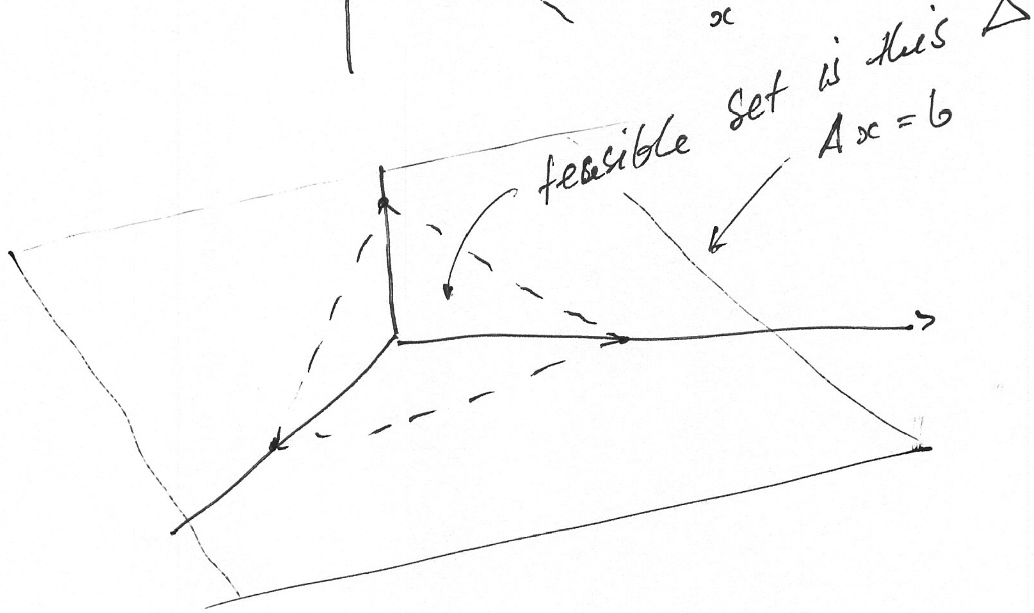
- Assume there is at least one x st
  $Ax = b$
- rows of A are linearly indep.

Notice geom of standard form
- take +ve orthant, slice w/ linear space

2D:



3D:



All this means that there must be <u>basic</u> solns.

- where "enough" of the vars are zero

$$m \updownarrow \overset{\longleftarrow n \longrightarrow}{A} \quad x = b$$

- basic soln.
  - m-element set of <u>indices</u> B st.
    $A_B = \{$cols of A indexed by $B\}$
    has full rank.
  - $x_j = 0$, $j \notin B$.

- Notice if we know B, we know $\underline{x}$ &
  (nonsingular A_B)
  - if there is a feasible ~~soln~~ point, ~~with~~ and ~~bound~~ objective is bounded above, then there is an optimal soln
  - If there is an optimal soln, then there is a <u>basic</u> optimal soln

Simplex by example
(Matousek + Gartner)

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{st} & -x_1 + x_2 \leq 1 \\ & x_1 \leq 3 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

<u>equational form</u>

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{st} & -x_1 + x_2 + x_3 = 1 \\ & x_1 + x_4 = 3 \\ & x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

Now write a tableau

$$\begin{aligned}
x_3 &= 1 + x_1 - x_2 \\
x_4 &= 3 - x_1 \\
x_5 &= 2 \qquad\quad - x_2 \\
\hline
z &= \quad x_1 + x_2
\end{aligned}$$

objective fn

interpret as $(0, 0, 1, 3, 2)$ ← basic feasible soln

value $= 0$

- improve z by making x_1 or x_2 non zero, some other zero
- choose x_2 - this can go to 1. (ei)
  then $x_3 \to 0$ → new tableau

$$\begin{aligned}
x_2 &= 1 + x_1 - x_3 \\
x_4 &= 3 - x_1 \\
x_5 &= 2 \qquad\quad - x_2
\end{aligned}$$

↑ subs $1 + x_1 - x_3$

$$\begin{aligned}
x_2 &= 1 + x_1 - x_3 \\
x_4 &= 3 - x_1 \\
x_5 &= 1 - x_1 + x_3 \\
\hline
z &= 1 + 2x_1 - x_3
\end{aligned}$$

$(0, 1, 0, 3, 1)$

Value 1

x_1 comes in, (+ve coeff in z)

- x_1 limited by equ 3 to $x_1 \leq 1$.
- so ~~to~~ rewrite e3 as

$$x_1 = 1 + x_3 - x_5$$

- insert in tableau

$$\left.\begin{array}{l} x_1 = 1 + x_3 - x_5 \\ x_2 = 2 - x_5 \\ x_4 = 2 - x_3 + x_5 \\ \hline z = 3 + x_3 - 2x_5 \end{array}\right] \to \begin{array}{l}(1,2,0,2,0) \\ \text{val. } 3\end{array}$$

- now insert x_3, remove x_4.

$$\left.\begin{array}{l} x_1 = 3 - x_4 \\ x_2 = 2 - x_5 \\ x_3 = 2 - x_4 + x_5 \\ \hline z = 5 - x_4 - x_5 \end{array}\right] \quad \begin{array}{l}(3,2,2,0,0) \\ 5.\end{array}$$

- <u>stop</u> (all − in z)

Notice we are moving along 1-faces of feasible polytope.

i.e. tableau $=$ $\underbrace{\text{0-face}}_{\text{vertex}}$

1 eqn goes out, another goes in $=$ move to another 0-face connected to last 0-face by a 1-face.

- and we do this along edges such that

$$c \cdot (\text{edge vector}) > 0$$

Killer algorithmic question:

- who goes in, who goes out?

This turns out to be complicated as methods can cycle.

Cycling

- all available 0-verts might have same value of objective.
- if this happens, we could come back to where we came from.

Various rules for pivoting:

- largest coefficient in objective
- largest increase in objective
- Steepest edge

$$\frac{c^T(x_{new} - x_{old})}{\|x_{new} - x_{old}\|} \text{ is best}$$

V. good: approx this test.

- random
  (best provable bounds)
- Bland

No cycling | . incoming var has smallest index
| . ditto outgoing, if choice

Very painful $\left\{\begin{array}{l}\text{practical}\\ \text{intellectual}\end{array}\right\}$ problem

→ Simplex is super-good on large problems

→ no pivot rule known can be proved polynomial

→ existence of a polynomial pivot rule is a <u>major</u> open qu. in geometry

One can prove various average case results (Spielman + Teng. etc).