Duals of linear programs

1) Assume we have a linear program

$$\min \quad -w^T f$$

$$\text{s.t.} \quad A_R f = 0$$

$$f \geq 0$$

$$c - f \geq 0$$

(This isn't the standard form — just convenient for flow)

its Lagrangian is

$$\mathcal{L}(f, \lambda_e, \lambda_i^1, \lambda_i^2)$$

$$= -w^T f + \lambda_e^T (A_R f) - \lambda_i^{1T} f - \lambda_i^{2T} (c - f)$$

we want the Lagrange dual.

$$\inf_{f} \ \mathcal{L}(f, \lambda_e, \lambda_i^1, \lambda_i^2)$$

$$= \begin{cases} -\lambda_i^{2T} c & \text{if } \left[-\omega^T + \lambda_e^T A_R - \lambda_i^{1T} + \lambda_i^{2T}\right] = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Now the dual problem is to <u>max</u> this

i.e.

$$\max \quad -\lambda_i^{2T} c$$

$$\text{s.t.} \quad -\omega^T + \lambda_e^T A_R - \lambda_i^{1T} + \lambda_i^{2T} = 0$$

$$\lambda_i^1 \geq 0$$

$$\lambda_i^2 \geq 0$$

We can clean this up a bit.

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max $\quad -\lambda_i^{2^T} c$

s.t. $\quad A_e^T \lambda_e + \lambda_i^2 \geq w^T$

$\lambda_i^2 \geq 0$

Now, notice:

- in augmenting path, we always had a flow
  - is this a cut? not necessarily
  - e.g.

$$s \xrightarrow{1\ (3)} \bullet \xrightarrow{1\ (3)} \bullet \xrightarrow{1\ (3)} d$$

  - flow that isn't a cut, BECAUSE there is an augmenting path.
  - c.f. discussion of aug paths.
    - a flow is ONLY a cut if there is no augmenting path.
      i.e. if its maximal.

In push-relabel, we have manipulating CUTS (by thm, above), and ONLY had a flow at optimality.

~~These two are dual~~

Max-flow, min. cut are DUAL

MAX FLOW	MIN CUT
min $-w^T f$	max $-\lambda_i^{2\,T} c$
s.t. $A_R f = 0$	s.t. $-w + A_R^T \lambda_e - \lambda_i^1 + \lambda_i^2 = 0$
$f \geq 0$	$\lambda_i^1 \geq 0$
$C - f \geq 0$	$\lambda_i^2 \geq 0$