Primal - Dual algorithms

recall

Primal LP

min $c^T x$

$Ax = b$

$x \geq 0$

Dual LP

max $b^T y$

$A^T y + \mu = c$

$\mu \geq 0$

(often written as

$A^T y \leq c$

which we'll use)

Now we wish to solve an integer L.P.

Recall for continuous, a pair (x, y) is a soln

$\Updownarrow$

- x is primal feasible (i.e $Ax = b$, $x \geq 0$)
- y is dual feasible (i.e. $A^T y \leq c$)
- x, y complementary (i.e $x_i \cdot ((A^T y)_i - c) = 0$)

This is unlikely to be true if x is an INTEGER point (because mostly LPs don't have integer vertices).

<u>But</u> we can still extract info from dual

$\hat{x}$ = integer point

x^* = best possible integer p.t.

imagine we have y, s.t.

$$c^T\hat{x} \leq f\, b^T y$$

f would have to be bigger than 1

Now $c^T\hat{x} \geq c^T x^*$ (because x^* is best pt)

and $c^T x^* \geq b^T y$ (because dual gives lower bound)

so $\dfrac{c^T\hat{x}}{c^T x^*} \leq \dfrac{c^T\hat{x}}{b^T y} \leq f$

<u>so</u> $\boxed{\dfrac{c^T\hat{x}}{c^T x^*} \leq f}$ ← quality estimate

This suggests a strategy

- Start w/ INTEGER primal feasible x^0
  dual feasible y^0

- Iterate
  - adjust x, w/ integer updates, to be more complementary w/ y
  - adjust y to be more complementary w x
- Stop when integer moves produce no change.
- now $\frac{c^T x}{b^T y}$ is a measure of goodness

Notice $\boxed{c^T x = b^T y \iff \text{exact soln.}}$

We can link this to the complementarity condition

write

$$L(x,y) = c^T x - y^T(Ax-b) - (c-A^T y)^T x$$

(Lagrangian)

notice by cancelling

$$L(x,y) = b^T y$$

now, imagine we have x, y, st

$$Ax = b \qquad c \geq A^T y$$

$$x \geq 0$$

AND if $x_i > 0$ then $(A^T y)_i \geq \frac{c_i}{f}$

This is a <u>relaxed</u> version of complementarity condition

$$x_i \cdot (c_i - (A^T y_i)) = 0$$

In this case,

$$L = c^T x - \cancel{y^T(Ax - b)}$$

0 constraints

$$- \underbrace{(c - A^T y)^T x}$$

ith entry: if $x_i = 0$, $= 0$

if $x_i > 0$

$g_i x_i$ where

~~$c_i(1-\frac{1}{f}) \leq \frac{g_i}{x_i} \leq c_i$~~

$c_i(1-\frac{1}{f}) \geq g_i \geq 0$

$$\therefore L \geq c^T x - (1-\tfrac{1}{f}) c^T x$$

$$= \frac{1}{f} c^T x$$

$$\therefore b^T y \geq \frac{1}{f} c^T x$$

<u>SO</u> This <u>relaxed</u> Complementarity cond guarantees an f-approx

$$\left(\frac{c^T x}{c^T x^*} \leq f \right)$$