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AN ANALYSIS OF APPROXIMATIONS FOR MAXIMIZING SUBMODULAR SET FUNCTIONS – II

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Let N be a finite set and $\mathscr{F}$ a nonempty collection of subsets of N which have the property that $F_1 \in \mathscr{F}$ and $F_2 \subset F_1$ imply $F_2 \in \mathscr{F}$. A real-valued function z defined on the subsets of N that satisfies $z(S) \leq z(T)$ for all $S \subset T \subseteq N$ and $z(S) + z(T) \geq z(S \cup T) + z(S \cap T)$ for all $S, T \subseteq N$ is called nondecreasing and submodular. We consider the problem $\max_{S \subseteq N}\{z(S)\colon S \in \mathscr{F}, z(S)$ submodular and nondecreasing$\}$ and several special cases of it.

We analyze greedy and local improvement heuristics, and a linear programming relaxation when $z(S)$ is linear. Our results are worst case bounds on the quality of the approximations. For example, when $(N, \mathscr{F})$ is described by the intersection of P matroids, we show that a greedy heuristic always produces a solution whose value is at least $1/(P+1)$ times the optimal value. This bound can be achieved for all positive integers P.

Key words: Heuristics, Greedy Algorithm, Linear Programming, Independence Systems, Matroids, Submodular Set Functions

1. Introduction

Let $N = \{1, \ldots, n\}$ be a finite set and z a real-valued function defined on the subsets of N that satisfies

$$z(S) + z(T) \geq z(S \cup T) + z(S \cap T)$$

for all S, T in N. Such a set function is called *submodular*. This paper is the third in a series dealing with approximate methods for maximizing submodular set functions. We additionally assume here that $z(S)$ is nondecreasing, i.e., $z(S) \leq z(T)$ for all $S \subset T \subseteq N$.

In [2] we studied the uncapacitated location problem

$$\max_{S \subseteq N} \left\{ z(S)\colon z(S) = \sum_{i \in I} \max_{j \in S} c_{ij}, |S| \leq K \right\},$$

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where $C = (c_{ij})$ is a nonnegative matrix with column index set N and row index set I and $z(\emptyset) = 0$. In [7] we generalized the results to the problem

$$\max_{S \subseteq N} \{z(S) \colon |S| \le K,\ z(S) \text{ submodular and nondecreasing}\}. \tag{1.1}$$

Since many combinatorial optimization problems, including the maximum m-cut problem [8], a storage allocation problem [1] and the matroid partition problem [3], require an optimal partition or packing, we were motivated to extend our results to the problem

$$\max_{S_1 \subseteq N, \dots, S_m \subseteq N} \left\{ \sum_{i=1}^{m} z_i(S_i) \colon \bigcup_{i=1}^{m} S_i \subseteq N,\ S_i \cap S_k = \emptyset,\ k \ne i, \right.$$
$$\left. z_i(S) \text{ submodular and nondecreasing},\ i = 1, \dots, m \right\}. \tag{1.2}$$

We like to think of (1.2) as the "m-box" model in which putting S_i in box i yields a value of $z_i(S_i)$ and the objective is to maximize the value summed over all boxes.

The m-box model can be used to describe a multiproduct version of the uncapacitated location problem. Here each box corresponds to a different product. Assigning the set of locations $S_i \subset N$ to box i means that these locations supply product i. The objective is to maximize $\sum_{i=1}^{m} \sum_{k \in I} \max_{j \in S_i} c_{kj}^{i}$.

By adding the restrictions $|S_i| \le 1$ to the m-box model we obtain the constraints of an assignment problem. Now by generalizing the objective function to include terms involving pairs of boxes we obtain a model of the quadratic assignment problem. Here the objective is no longer a sum of set functions but a multidimensional set function of the form $v(S_1, \dots, S_m)$. We can treat these multidimensional set functions directly by defining a multidimensional version of submodularity, i.e.,

$$v(S_1, \dots, S_m) + v(T_1, \dots, T_m)$$
$$\ge v(S_1 \cup T_1, \dots, S_m \cup T_m) + v(S_1 \cap T_1, \dots, S_m \cap T_m).$$

However, an alternative viewpoint of the box model renders this multidimensional construct unnecessary and provides a more general and unified framework for the extensions of (1.1) that we consider here.

Let M be the set of boxes, rename the set of elements to be put into the boxes E, let $N = \{(i, j) \colon i \in M, j \in E\}$ and $N_j = \{(i, j) \colon i \in M\}$, $j \in E$. There is a one-to-one correspondence between packings $(S_1, \dots, S_m)$ of E and subsets $S \subseteq N$ that satisfy $|S \cap N_j| \le 1$, $j \in E$. The correspondence is given by $S_i = \{j \colon (i, j) \in S\}$ and $S = \{(i, j) \colon j \in S_i, i \in M\}$. Therefore a generalized version of the m-box problem (1.2) is

$$\max_{S \subseteq N} \{z(S) \colon |S \cap N_j| \le 1,\ j \in E,$$
$$z(S) \text{ submodular and nondecreasing}\}. \tag{1.3}$$

Now comparing (1.1) and (1.3), we see that they differ only in their constraints. However in each case the family $\mathcal{F}$ of feasible or independent sets forms a matroid $\mathcal{M} = (N, \mathcal{F})$; i.e., $F_1 \in \mathcal{F}$ and $F_2 \subset F_1 \Rightarrow F_2 \in \mathcal{F}$ [$(N, \mathcal{F})$ is an independence system] and for all $N' \subseteq N$ every maximal member of $\mathcal{F}(N') = \{F: F \in \mathcal{F}, F \subseteq N'\}$ has the same cardinality. In (1.1) $\mathcal{M}$ is the matroid in which all subsets of cardinality K or smaller are independent and in (1.3) $\mathcal{M}$ is a partition matroid. Thus a natural generalization of (1.1) and (1.3) is

$$\max_{S \subseteq N} \{z(S): S \in \mathcal{F}, \mathcal{M} = (N, \mathcal{F}) \text{ a matroid, } z(S) \text{ submodular and nondecreasing}\} \tag{1.4}$$

and an obvious generalization of (1.4) is

$$\max_{S \subseteq N} \{z(S): S \in \bigcap_{p=1}^{P} \mathcal{F}_p, \mathcal{M}_p = (N, \mathcal{F}_p) \text{ are matroids, } p = 1, \dots, P, z(S) \text{ submodular and nondecreasing}\}. \tag{1.5}$$

Note that any independence system can be described as the intersection of P matroids for suitably large P.

Finally, a different generalization of problem (1.3) is

$$\max_{S \subseteq N} \left\{z(S): N = \bigcup_{j=1}^{n} N_j, N_j \cap N_k = \emptyset, j \neq k, S \cap N_j \in \mathcal{F}^j, z(S) \text{ submodular and nondecreasing}\right\} \tag{1.6}$$

where $(N_j, \mathcal{F}^j)$ $j = 1, \dots, n$ are independence systems, each the intersection of P or fewer matroids. Note that combining the disjoint independence systems gives a problem over N of the form (1.5) involving the intersection of P matroids. Alternatively we can view (1.6) as a generalization of (1.5), where (1.5) is obtained from (1.6) by taking $n = 1$.

We now summarize our results. In Section 2 we consider a greedy heuristic for problem (1.5). The greedy heuristic first solves (1.5) with the constraint $|S| = 1$ to obtain a set S^1 and then iteratively builds a nested sequence of sets $\{S^t\}$, $t = 2, 3, \dots$, where $S^t \in \bigcap_{i=1}^{P} \mathcal{F}_i$ and $|S^t| = t$. S^{t+1} is determined by adding to S^t (if possible) a j^* such that

$$z(S^t \cup \{j^*\}) = \max\left\{z(S^t \cup \{j\}): S^t \cup \{j\} \in \bigcap_{i=1}^{P} \mathcal{F}_i, j \notin S^t\right\}.$$

We obtain the tight bound

$$\frac{\text{value of greedy approximation}}{\text{value of optimal solution}} \geq \frac{1}{P+1}.$$

We also show that without regard to P, if K is the cardinality of a largest independent set, and $k+1$ the cardinality of a smallest dependent (not independent) set,

$$\frac{\text{value of greedy approximation}}{\text{value of optimal solution}} \geq 1 - \left(\frac{K-1}{K}\right)^k.$$

Problem (1.1) is the special case of this model with $k = K$.

In Section 3 we assume that $z(S)$ is linear, in which case (1.5) can be represented as an integer program. We study the linear programming relaxation of this integer program, which is obtained by suppressing the integrality restrictions. Our result is

$$\frac{\text{value of greedy approximation}}{\text{value of linear programming solution}} \geq \frac{1}{P},$$

which is a bound on the duality gap and also implies the bound obtained by Jenkyns [5] and Korte and Hausmann [6] on the ratio of the greedy and integer solutions.

In Section 4 we examine problem (1.6) and show that the greedy heuristic can be simplified and the bound of $1/(P+1)$ maintained. Also, for problem (1.3) when $z(S)$ has a certain symmetry with respect to the boxes, the bound of $\frac{1}{2}$ can be improved to $m/(2m-1)$, where m is the number of boxes.

In Section 5 we examine a local improvement heuristic for model (1.5). We show that when $P = 1$

$$\frac{\text{value of local improvement approximation}}{\text{value of optimal solution}} \geq \frac{1}{2},$$

but that the heuristic is arbitrarily bad when $P \geq 2$.

We close this section by giving two other equivalent definitions of submodularity that are proved in [7]. Although this paper can be read independently, we strongly recommend the prior reading of [7].

Let $\rho_j(S) = z(S \cup \{j\}) - z(S)$.

Proposition 1.1. *Each of the following statements is equivalent and defines a nondecreasing submodular set function.*

(i) $z(S) + z(T) \geq z(S \cup T) + z(S \cap T), \quad \forall S, T \subseteq N$

$z(S) \leq z(T), \quad \forall S \subseteq T \subseteq N.$

(ii) $\rho_j(S) \geq \rho_j(T) \geq 0, \quad \forall S \subset T \subset N$ *and* $j \in N - T$.

(iii) $z(T) \leq z(S) + \sum_{j \in T-S} \rho_j(S), \quad \forall S, T \subseteq N.$

Finally, we assume throughout the paper that $z \neq z(\emptyset)$ and therefore exclude the trivial possibility of $\emptyset$ being an optimal solution.

2. The greedy heuristic

We first describe the greedy heuristic and then obtain two worst case bounds for problem (1.5).

The greedy heuristic for nondecreasing set functions on independence systems $(N, \mathcal{F})$.

Initialization. Let $S^0 = \emptyset$, $N^0 = N$ and set $t = 1$.

Iteration t

Step 0. If $N^{t-1} = \emptyset$, stop with S^{t-1} the greedy solution.

Step 1. Select $i(t) \in N^{t-1}$ for which $\rho_{i(t)}(S^{t-1}) = \max_{i \in N^{t-1}} \rho_i(S^{t-1})$, with ties settled arbitrarily.

Step 2a. If $S^{t-1} \cup \{i(t)\} \notin \mathcal{F}$, set $N^{t-1} = N^{t-1} - \{i(t)\}$ and return to Step 0.

Step 2b. If $S^{t-1} \cup \{i(t)\} \in \mathcal{F}$, set $\rho_{t-1} = \rho_{i(t)}(S^{t-1})$, $S^t = S^{t-1} \cup \{i(t)\}$ and $N^t = N^{t-1} - \{i(t)\}$.

Step 3. Set $t \to t+1$ and continue.

Let U^t be the set of elements considered in the first $t+1$ iterations of the greedy heuristic before the addition of a $(t+1)$st element. Suppose $\mathcal{F} = \bigcap_{p=1}^{P} \mathcal{F}_p$, where $\mathcal{M}_p = (N, \mathcal{F}_p)$ are matroids $p = 1, \dots, P$. Define $r_p(S)$, called the rank of S in matroid p, to be the cardinality of a largest independent set contained in S in matroid p, and define $\mathrm{sp}^p(S)$, called the span of S in matroid p, by

$$\mathrm{sp}^p(S) = \{j \in N : r_p(S \cup \{j\}) = r_p(S)\}.$$

Before stating our first result we need two simple propositions.

Proposition 2.1. $U^t \subseteq \bigcup_{p=1}^{P} \mathrm{sp}^p(S^t)$, $t = 0, 1, \dots$.

Proof. If $j \in U^t$, then either $j \in S^t \subseteq \mathrm{sp}^p(S^t)$ for all p, or j failed the independence test at Step 2a, which implies $j \in \mathrm{sp}^p(S^t)$ for some p.

Proposition 2.2. *If* $\sum_{i=0}^{t-1} \sigma_i \le t$ *for* $t = 1, \dots, K$, *and* $\rho_{i-1} \ge \rho_i$, $i = 1, \dots, K-1$ *with* $\rho_i, \sigma_i \ge 0$, *then* $\sum_{i=0}^{K-1} \rho_i \sigma_i \le \sum_{i=0}^{K-1} \rho_i$.

Proof. Consider the linear program

$$V = \max_{\sigma} \left\{ \sum_{i=0}^{K-1} \rho_i \sigma_i : \sum_{i=0}^{t-1} \sigma_i \le t,\ t = 1, \dots, K,\ \sigma_i \ge 0,\ i = 0, \dots, K-1 \right\}$$

with dual

$$W = \min_{u} \left\{ \sum_{t=1}^{K} t u_{t-1} : \sum_{t=i}^{K-1} u_t \ge \rho_i,\ i = 0, \dots, K-1,\ u_t \ge 0,\ t = 0, \dots, K-1 \right\}.$$

As $\rho_i \geq \rho_{i+1}$, the solution $u_i = \rho_i - \rho_{i+1}$, $i = 0, \dots, K-1$ (where $\rho_K = 0$) is dual feasible with value $\sum_{t=1}^{K} t(\rho_{t-1} - \rho_t) = \sum_{i=0}^{K-1} \rho_i$. By weak linear programming duality, $\sum_{i=0}^{K-1} \rho_i \sigma_i \leq V \leq W \leq \sum_{i=0}^{K-1} \rho_i$.

Let Z denote the optimal value and Z^G the value of a greedy solution to problem (1.5).

Theorem 2.1.[1] *If the greedy heuristic is applied to problem* (1.5), *then*

$$\frac{Z - Z^G}{Z - z(\emptyset)} \leq \frac{P}{P+1}.$$

The bound is tight for all P.

Proof.[2] Let T and S be optimal and greedy solutions respectively, with $|S| = K$. For $t = 1, \dots, K$ let $s_{t-1} = |T \cap (U^t - U^{t-1})|$, where U^{t-1} is the set of all elements considered during the first t iterations before the addition of a tth element to S^{t-1}. Below we show

(a) $\sum_{j \in T-S} \rho_j(S) \leq \sum_{i=1}^{K} \rho_{i-1} s_{i-1}$, and

(b) $\sum_{i=1}^{t} s_{i-1} \leq Pt$ for $t = 1, \dots, K$, which implies via Proposition 2.2 that $\sum_{i=1}^{K} \rho_{i-1} s_{i-1} \leq P \sum_{i=1}^{K} \rho_{i-1}$. The result then follows from Proposition 1.1 since

$$Z = z(T) \leq z(S) + \sum_{j \in T-S} \rho_j(S) \leq Z^G + P \sum_{i=1}^{K} \rho_{i-1} = Z^G + P(Z^G - z(\emptyset)).$$

(a) $$\sum_{j \in T-S} \rho_j(S) \leq \sum_{j \in T} \rho_j(S) = \sum_{t=1}^{K} \sum_{j \in T \cap (U^t - U^{t-1})} \rho_j(S) \leq \sum_{t=1}^{K} \rho_{t-1} s_{t-1}$$

as $\rho_j(S) \leq \rho_{t-1}$ for $j \in U^t - U^{t-1}$ by the nature of the greedy heuristic.

(b) From the definition of s_{t-1}, and assuming without loss of generality that $U^0 = \emptyset$, we have $\sum_{i=1}^{t} s_{i-1} = |T \cap U^t|$. Now by Proposition 2.1 $U^t \subseteq \bigcup_{p=1}^{P} \mathrm{sp}^p(S^t)$ so that $|T \cap U^t| \leq \sum_{p=1}^{P} |T \cap \mathrm{sp}^p(S^t)|$. But as T is independent in matroid $(N, \mathcal{F}^p)$ and $r_p(\mathrm{sp}^p(S^t)) = t$, we obtain $|T \cap \mathrm{sp}^p(S^t)| \leq t$. Therefore $\sum_{i=1}^{t} s_{i-1} \leq Pt$.

To show that the bound is tight we exhibit a family of problems with $N = \{1, \dots, P+2\}$, $Z^G = 1$ and $Z = P+1$. We associate the variable $x_i \in \{0, 1\}$ with element i and represent $S \subseteq N$ by its characteristic vector; $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$. Consider the problem

$$\begin{array}{llll}
Z = \max & x_1 + x_2 + \cdots + x_{P+1} + x_{P+2} - x_1 x_{P+2} & & \\
& x_1 + x_2 & \leq 1, & \mathcal{M}_1 \\
& x_1 \quad + x_3 & \leq 1, & \mathcal{M}_2 \\
& \vdots & \vdots & \\
& x_1 \qquad + x_P & \leq 1, & \mathcal{M}_{P-1} \\
& x_1 \qquad\quad + x_{P+1} & \leq 1, & \\
& \qquad\qquad\qquad + x_{P+2} & \leq 1, & \mathcal{M}_P \\
\end{array}$$

$$x_i \in \{0, 1\}, \quad i = 1, \dots, P+2.$$

It is easy to verify that this quadratic objective function is submodular and nondecreasing. An optimal solution is given by $x_1 = 0$ and $x_i = 1$, $i = 2, \ldots, P+2$ so that $Z = P+1$. However the greedy heuristic can select elements 1 and $P+2$ in the given order, which yields $Z^G = 1$.

Now we turn to another bound that can be used whenever the greedy heuristic is applied. This bound is independent of P. It, instead, depends on the cardinality of a largest independent set in $(N, \mathcal{F})$, denoted by K, and the cardinality of a smallest dependent set in $(N, \mathcal{F})$, denoted by $k+1$.

Theorem 2.2. *If the greedy heuristic is applied to problem* (1.5), *then*

$$\frac{Z - Z^G}{Z - z(\emptyset)} \le \left(\frac{K-1}{K}\right)^k.$$

The bound is tight for all $k \le K$ and all K.

Proof. Let T be an optimal solution to (1.5). By Proposition 1.1 $z(T) \le z(S^t) + \sum_{j \in T - S^t} \rho_j(S^t)$. Now $\forall t < k$, $S^t \cup \{j\} \in \mathcal{F}$, $\forall j \in T - S^t$, and therefore $\rho_j(S^t) \le \rho_t$. Since $|T - S^t| \le K$, $\sum_{j \in T - S^t} \rho_j(S^t) \le K\rho_t$ and we obtain

$$Z \le z(\emptyset) + \sum_{i=0}^{t-1} \rho_i + K\rho_t, \quad t = 0, \ldots, k-1. \tag{2.1}$$

Now

$$\frac{Z - Z^G}{Z - z(\emptyset)} \le \frac{Z - z(S^k)}{Z - z(\emptyset)} = \frac{Z - z(\emptyset) - \sum_{i=0}^{k-1} \rho_i}{Z - z(\emptyset)}.$$

We can assume, without loss of generality, that all nondecreasing submodular functions with $Z > z(\emptyset)$ have been normalized so that $Z - z(\emptyset) = 1$. We then obtain a bound on $(Z - Z^G)/(Z - z(\emptyset))$ by maximizing $1 - \sum_{i=0}^{k-1} \rho_i$ subject to the inequalities (2.1). This linear program has maximum value $[(K-1)/K]^k$ – see [7, Lemma 4.1] – and the result follows.

For $k = K$ the bound has been shown to be tight for all K in [2]. The constraints of these problems can be modified to imply the tightness of the bound for all $k \le K$ and all K. We simply redefine the independence system in the example of [2] so that the first k elements chosen by the greedy heuristic are a maximal independent set.

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[1] This theorem can be easily generalized to "p-systems" [5], which are a family of independence systems that include those given in problem (1.5).

[2] This proof is similar to the proof of Theorem 1 of [5], where it is assumed that z is linear and the independence system is a p-system.

3. The greedy heuristic and linear objective functions

When $z(S)$ is linear, problem (1.5) can be written as the integer program

$$\begin{aligned} Z = \max \sum_{e\in N} c_e x_e, & \\ \sum_{e\in F} x_e \le r_p(F), \quad \forall F \subseteq N \quad \text{and} \quad p = 1, \dots, P & \qquad (3.1) \\ x_e \in \{0, 1\}, \quad \forall e \in N. & \end{aligned}$$

Let Z^{LP} be the optimal value of the linear programming problem obtained by dropping the integrality constraints in (3.1).

Theorem 3.1. *If the greedy heuristic is applied to problem* (3.1), *then*

$$\frac{Z^{\mathrm{LP}} - Z^{\mathrm{G}}}{Z^{\mathrm{LP}}} \le \frac{P-1}{P},$$

and the bound is tight for all P.

Proof. From duality

$$Z^{\mathrm{LP}} = \min_{u\ge 0} \max_{x_e \in \{0,1\}} \sum_{e\in N} \Big(c_e - \sum_{p=1}^{P} \sum_{F\ni e} u_p(F)\Big) x_e + \sum_{p=1}^{P} \sum_{F\subseteq N} u_p(F) r_p(F)$$

where u is a vector of dual variables with elements $u_p(F)$, $\forall F \subseteq N$ and $p = 1, \dots, P$.

Suppose the greedy heuristic terminates with the set S^K. We choose dual variables u as follows:

$$u_p(\mathrm{sp}^p(S^{k+1})) = \rho_k - \rho_{k+1} \quad \text{for } p = 1, \dots, P, \quad k = 0, \dots, K-1, \quad \text{where } \rho_K = 0,$$

$$u_p(F) = 0, \quad \text{otherwise.}$$

We claim that this choice of dual variables yields $c_e - \sum_{p=1}^{P} \sum_{F\ni e} u_p(F) \le 0$, $\forall e \in N$. To prove this result suppose that e is examined when the greedy heuristic has chosen S^k. We now consider two cases depending on whether e is selected by the greedy heuristic.

(a) If e is selected then $\sum_{F\ni e} u_p(F) = (\rho_k - \rho_{k+1}) + (\rho_{k+1} - \rho_{k+2}) + \cdots + \rho_{K-1} = \rho_k$ and $c_e = \rho_k$. Hence $c_e - \sum_{p=1}^{P} \sum_{F\ni e} u_p(F) \mp \rho_k - P\rho_k \le 0$.

(b) If e is not selected then there exists p^* such that $e \in \mathrm{sp}^{p^*}(S^k)$. Hence $\sum_{F\ni e} u_{p^*}(F) \ge \sum_{j=k}^{K} u_{p^*}(\mathrm{sp}^{p^*}(S^j)) = (\rho_{k-1} - \rho_k) + (\rho_k - \rho_{k+1}) + \cdots + \rho_{K-1} = \rho_{k-1}$ and $\rho_{k-1} \ge c_e$ from the order in which the variables are considered. Thus

$$c_e - \sum_{p=1}^{P} \sum_{F\ni e} u_p(F) \le c_e - \sum_{F\ni e} u_{p^*}(F) \le c_e - \rho_{k-1} \le 0.$$

Finally, we note that

$$\sum_{F\subseteq N} u_p(F) r_p(F) = (\rho_0 - \rho_1) + 2(\rho_1 - \rho_2) + \cdots + (K-1)\rho_{K-1}$$
$$= \sum_{i=0}^{K-1} \rho_i = Z^{\mathrm{G}}.$$

Therefore $Z^{\mathrm{LP}} \leq PZ^{\mathrm{G}}$ and the result follows.

To see that the bound is tight for all P consider

$$\begin{array}{llll} Z = \max & x_1 + x_2 + \cdots + x_{P+1}, & & \\ & x_1 + x_2 & \leq 1, & \mathcal{M}_1 \\ & x_1 \quad + x_3 & \leq 1, & \mathcal{M}_2 \\ & \vdots & \vdots & \\ & x_1 \qquad\quad + x_{P+1} & \leq 1, & \mathcal{M}_P \\ & x_i \in \{0, 1\}, \quad i = 1, \ldots, P+1. & & \end{array}$$

A greedy solution is given by $x_1 = 1$, $x_i = 0$, $i \geq 2$, $Z^{\mathrm{G}} = 1$, and an optimal LP solution is given by $x_1 = 0$, $x_i = 1$, $i \geq 2$, $Z^{\mathrm{LP}} = P$.

Corollary 3.1[3] [5, 6]. *If the greedy heuristic is applied to problem* (3.1), *then* $(Z - Z^{\mathrm{G}})/Z \leq (P-1)/P$ *and the bound is tight for all* P.

Proof. This bound is implied by the bound of Theorem 3.1 and the example given above has $Z^{\mathrm{LP}} = Z$.

Problem (3.1) has the alternative representation

$$\max_{S\subseteq N} \{v(S): S \in \bigcap_{p=2}^{P} \mathcal{F}_p\} \tag{3.2}$$

where $v(S) = \max_{F\subseteq S}\{\sum_{e\in F} c_e: F \in \mathcal{F}_1\}$ is submodular and nondecreasing (Proposition 3.1 of [7]). It then can be shown that applying the greedy heuristic to problems (3.1) and (3.2) can lead to identical solutions. Thus Corollary 3.1 is also a consequence of Theorem 2.1.

Corollary 3.2. *For problem* (3.1),

$$\frac{Z^{\mathrm{LP}} - Z}{Z^{\mathrm{LP}}} \leq \frac{P-1}{P}.$$

Although it can be shown that the ratio $(Z^{\mathrm{LP}} - Z)/Z^{\mathrm{LP}}$ has a limit of one as P approaches infinity, we do not know whether the bound of $(P-1)/P$ is tight for $P > 2$. Edmonds [4] has proved that there is no duality gap for $P = 2$.

[3] In [5] this result is obtained for p-systems.

4. The locally greedy heuristic

Here we study problem (1.6) and then problem (1.3) when z has a certain symmetry. In problem (1.6) N is partitioned into n smaller sets $\{N_j\}_{j=1}^n$ and the computation required by the greedy heuristic can be reduced by considering these sets separately.

4.1. The locally greedy heuristic for Problem (1.6)

Initialization. Arbitrarily index the sets $\{N_j\}_{j=1}^n$, set $S^0 = \emptyset$ and $t = 1$.

Iteration t. Apply the greedy heuristic to the problem

$$\max_{I \subseteq N_t} \{z(S^{t-1} \cup I) : I \in \mathcal{F}^t\}$$

and let I_t be a greedy solution. Set $S^t = S^{t-1} \cup I_t$. If $t = n$ stop, $S = S^n$ is a locally greedy solution. Otherwise set $t \to t+1$ and continue.

If $n_j = |N_j|$, the locally greedy heuristic requires $\sum_{j=1}^n \mathcal{O}(n_j^2)$ evaluations while the greedy heuristic requires $\mathcal{O}(\sum_{j=1}^n n_j)^2$ evaluations.

Let Z^{LG} be the value of a locally greedy solution.

Theorem 4.1. *If the locally greedy heuristic is applied to problem* (1.6) *then*

$$\frac{Z - Z^{\mathrm{LG}}}{Z - z(\emptyset)} \leq \frac{P}{P+1}.$$

The bound is tight for all P and n.

Proof. Let T be an optimal solution, $T^t = T \cap N_t$, $S = S^n$ a locally greedy solution and $S \cap N_t = I_t$. By Proposition 1.1

$$\begin{aligned} z(T) &\leq z(S) + \sum_{q \in T-S} \rho_q(S) = z(S) + \sum_{t=1}^n \sum_{q \in T^t - I_t} \rho_q(S) \\ &\leq z(S) + \sum_{t=1}^n \sum_{q \in T^t} \rho_q(S) \leq z(S) + \sum_{t=1}^n \sum_{q \in T^t} \rho_q(S^t). \end{aligned}$$

Since $(N_t, \mathcal{F}^t)$ is the intersection of at most P matroids, exactly as in the proof of Theorem 2.1 it can be argued that $\sum_{q \in T^t} \rho_q(S^t) \leq P[z(S^t) - z(S^{t-1})]$. Thus

$$\begin{aligned} Z = z(T) &\leq z(S) + \sum_{t=1}^n P[z(S^t) - z(S^{t-1})] \\ &= Z^{\mathrm{G}} + P[Z^{\mathrm{G}} - z(\emptyset)]. \end{aligned}$$

For $n = 1$ the worst case example of Theorem 2.1 shows that the bound is tight for all P. For arbitrary n, we obtain a worst case example by replicating n times the variables and constraints of the example of Theorem 2.1.

Specializing further, we consider problem (1.3). Recall that this model is a special case of problem (1.6) in which $P = 1$, $M = \{1, \dots, m\}$ is a set of boxes, $E = \{1, \dots, n\}$ is a set of elements to be packed into the boxes, $N = \{(i, j): i \in M, j \in E\}$ and $N_j = \{(i, j): i \in M\}$, $j \in E$. Since $z(S)$ is nondecreasing there is an optimal packing in which all of the elements are included and we can replace the packing constraints $|S \cap N_j| \leq 1$ by the partition constraints $|S \cap N_j| = 1$. These partition constraints are equivalent to $S_i \cap S_k = \emptyset$, $i \neq k$, $\bigcup_{i \in M} S_i = E$, where $S_i = \{j: (i, j) \in S\}$ is the subset of elements placed in box i. Defining $v(S_1, \dots, S_m) = z(S)$ we can restate problem (1.3) as

$$\max_{S_1 \subseteq E, \dots, S_M \subseteq E} \{v(S_1, \dots, S_m): S_i \cap S_k = \emptyset, i \neq k, \bigcup_{i \in M} S_i = E\}. \tag{4.1}$$

Let $\rho_j^i(S_1, \dots, S_m) = v(S_1, \dots, S_i \cup \{j\}, \dots, S_m) - v(S_1, \dots, S_i, \dots, S_m) = z(S \cup \{i, j\}) - z(S)$. Then $z(S)$ is submodular and nondecreasing if and only if $\rho_j^i(S_1, \dots, S_m) \geq \rho_j^i(T_1, \dots, T_m) \geq 0$ for $S_k \subseteq T_k$, $\forall k \in M$, which is equivalent to

$$v(T_1, \dots, T_m) \leq v(S_1, \dots, S_m) + \sum_{i \in M} \sum_{j \in T_i - S_i} \rho_j^i(S_1, \dots, S_m).$$

In the context of problem (4.1) the locally greedy heuristic considers the elements of E in any order and puts each element in whichever box gives the greatest immediate gain.

4.2. The locally greedy heuristic for Problem (4.1)

Initialization. Assume the elements of E are ordered $1, \dots, n$, set $S_i^0 = \emptyset$, $i \in M$ and $t = 1$.

Iteration t. Find $\rho_t = \rho_t^{i^*}(S_1^{t-1}, \dots, S_m^{t-1}) = \max_i \rho_t^i(S_1^{t-1}, \dots, S_m^{t-1})$ with ties settled arbitrarily. Set $S_{i^*}^t = S_{i^*}^{t-1} \cup \{t\}$ and $S_i^t = S_i^{t-1}$ for $i \neq i^*$. If $t = n$ stop. Otherwise set $t \to t + 1$ and continue.

We know that setting $P = 1$ in Theorem 4.1 yields $(Z - Z^{\mathrm{LG}})/(Z - z(\emptyset)) \leq \frac{1}{2}$ for problem (4.1).

Suppose now that $z(S) = v(S_1, \dots, S_m)$ is *symmetric*, i.e.,

$$\begin{aligned} &v(S_1, \dots, S_{i-1}, S_i, S_{i+1}, \dots, S_{k-1}, S_k, S_{k+1}, \dots, S_m) \\ &\quad = v(S_1, \dots, S_{i-1}, S_k, S_{i+1}, \dots, S_{k-1}, S_i, S_{k+1}, \dots, S_m) \quad \text{for all pairs } i, k. \end{aligned}$$

This condition is satisfied when $v(S_1, \dots, S_m) = \sum_{i \in M} \tilde{v}(S_i)$, where $\tilde{v}$ is a nondecreasing submodular function on E and more generally means that the boxes are identical.

Theorem 4.2. *If the locally greedy heuristic is applied to problem* (4.1) *and* v *is* *symmetric, then*

$$\frac{Z - Z^{\mathrm{LG}}}{Z - z(\emptyset)} \le \frac{m-1}{2m-1}.$$

The bound is tight for all m.

Proof. Let $(T_1, \dots, T_m)$ be an optimal partition of E and $(S_1, \dots, S_m)$ a locally greedy partition. Submodularity of v implies

$$v(T_1, \dots, T_m) \le v(S_1, \dots, S_m) + \sum_{i \in M} \sum_{t \in T_i - S_i} \rho_t^i(S_1, \dots, S_m).$$

Furthermore $\rho_t \ge \rho_t^i(S_1^{t-1}, \dots, S_m^{t-1}) \ge \rho_t^i(S_1, \dots, S_m)$, where the first inequality is implied by the locally greedy heuristic and the second by submodularity. Therefore

$$\begin{aligned} Z = v(T_1, \dots, T_m) &\le v(S_1, \dots, S_m) + \sum_{i \in M} \sum_{t \in T_i - S_i} \rho_t \\ &= v(S_1, \dots, S_m) + \sum_{i \in M} \sum_{t \in T_i} \rho_t - \sum_{i \in M} \sum_{t \in T_i \cap S_i} \rho_t \\ &= v(S_1, \dots, S_m) + \sum_{t=1}^{n} \rho_t - \sum_{i \in M} \sum_{t \in T_i \cap S_i} \rho_t \\ &= 2Z^{\mathrm{LG}} - z(\emptyset) - \sum_{i \in M} \sum_{t \in T_i \cap S_i} \rho_t, \end{aligned}$$

where $z(\emptyset) = v(\emptyset, \dots, \emptyset)$ and $Z^{\mathrm{LG}} = v(S_1, \dots, S_m) = \sum_{t=1}^{n} \rho_t + z(\emptyset)$.

Using the fact that v is symmetric, we note that $v(T_1, \dots, T_m) = v(T_{(1+k) \bmod m}, \dots, T_{(m+k) \bmod m})$ for $k = 0, 1, \dots, m-1$. Hence

$$Z \le 2Z^{\mathrm{LG}} - z(\emptyset) - \sum_{i \in M} \sum_{t \in T_{(i+k) \bmod m} \cap S_i} \rho_t, \quad k = 0, 1, \dots, m-1.$$

Since $\bigcup_{k=0}^{m-1} T_{(i+k) \bmod m} \cap S_i = S_i$, summing these last inequalities over $k = 0, \dots, m-1$ yields

$$\begin{aligned} mZ &\le m[2Z^{\mathrm{LG}} - z(\emptyset)] - \sum_{i \in M} \sum_{t \in S_i} \rho_t \\ &= m[2Z^{\mathrm{LG}} - z(\emptyset)] - \sum_{t=1}^{n} \rho_t = m[2Z^{\mathrm{LG}} - z(\emptyset)] - [Z^{\mathrm{LG}} - z(\emptyset)], \end{aligned}$$

or $(2m-1)(Z - Z^{\mathrm{LG}}) \le (m-1)(Z - z(\emptyset))$, and the result follows.

We now show that the bound is tight by considering a matroid partition problem for which Edmonds [3] has given a good algorithm. We are given a multigraph $G = (V, E)$ that is to be partitioned into m disjoint forests containing a maximum number of edges. An equivalent statement of this problem is to partition the edges among m boxes, where the value of the ith box is the cardinality of a largest forest that can be constructed from the edges in it.

Take the multigraph G on $2m$ nodes, consisting of a complete bipartite graph $(V_1 \cup V_2, V_1 \times V_2)$, where $V_1 = \{1, \dots, m\}$ and $V_2 = \{m+1, \dots, 2m\}$, and m copies of the edge $(i, i+1)$ $i = 1, \dots, m-1$. An example for $m = 3$ is given in Fig. 1.

Consider the disjoint trees $\{T_i\}_{i=1}^m$, where T_i has $2m-1$ edges: $(j, j+1)$ for $1 \le j \le m-1$, and $(i, m+j)$ for $1 \le j \le m$. Since these trees use all of the edges in G, they represent an optimal solution and $Z = m(2m-1)$. On the other hand the locally greedy heuristic can choose the m disjoint trees $\{S_i^{m^2}\}_{i=1}^m$ during the first m^2 iterations where $S_i^{m^2}$ has m edges $(j, m+i)$ for $1 \le j \le m$. The succeeding iterations contribute nothing to the value of this solution. Hence $Z^{\mathrm{LG}} = m^2$ and $(Z - Z^{\mathrm{LG}})/Z = (m-1)/(2m-1)$.

5. The interchange heuristic

Finally, we consider the interchange heuristic, which is a local improvement procedure. For problem (1.5), we show that for $P = 1$ the interchange and greedy heuristics have the same worst case behavior, but for $P \ge 2$ the interchange heuristic can behave arbitrarily badly.

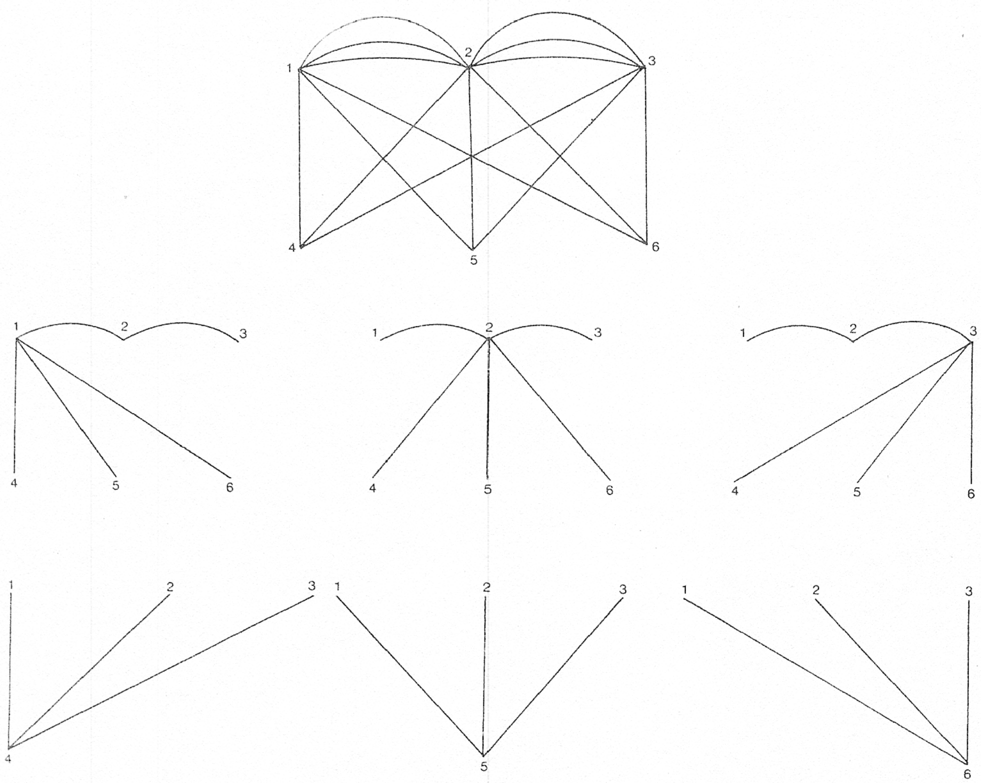

Fig. 1.

5.1. The interchange heuristic for set functions on independence systems $(N, \mathscr{F})$

Initialization. Pick an arbitrary independent set $S^0 \subseteq N$. Set $t = 1$.

Iteration t. Given a set S^{t-1} try to find an independent set $Q \subseteq N$ such that $|Q - S^{t-1}| \leq 1, |S^{t-1} - Q| \leq 1$ and $z(Q) > z(S^{t-1})$. If such a Q exists set $S^t = Q$, $t \to t+1$ and continue. If not $S = S^{t-1}$ is an interchange solution.

Let Z^{I} denote the value of an interchange solution.

Theorem 5.1. *If the interchange heuristic is applied to problem* (1.5), *then*

(i) $(Z - Z^{\mathrm{I}})/(Z - z(\emptyset)) \leq \frac{1}{2}$ *if* $P = 1$, *and this bound is tight,*

(ii) $(Z - Z^{\mathrm{I}})/(Z - z(\emptyset)) \leq 1$ *if* $P \geq 2$, *and this bound is tight.*

Proof. (i) As the independence system is a matroid and z is nondecreasing we can assume, without loss of generality, that an interchange solution S and an optimal solution T are both bases of the matroid. Let $e_1, \dots, e_K$ be any ordering of the elements of S, $S_k = \{e_1, \dots, e_{k-1}\}$ and $\rho_{k-1} = \rho_{e_k}(S_{k-1})$, $k = 1, \dots, K$. It is shown in von Randow [9, p. 82, Th. 25] that there exists a bijection $f: S \to T$ such that $(S - \{e\}) \cup \{f(e)\} \in \mathscr{F}, \forall e \in S$. Now

$$\begin{aligned}\sum_{j \in T} \rho_j(S) &= \sum_{i=1}^{K} \rho_{f(e_i)}(S) \leq \sum_{i=1}^{K} \rho_{f(e_i)}(S - \{e_i\}) \\ &\leq \sum_{i=1}^{K} \rho_{e_i}(S - \{e_i\}) \leq \sum_{i=1}^{K} \rho_{e_i}(S_{i-1}) = \sum_{i=0}^{K-1} \rho_i,\end{aligned}$$

where the first and third inequalities follow from submodularity (Proposition 1.1), and the second from the fact that S is an interchange solution. Therefore

$$\begin{aligned}Z = z(T) &\leq z(S) + \sum_{j \in T-S} \rho_j(S) \\ &\leq z(S) + \sum_{i=0}^{K-1} \rho_i = 2z(S) - z(\emptyset) = 2Z^{\mathrm{I}} - z(\emptyset).\end{aligned}$$

To show that this bound is tight consider the problem

$$\begin{aligned}\text{maximize} \quad & x_1 + x_2 + x_3 - x_1 x_3, \\ \text{subject to} \quad & x_1 + x_2 \leq 1, \\ & x_3 \leq 1, \\ & x_i \in \{0, 1\}, \quad i = 1, 2, 3,\end{aligned}$$

which is the example of Theorem 2.1 for $P = 1$. The optimal solution is given by $x_1 = 0$, $x_2 = x_3 = 1$ and an interchange solution is given by $x_1 = 1$, $x_2 = x_3 = 0$. Hence $(Z - Z^{\mathrm{I}})/Z = \frac{1}{2}$.

(ii) To show that the bound is tight for $P \geq 2$, consider the family of problems

$$\begin{aligned}\text{maximize} \quad & x_{P+1}, \\ \text{subject to} \quad & x_i + x_{P+1} \leq 1, \quad i = 1, \dots, P, \\ & x_i \in \{0, 1\}, \quad i = 1, \dots, P+1.\end{aligned}$$

The optimal solution is given by $x_i = 0$, $i = 1, \dots, P$, $x_{P+1} = 1$ and an interchange solution is given by $x_i = 1$, $i = 1, \dots, P$, $x_{P+1} = 0$. Therefore $(Z - Z^{\mathrm{I}})/Z = 1$.

6. Conclusions

A number of other details that we studied in [7] for problem (1.1) could also be considered for problem (1.5). These include (1) elimination of the nondecreasing assumption, (2) more general heuristics, such as R-step greedy and interchange, and combining a heuristic with partial enumeration and (3) a linear programming relaxation for (1.5) that yields $(Z^{\mathrm{LP}} - Z^{\mathrm{G}})/Z^{\mathrm{LP}} \leq P/(P+1)$ for $P \leq 2$ when $z(S)$ is generated from a matroid as indicated in [7, Section 6]. However, since the bounds obtained here are already poor, none of these generalizations seem to be worthy of a detailed development, much of which would parallel that in [7]. We believe, instead, that the value of the development here is its use as a framework for studying narrower classes of problems and other heuristics for which sharper results might be obtained.

We are also interested in determining optimal heuristics, i.e., a heuristic that guarantees the best worst case bound for a fixed amount of computation. This problem for the maximization of submodular functions will be the subject of a forthcoming paper.

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